Dark solitons in external potentials

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References:

D.P., D. Frantzeskakis, P. Kevrekidis, Phys. Rev. E 72, 016615 (2005)

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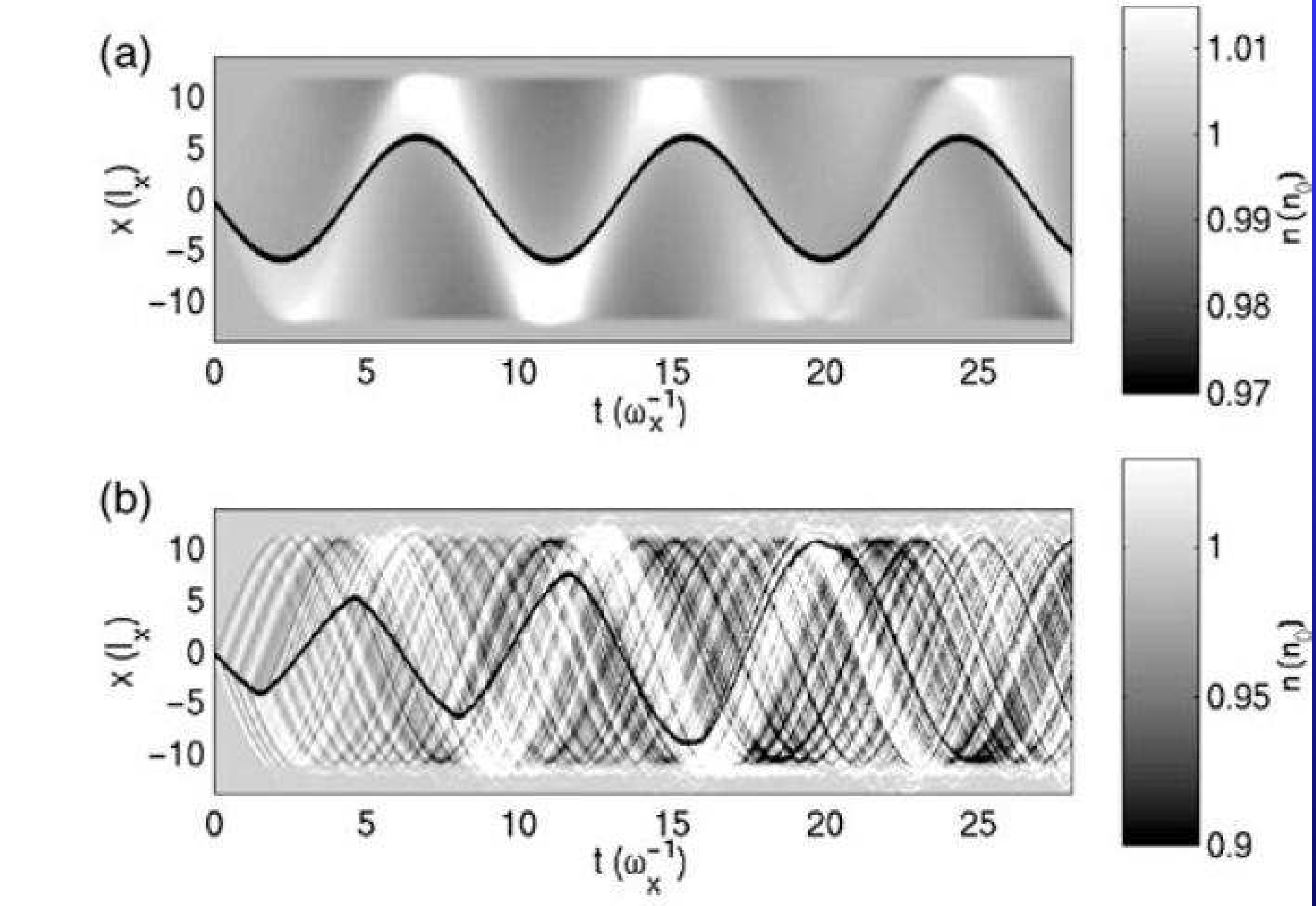
Numerical simulations



Questions:

Find approximations of the frequency of oscillations of a dark soliton and study long-time changes in the amplitude of oscillations.

