Well-posedness and stability in the integrable systems

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Structure of Talk

I will speak on two particular problems for nonlinear PDEs:

- Global well-posedness for the derivative NLS equation

$$\begin{cases} iu_t + u_{xx} + i(|u|^2 u)_x = 0, & t > 0, \\ u|_{t=0} = u_0. \end{cases}$$

- Orbital stability for the massive Thirring model (MTM)

$$\begin{cases} i(u_t + u_x) + v = u|v|^2, & t > 0, \\ i(v_t - v_x) + u = v|u|^2, \\ (u, v)|_{t=0} = (u_0, v_0). \end{cases}$$

Both nonlinear PDEs belong to the class of integrable systems with the inverse scattering transform method.

Inverse scattering (for the DNLS equation)

Denote $Q(u) = \begin{bmatrix} 0 & u \\ -\overline{u} & 0 \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and consider two linear equations for $\psi(x,t) \in \mathbb{C}^2$:

$$\partial_x \psi = \left[-i\lambda^2 \sigma_3 + \lambda Q(u)\right] \psi$$

and

$$\partial_t \psi = \left[-2i\lambda^4 \sigma_3 + 2\lambda^3 Q(u) + i\lambda^2 |u|^2 \sigma_3 - \lambda |u|^2 Q(u) + i\lambda \sigma_3 Q(u_x)\right] \psi,$$

where $\lambda \in \mathbb{C}$ is the (x,t)-independent spectral parameter.

Lax representation:

Consider smooth ψ and u as functions of (x,t). Then, $\partial_x \partial_t \psi = \partial_t \partial_x \psi$ if and only if $iu_t + u_{xx} + i(|u|^2 u)_x = 0$.

Zakharov-Shabat (1972), Ablowitz-Kaup-Newell-Segur (1974), Kaup-Newell (1976), and many more...

Miracles of the integrable nonlinear PDEs

- A countable set of time-conserved quantities in some Sobolev spaces
- A rich set of exact analytic solutions given by elementary and elliptic functions (solitary waves, periodic waves, rogue waves, etc.)
- Bäcklund and Darboux transformations to add or to remove solitons
- The inverse scattering transform as a nonlinear Fourier transform

Quick Review: Well-posedness for dispersive PDEs

For the general Cauchy problem:

$$\begin{cases} iu_t + \Delta u + N(u) = 0, \\ u|_{t=0} = u_0 \in X, \end{cases}$$

where X is some Banach space and $N(u)$ is a nonlinear term.

The Cauchy problem is locally well-posed in X if there exists an unique solution $u(t,\cdot) \in X$ for $t \in (-T, T)$ with finite $T > 0$ and the solution map $u_0 \mapsto u(t,\cdot)$ is continuous.

The Cauchy problem is globally well-posed if T can be arbitrarily large.

The proof relies usually on the integral form obtained by Duhamel's formula:

$$u = U(t)u_0 + i\int_0^t U(t-s)N(u(s))ds, \quad U(t)u_0 := \mathcal{F}^{-1}(e^{-i|\xi|^2 t}\hat{u}_0).$$

Fixed-point argument: Define the map $\mathcal{M}$ on some Banach space X

$$\mathcal{M}u := U(t)u_0 + \int_0^t U(t-s)N(u(s))ds.$$

We need to prove for some $\|u_0\|_X$-dependent $T > 0$ that
(a) $\mathcal{M}$ maps $L^\infty((-T,T),X_0)$ to itself, where X_0 is a closed subset of X
(b) $\mathcal{M}$ is a contraction in $L^\infty((-T,T),X_0)$.
Then, local well-posedness holds with the solution $u \in L^\infty((-T,T),X)$.

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Continuation argument by energy:
If there exists a time-independent quantity $E(u)$, defined for $u(t,\cdot) \in X$ and $t \in (-T,T)$, such that

$$\|u(t,\cdot)\|_X \leq C(E(u)) = C(E(u_0)),$$

then the norm of u in X is bounded by a t-independent constant.
Repeating fixed-point arguments k times, we extend solutions for $t \in (-kT^* - T, T + kT^*)$. This implies global well-posedness.

Global well-posedness of the DNLS equation

For the Cauchy problem related to the derivative NLS equation

$$\begin{cases} iu_t + u_{xx} + i(|u|^2 u)_x = 0, & t > 0, \\ u|_{t=0} = u_0 \in X = H^s(\mathbb{R}), \end{cases}$$

- Tsutsumi & Fukuda (1980) established local well-posedness in H^s with $s > \frac{3}{2}$ and extended solutions globally in H^2 for small data in H^1
- Hayashi (1993) used gauge transformation of DNLS to a system of semi-linear NLS and established local and global well-posedness in H^1 under the condition $\|u_0\|_{L^2} < \sqrt{2\pi}$.
- Takaoka (1999) proved local well-posedness in H^s with $s \geq \frac{1}{2}$ by using Fourier restriction method.
- Global existence was proved in H^s for $s > \frac{32}{33}$ (Takaoka, 2001), $s > \frac{1}{2}$ (Colliander et al, 2002), and $s = \frac{1}{2}$ (Mio-Wu-Xu, 2011), under the same constraint $\|u_0\|_{L^2} < \sqrt{2\pi}$.

Why constraint $\|u_0\|_{L^2} < \sqrt{2\pi}$?

First three conserved quantities of the DNLS equation:

$$
\begin{aligned}
I_0 &= \int_{\mathbb{R}} |u|^2 dx, \\
I_1 &= i \int_{\mathbb{R}} (\bar{u} u_x - u \bar{u}_x) dx - \int_{\mathbb{R}} |u|^4 dx, \\
I_2 &= \int_{\mathbb{R}} |u_x|^2 dx + \frac{3i}{4} \int_{\mathbb{R}} |u|^2 (u \bar{u}_x - u_x \bar{u}) dx + \frac{1}{2} \int_{\mathbb{R}} |u|^6 dx.
\end{aligned}
$$

By the gauge transformation $u = v e^{-\frac{3i}{4} \int_{-\infty}^{x} |v(y)|^2 dy}$ and the Gagliardo–Nirenberg inequality $\|f\|_{L^6}^6 \leq \frac{4}{\pi^2} \|f\|_{L^2}^4 \|f_x\|_{L^2}^2$,

$$
I_2 = \|v_x\|_{L^2}^2 - \frac{1}{16} \|v\|_{L^6}^6 \geq \left(1 - \frac{1}{4\pi^2} \|v\|_{L^2}^4 \right) \|v_x\|_{L^2}^2.
$$

Hence, we must require $1 - \frac{1}{4\pi^2} \|v\|_{L^2}^4 > 0$.

Open question:

Is $\|u_0\|_{L^2} < \sqrt{2\pi}$ optimal? Is there a blowup in a finite time for large data?

Analogy is the quintic NLS equation

$$\left\{ \begin{array}{l} iu_t + u_{xx} + |u|^4 u = 0, \quad t > 0, \\ u|_{t=0} = u_0 \in X = H^1(\mathbb{R}), \end{array} \right.$$

There is a finite C_0 such that the solution is global if $\|u_0\|_{L^2} < C_0$ and blowup in a finite time if $\|u_0\|_{L^2} > C_0$.

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The answer to the open question may be NO!

- Colin-Ohta (2006) proved orbital stability of solitons, for which $\|u_0\|_{L^2}$ may exceed $\sqrt{2\pi}$.
- Wu (2014) shows global well-posedness in H^1 with $\|u_0\|_{L^2} < 2\sqrt{\pi}$.
- Liu-Simpson-Sulem (2013) found no blowup in numerical studies of the Cauchy problem.

Why inverse scattering transform?

Because it is a nonlinear Fourier transform which requires no use of energy.

The linear case $iu_t + u_{xx} = 0$:
The Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is bijective and

$$u(x,t) = \mathcal{F}^{-1}(\mathcal{F}(u_0)e^{it\xi^2}), \quad u_0 \in L^2(\mathbb{R}), \quad t \in \mathbb{R}.$$

Moreover, $\mathcal{F} : H^s(\mathbb{R}) \cap L^{2,s}(\mathbb{R}) \to H^s(\mathbb{R}) \cap L^{2,s}(\mathbb{R})$ is also bijective.