Set-up

Dark solitons in Bose–Einstein condensates

$$iu_t = -u_{xx} + |u|^2 u + \epsilon V(x) u,$$

where ϵ is small and $V(x) : \mathbb{R} \mapsto \mathbb{R}$ is a smooth, exponentially decaying function such that

$$\exists C > 0,\ \kappa > 0 : \quad |V(x)| \leq C e^{-\kappa |x|}, \quad \forall x \in \mathbb{R}$$

Example: symmetric external potentials

$$V_1(x) = -\mathrm{sech}^2\left(\frac{\kappa x}{2}\right), \qquad V_2(x) = x^2 e^{-\kappa |x|}, \qquad x \in \mathbb{R}.$$

More general context: periodic and confining potentials $V(x)$.

Approaches to the solution at glance

$$iu_t = -u_{xx} + |u|^2 u + \epsilon V(x) u,$$

- $\epsilon = 0$ - existence and stability of dark solitons is known
- $\epsilon \ll 1$ - persistence of solutions by using the method of Lyapunov–Schmidt reductions
- $\epsilon \ll 1$ - stability of solutions using the Evans function method and the negative index theory
- $\epsilon \neq 0$ - long-time dynamics by using Newton's law of motion and central manifold reductions

Main results

1. A black soliton $u = \phi_0(x-s)e^{-it}$ with $\phi_0 \to \pm 1$ as $x \to \pm\infty$ persists for small $\epsilon \neq 0$ if $M'(s) = 0$ and $M''(s) \neq 0$, where

$$M'(s) = \int_{\mathbb{R}} V'(x) \left[1 - \phi_0^2(x-s)\right] dx.$$

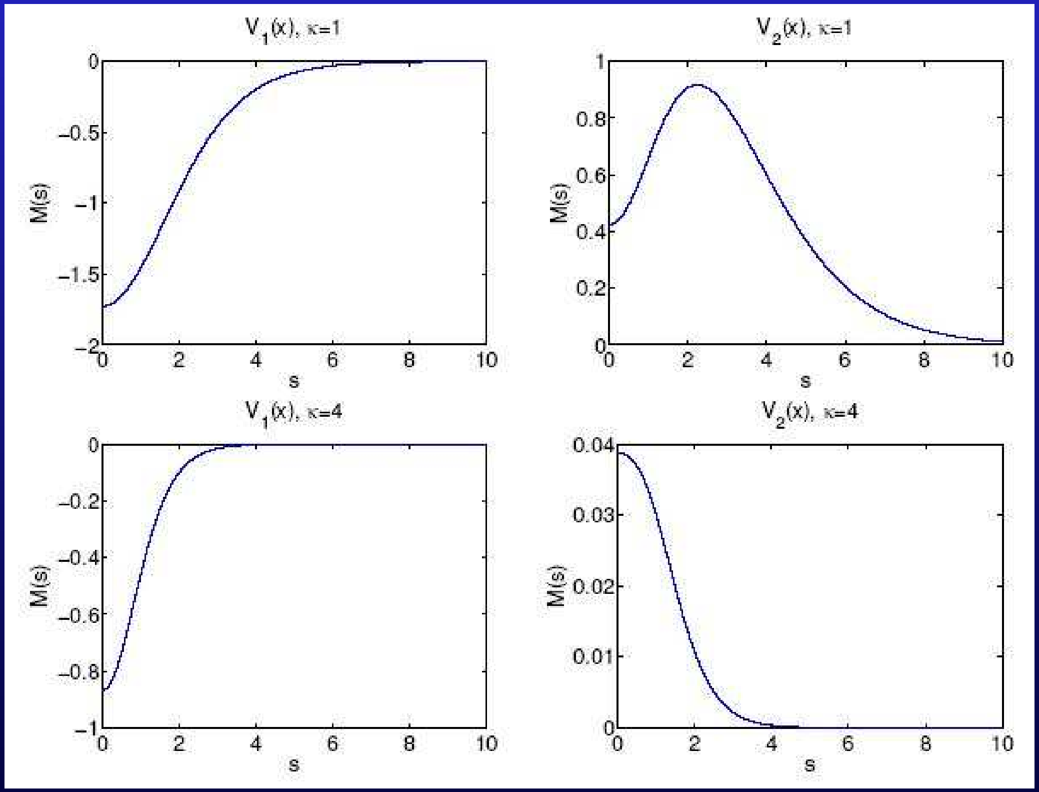
2. If a black soliton is spectrally stable for $\epsilon = 0$, then it is spectrally unstable for small $\epsilon \neq 0$ with one real positive eigenvalue if $M''(s) < 0$ and two complex-conjugate eigenvalues if $M''(s) > 0$.