The nonlinear case:
Bijectivity of the inverse scattering was studied by Deift–Zhou (1998,2003) for focusing/defocusing cubic NLS equation and modified KdV equation. All works on derivative NLS were formal so far, including Lee (1989), Kitaev-Vartanian (1997), Xu-Fan (2012).

Main result

Recall the Kaup-Newel spectral problem for derivative NLS:

$$\text{(KN)} \quad \partial_x \psi = \left[-i\lambda^2 \sigma_3 + \lambda Q(u) \right] \psi, \quad \psi \in \mathbb{C}^2.$$

Theorem (P-S, 2015)

For every $u_0 \in H^2(\mathbb{R}) \cap L^{2,2}(\mathbb{R})$ such that (KN) admits no eigenvalues or resonances, there exists a unique global solution $u(t, \cdot) \in H^2(\mathbb{R}) \cap L^{2,2}(\mathbb{R})$ of the Cauchy problem for every $t \in \mathbb{R}$. Furthermore, the map

$$H^2(\mathbb{R}) \cap L^{2,2}(\mathbb{R}) \ni u_0 \mapsto u \in C(\mathbb{R}, H^2(\mathbb{R}) \cap L^{2,2}(\mathbb{R}))$$

is Lipschitz.

- Eigenvalues of (KN) are related to solitons, excluded for simplification.
- Resonances of (KN) are non-generic and require special study.
- A parallel ongoing work is by Liu-Perry-Sulem (2015).

Direct scattering problem

Kaup-Newel spectral problem for derivative NLS:

$$\partial_x \psi = (-i\lambda^2 \sigma_3 + \lambda Q(u))\psi, \quad Q(u) = \begin{bmatrix} 0 & u \\ -\overline{u} & 0 \end{bmatrix}$$

Jost functions with asymptotical values from the case $Q(u) \equiv 0$:

$$\Psi_\pm(x;\lambda) \to e^{-i\lambda^2 x \sigma_3} \quad \text{as } x \to \pm\infty.$$

They are bounded for every $\lambda^2 \in \mathbb{R}$.

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Jost functions in $\Psi_\pm := e^{-i\lambda^2 x\sigma_3}[\varphi_\pm, \phi_\pm]$ satisfy Volterra's equations

$$\varphi_\pm(x;\lambda) = e_1 + \lambda \int_{\pm\infty}^{x} \begin{bmatrix} 1 & 0 \\ 0 & e^{2i\lambda^2(x-y)} \end{bmatrix} Q(u(y))\varphi_\pm(y;\lambda)dy,$$

$$\phi_\pm(x;\lambda) = e_2 + \lambda \int_{\pm\infty}^{x} \begin{bmatrix} e^{-2i\lambda^2(x-y)} & 0 \\ 0 & 1 \end{bmatrix} Q(u(y))\phi_\pm(y;\lambda)dy.$$

Fixed point arguments are not uniform in λ as $|\lambda| \to \infty$ if $Q(u) \in L^1(\mathbb{R})$.

The way around this obstacle

Introduce transformations $m_\pm := T_1\varphi_\pm$ and $n_\pm := T_2\phi_\pm$, where

$$T_1(x;\lambda) = \begin{bmatrix} 1 & 0 \\ -\overline{u}(x) & 2i\lambda \end{bmatrix}, \quad T_2(x;\lambda) = \begin{bmatrix} 2i\lambda & -u(x) \\ 0 & 1 \end{bmatrix},$$

Then, Volterra's equations become

$$m_\pm(x;z) = e_1 + \int_{\pm\infty}^{x} \begin{bmatrix} 1 & 0 \\ 0 & e^{2iz(x-y)} \end{bmatrix} Q_1(u(y)) m_\pm(y;z) dy,$$

$$n_\pm(x;z) = e_2 + \int_{\pm\infty}^{x} \begin{bmatrix} e^{-2iz(x-y)} & 0 \\ 0 & 1 \end{bmatrix} Q_2(u(y)) n_\pm(y;z) dy,$$

where $z := \lambda^2$ and

$$Q_1(u) = \frac{1}{2i} \begin{bmatrix} |u|^2 & u \\ -2i\overline{u}_x - \overline{u}|u|^2 & -|u|^2 \end{bmatrix}, \quad Q_2(u) = \frac{1}{2i} \begin{bmatrix} |u|^2 & -2iu_x + u|u|^2 \\ -\overline{u} & -|u|^2 \end{bmatrix}.$$

Instead of one Kaup-Newell spectral problem,
we have two Zakharov-Shabat-type spectral problems!