3. If $u(x,0)$ is close to $\phi_\epsilon(x - s(0))$, then $u(x,t)$ remains close to $\phi_\epsilon(x-s(t))e^{-it}$, where $s(t)$ solves for $0 < t < C/\epsilon$

$$\mu_0 \ddot{s} - \epsilon \lambda_0 M''(s) \dot{s} + \epsilon M'(s) = \mathrm{O}(\epsilon^2), \qquad \lambda_0, \mu_0 > 0.$$

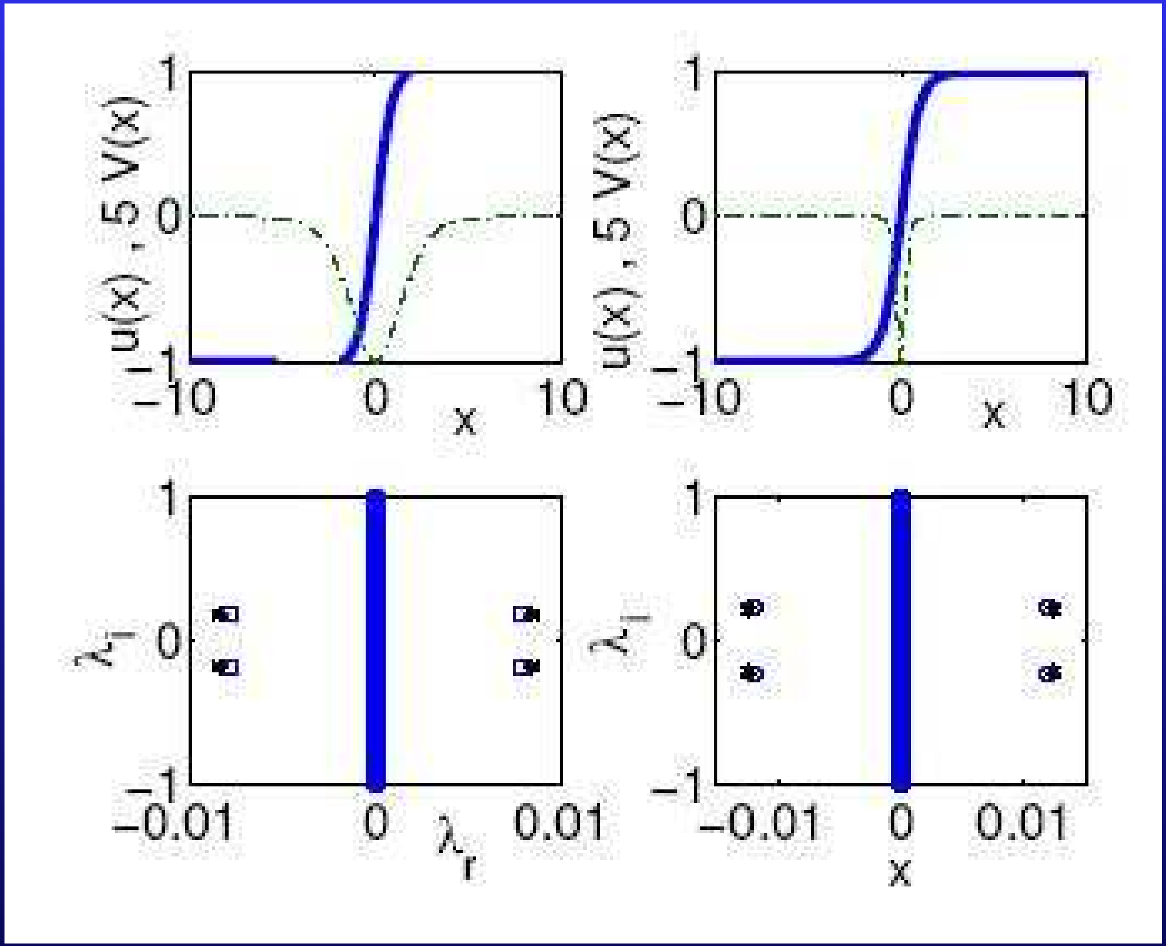
Applications

- If $V(-x) = V(x)$, then $M'(0) = 0$ and the black soliton with $s = 0$ persists for $\epsilon \neq 0$.
- Additional roots $s = \pm s_0$ may exist if $\mathrm{sign}(M(0)M''(0)) = 1$ since $M(s) \to 0$ as $s \to \infty$.



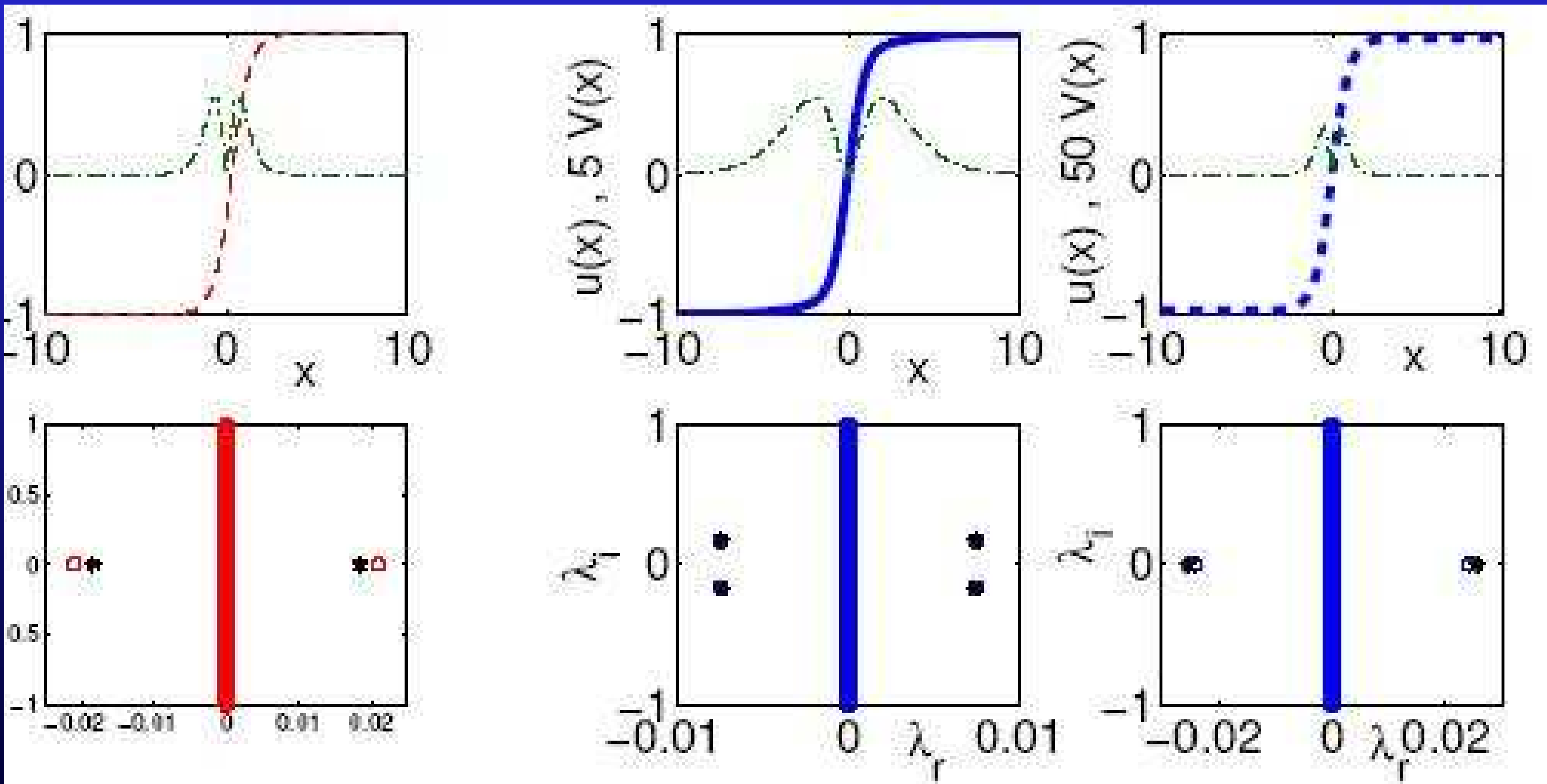
Example: $V_1 = -\text{sech}^2\left(\frac{\kappa x}{2}\right)$