Choice of spaces

From the condition $Q_{1,2}(u) \in L^1(\mathbb{R})$, where

$$Q_1(u) = \frac{1}{2i}\begin{bmatrix} |u|^2 & u \\ -2i\overline{u}_x - \overline{u}|u|^2 & -|u|^2 \end{bmatrix}, \quad Q_2(u) = \frac{1}{2i}\begin{bmatrix} |u|^2 & -2iu_x + u|u|^2 \\ -\overline{u} & -|u|^2 \end{bmatrix},$$

we realize that $u \in L^1(\mathbb{R}) \cap L^3(\mathbb{R})$ and $\partial_x u \in L^1(\mathbb{R})$ is the best choice for the potential u. With $u \in L^\infty(\mathbb{R})$, it only gets better!

- There exist unique L^∞ solutions $m_\pm(\cdot; z)$ for every $z \in \mathbb{R}$.
- For every $x \in \mathbb{R}$, $m_\mp(x; \cdot)$, $n_\pm(x; \cdot)$ are continued analytically in $\mathbb{C}^\pm$.
- Limits of $m_\mp(x; z)$, $n_\pm(x; z)$ as $|z| \to \infty$ are defined in $\mathbb{C}^\pm$.

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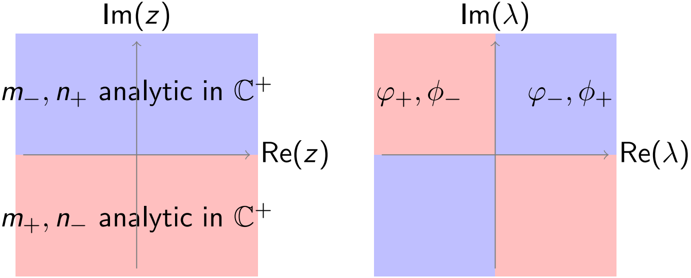
To use Fourier theory, it is better to work in $H^{1,1}(\mathbb{R})$ with $u, \partial_x u \in L^{2,1}(\mathbb{R})$.

Summary on Jost functions

Assume $u_0 \in H^{1,1}(\mathbb{R})$ for initial data of DNLS.



$$z := \lambda^2$$

Spectral data

From basic ODE theory, it follows that each Jost function is spanned by the two others:

$$\begin{bmatrix} \varphi_-(x;\lambda) \\ \phi_-(x;\lambda) \end{bmatrix} = \begin{bmatrix} a(\lambda) & b(\lambda)e^{2i\lambda^2 x} \\ -\overline{b(\bar{\lambda})}e^{-2i\lambda^2 x} & \overline{a(\bar{\lambda})} \end{bmatrix} \begin{bmatrix} \varphi_+(x;\lambda) \\ \phi_+(x;\lambda) \end{bmatrix},$$

where the scattering coefficients are x-independent from Wronskian determinants:

$$a(\lambda) = W(\varphi_-, \phi_+), \quad b(\lambda) = e^{-2i\lambda^2 x} W(\varphi_+, \varphi_-).$$

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- a is continued analytically in $\mathbb{C}^+$ for $z := \lambda^2$
- a converges to a limit a_∞ as $|z| \to \infty$
- $a - a_\infty$, $\lambda b(\lambda)$, and $\lambda^{-1} b(\lambda)$ are $H^1(\mathbb{R})$ w.r.t. z.

What do our assumptions give?

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What do our assumptions give?

- No eigenvalues in (KN): $a(\lambda) \neq 0$ for every $\lambda^2 \in \mathbb{C}^+$.
- No resonances in (KN): $a(\lambda) \neq 0$ for every $\lambda^2 \in \mathbb{R}$.

Time evolution of spectral data

Since $a - a_\infty$, $\lambda b(\lambda)$, and $\lambda^{-1} b(\lambda)$ are $H^1(\mathbb{R})$ w.r.t. z, and a does not vanish on $\mathbb{R}$, we define the spectral data by

$$r_+(z) := -\frac{b(\lambda)}{2i\lambda a(\lambda)}, \quad r_-(z) := \frac{2i\lambda b(\lambda)}{a(\lambda)},$$

so that $r_\pm(z) \in H^1(\mathbb{R})$.

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Time evolution is found from the Lax system of linear equations:

$$r_\pm(z,t) = r_\pm(z,0) e^{4iz^2 t},$$

since the Cauchy problem for derivative NLS equation is locally well-posed. However, if $r_\pm(z,0) \in H^1(\mathbb{R})$, then $r_\pm(z,t) \notin H^1(\mathbb{R})$, because

$$\partial_z r_\pm(z,t) = \left[\partial_z r_\pm(z,0) + 8itz r_\pm(z,0)\right] e^{4iz^2 t}.$$

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The way around is to require $u_0 \in H^2(\mathbb{R}) \cap L^{2,2}(\mathbb{R})$, which result in $r_\pm \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$.

Inverse scattering transform

The scattering relations

$$\begin{bmatrix} \varphi_-(x;\lambda) \\ \phi_-(x;\lambda) \end{bmatrix} = \begin{bmatrix} a(\lambda) & b(\lambda)e^{2i\lambda^2 x} \\ -\overline{b(\bar{\lambda})}e^{-2i\lambda^2 x} & \overline{a(\bar{\lambda})} \end{bmatrix} \begin{bmatrix} \varphi_+(x;\lambda) \\ \phi_+(x;\lambda) \end{bmatrix},$$

can be written as the Riemann–Hilbert problem in the λ complex plane

$$\Phi_+(x;\lambda) - \Phi_-(x;\lambda) = \Phi_-(x;\lambda) S(x;\lambda),$$

where

$$\Phi_+(x;\lambda) := \left[\frac{\varphi_-(x;\lambda)}{a(\lambda)}, \phi_+(x;\lambda) \right], \quad \Phi_-(x;\lambda) := \left[\varphi_+(x;\lambda), \frac{\phi_-(x;\lambda)}{\bar{a}(\lambda)} \right].$$

are analytically continued in the upper and lower half plane of $z := \lambda^2$ with the jump on the line $z \in \mathbb{R}$ and the limits as $|z| \to \infty$:

$$\Phi_\pm(x;\lambda) \to \Phi_\infty(x) := \left[e^{\frac{1}{2i}\int_{+\infty}^x |u(y)|^2 dy} e_1, \quad e^{-\frac{1}{2i}\int_{+\infty}^x |u(y)|^2 dy} e_2 \right].$$

Interesting facts about the jump matrix

The jump matrix in the Riemann–Hilbert problem:

$$S(x;\lambda) := \begin{bmatrix} r(\lambda)\overline{r(\bar{\lambda})} & \overline{r(\bar{\lambda})}e^{-2i\lambda^2 x} \\ r(\lambda)e^{2i\lambda^2 x} & 0 \end{bmatrix}.$$

For $\lambda \in \mathbb{R}$, the matrix is Hermitian:

$$S(x;\lambda) := \begin{bmatrix} |r(\lambda)|^2 & \overline{r(\lambda)}e^{-2i\lambda^2 x} \\ r(\lambda)e^{2i\lambda^2 x} & 0 \end{bmatrix}.$$

For $\lambda \in i\mathbb{R}$, the matrix is not Hermitian but $1 - |r(\lambda)|^2 > 0$:

$$S(x;\lambda) := \begin{bmatrix} -|r(\lambda)|^2 & -\overline{r(\lambda)}e^{-2i\lambda^2 x} \\ r(\lambda)e^{2i\lambda^2 x} & 0 \end{bmatrix}.$$

In both cases, $I + S(x;\lambda)$ defines a positive quadratic form.

Under these conditions, the Riemann–Hilbert problem has a unique solution in $L^2(\mathbb{R})$ (Zhou, 1989).