Only one solution persists with $s_0 = 0$ and $M''(0) > 0$



Example: $V_2 = x^2 e^{-\kappa|x|}$

For $\kappa < 3.21$, three solutions persist with $s_0 = 0$ ($M''(0) > 0$) and $s_0 = \pm s_*$ ($M''(s_*) < 0$). For $\kappa > 3.21$, only one solution persists with $s_0 = 0$ and $M''(0) < 0$

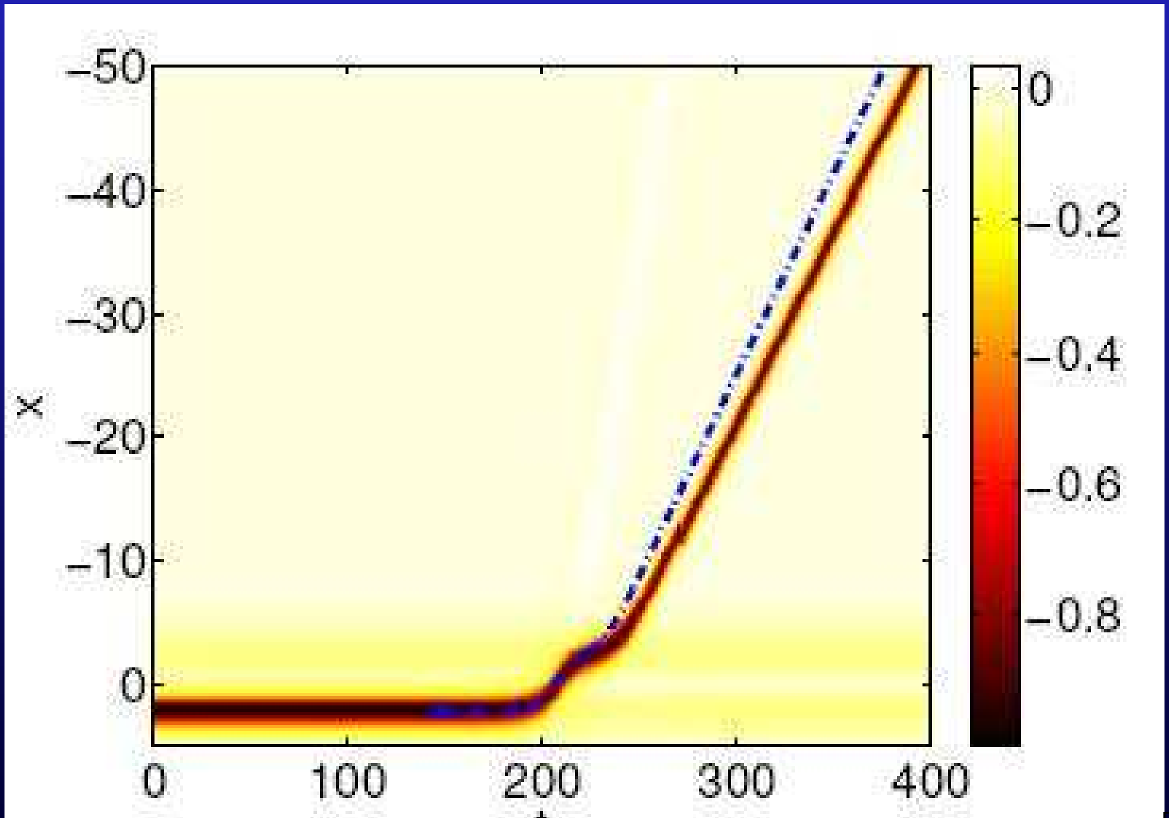


Nonlinear dynamics of instability

Newton's particle equation