Going back to the solution $u(x,t)$

- Reformulation of the Riemann–Hilbert problem in z complex plane with the jump on the real axis:

$$P_+(x;z) - P_-(x;z) = P_-(x;z)R(x;z), \quad R := \begin{bmatrix} \bar{r}_+(z)r_-(z) & \bar{r}_+(z)e^{-2izx} \\ r_-(z)e^{2izx} & 0 \end{bmatrix},$$

where

$$P_+(x;z) = \left[\frac{m_-(x;z)}{a(z)},\ p_+(x;z)\right], \quad P_-(x;z) = \left[m_+(x;z),\ \frac{p_-(x;z)}{\bar{a}(z)}\right].$$

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- Reconstruction formulas:

$$\partial_x \left(\bar{u}(x) e^{\frac{1}{2i}\int_{\pm\infty}^x |u(y)|^2 dy} \right) = 2i \lim_{|z|\to\infty} z m_\pm^{(2)}(x;z)$$

and

$$u(x) e^{-\frac{1}{2i}\int_{\pm\infty}^x |u(y)|^2 dy} = -4 \lim_{|z|\to\infty} z p_\pm^{(1)}(x;z).$$

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The rest is estimates, estimates, and more estimates ...

What other problems can we study with inverse scattering?

For solitary wave solutions, we can study

- spectral stability
- orbital stability
- asymptotic stability
- long-time asymptotics

What other problems can we study with inverse scattering?

For solitary wave solutions, we can study

- spectral stability
- orbital stability
- asymptotic stability
- long-time asymptotics

The rest of the talk is concerned with orbital stability of solitons in the massive Thirring model (MTM)

$$\begin{cases} i(u_t + u_x) + v = u|v|^2, \quad t > 0, \\ i(v_t - v_x) + u = v|u|^2, \\ (u, v)|_{t=0} = (u_0, v_0). \end{cases}$$

Definition of orbital stability

A family of the stationary MTM solitons is known

$$\begin{cases} u_\omega(x,t) = i\alpha \,\mathrm{sech}\left[\alpha x - i\frac{\gamma}{2}\right] e^{-i\omega t}, \\ v_\omega(x,t) = -i\alpha \,\mathrm{sech}\left[\alpha x + i\frac{\gamma}{2}\right] e^{-i\omega t}, \end{cases}$$

where $\alpha = \sin(\gamma)$, $\omega = \cos(\gamma)$ with $\gamma \in (0, \pi)$.

Definition

A soliton solution $\mathbf{u}_\omega(t,x)$ is said to be orbitally stable in X if for any $\epsilon > 0$ there is a $\delta > 0$ such that if $\|\mathbf{u}(0,\cdot) - \mathbf{u}_\omega(0,\cdot)\|_X < \delta$ then

$$\inf_{\theta, x_0 \in \mathbb{R}} \|\mathbf{u}(t,\cdot) - e^{i\theta}\mathbf{u}_\omega(t, \cdot + x_0)\|_X < \epsilon$$

for all $t \in \mathbb{R}_+$.

Notations: $\mathbf{u} \equiv (u,v)$, $\|\mathbf{u}\|_X \equiv \|u\|_X + \|v\|_X$ for some Hilbert space X.

Why the massive Thirring model (MTM)?

The energy functional is sign-indefinite near $(0,0)$:

$$E(u,v) = \frac{i}{2}\int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u} - v\bar{v}_x + v_x\bar{v})\, dx + \int_{\mathbb{R}} \left(-v\bar{u} - u\bar{v} + 2|u|^2|v|^2\right) dx.$$

No literature on orbital stability result for a class of nonlinear Dirac equations, except for the MTM with several recent results:

- L^2 global well-posedness (Candy, 2011).
- orbital stability of solitons for H^1 solution (P-S, 2014)
  - by finding a Lyapunov functional from a higher-order conserved energy
- orbital stability of solitons for L^2 solution (Contreras-P-S, 2015)
  - by using the auto-Bäcklund transformation between solutions

Orbital stability via Bäcklund transformation

The (auto) Bäcklund transformation is a black box that takes a solution of the equation to a new solution of the same equation.