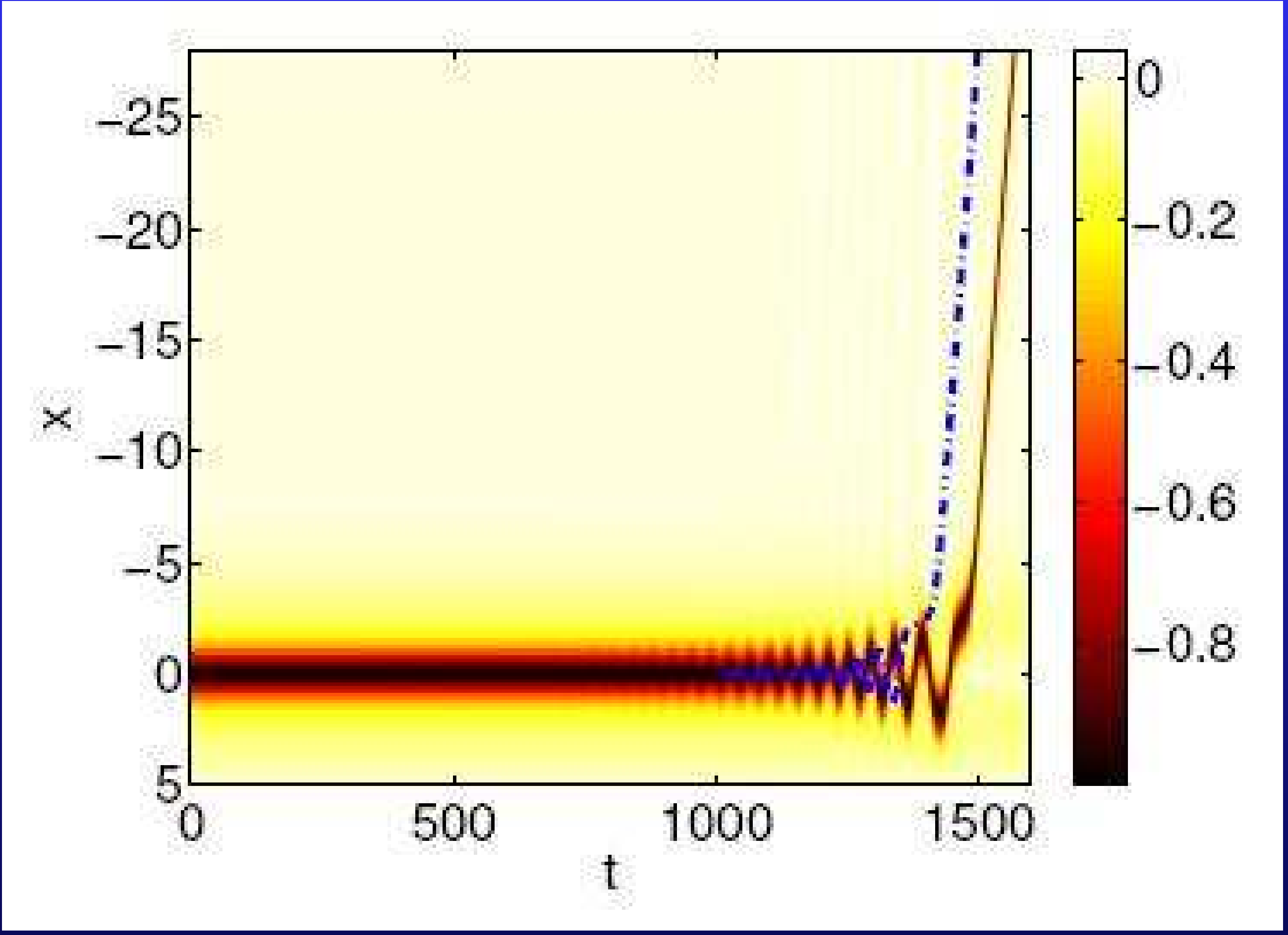
$$\mu_0 \ddot{s} - \epsilon \lambda_0 M''(s) \dot{s} + \epsilon M'(s) = \mathrm{O}(\epsilon^2), \qquad \lambda_0, \mu_0 > 0.$$

Real instability for $V_2(x)$, $\kappa < 3.21$ and $s_0 = s_* \neq 0$



Nonlinear dynamics of instability

Complex instability for $V_2(x)$, $\kappa < 3.21$ and $s_0 = 0$



Nonlinear dynamics of instability

Real instability for $V_2(x)$, $\kappa > 3.21$ and $s_0 = 0$