Let the Bäcklund transform $\mathcal{B}$ be the map that takes (u, v) of the MTM to $(\tilde{u}, \tilde{v})$ of the MTM,

$$\mathcal{B} : (u, v) \mapsto (\tilde{u}, \tilde{v}),$$

In particular, the Bäcklund transformation relates zero $\leftrightarrow$ one soliton:

$$(0, 0) \overset{\mathcal{B}}{\longleftrightarrow} (u_\omega, v_\omega)$$

Heuristic stability argument by Bäcklund transform

$\mathcal{B}$: stable small solution $\longleftrightarrow$ solution around stable one soliton.

–Merle-Vega-2003 (KdV solitons)
–Mizumachi-P-2012 (NLS solitons)

Lax operators for the MTM

The MTM is obtained from the compatibility condition of the linear system

$$\vec{\phi}_x = L\vec{\phi} \quad \text{and} \quad \vec{\phi}_t = A\vec{\phi},$$

where

$$L = \frac{i}{2}(|v|^2 - |u|^2)\sigma_3 - \frac{i\lambda}{\sqrt{2}}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{\sqrt{2}\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\frac{1}{\lambda^2} - \lambda^2\right)\sigma_3$$

and

$$A = -\frac{i}{4}(|u|^2 + |v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\sigma_3$$

References:
Kaup–Newell (1977); Kuznetsov–Mikhailov (1977).

Bäcklund transformation for the MTM

- Let (u, v) be a C^1 solution of the MTM system.
- Let $\vec{\phi} = (\phi_1, \phi_2)^t$ be a C^2 nonzero solution of the linear system associated with (u, v) and $\lambda = e^{i\gamma/2}$.

A new C^1 solution of the MTM system is given by

$$\tilde{u} = -u\frac{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2} + \frac{2i\sin\gamma\overline{\phi}_1\phi_2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}$$

$$\tilde{v} = -v\frac{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2} - \frac{2i\sin\gamma\overline{\phi}_1\phi_2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2},$$

A new C^2 nonzero solution $\vec{\psi} = (\psi_1, \psi_2)^t$ of the linear system associated with $(\tilde{u}, \tilde{v})$ and same λ is given by

$$\psi_1 = \frac{\overline{\phi}_2}{|e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2|}, \quad \psi_2 = \frac{\overline{\phi}_1}{|e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2|}.$$

Orbital stability of MTM solitons in L^2

Well-posedness (Candy, 2011): For any $(u_0, v_0) \in L^2(\mathbb{R})$, there exists a global solution of the MTM $(u, v) \in C(\mathbb{R}, L^2(\mathbb{R}))$:

$$\|u(\cdot,t)\|_{L^2}^2 + \|v(\cdot,t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2.$$

Moreover, the solution is unique in a subspace of $C(\mathbb{R}, L^2(\mathbb{R}))$ and depends continuously on initial data.

Theorem

Let $(u, v) \in C(\mathbb{R}; L^2(\mathbb{R}))$ be a solution of the MTM system and λ_0 be a complex non-zero number. There exist a real positive constant ϵ such that if the initial value $(u_0, v_0) \in L^2(\mathbb{R})$ satisfies

$$\|u_0 - u_{\lambda_0}(\cdot,0)\|_{L^2} + \|v_0 - v_{\lambda_0}(\cdot,0)\|_{L^2} \leq \epsilon,$$

then for every $t \in \mathbb{R}$, there exists $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_0| \leq C\epsilon$,

$$\inf_{a,\theta\in\mathbb{R}} \left(\|u(\cdot + a, t) - e^{-i\theta} u_\lambda(\cdot,t)\|_{L^2} + \|v(\cdot + a, t) - e^{-i\theta} v_\lambda(\cdot,t)\|_{L^2}\right) \leq C\epsilon,$$

Steps in the proof of the stability result

Fix $\gamma \in (0, \pi)$ for a soliton $\mathbf{u}_\omega$. Take initial data $\mathbf{u}_0 \in H^2(\mathbb{R})$ s.t. $\|\mathbf{u}_0 - \mathbf{u}_\omega\|_{L^2} < \epsilon$ for $\epsilon > 0$ sufficiently small.

Steps in the proof of the stability result

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- Step 1: From a perturbed one-soliton to a small solution at $t = 0$:

  There exists $\lambda_0 \in \mathbb{C}$ and the corresponding L^2-solution $\vec{\phi}$ of $\partial_x \vec{\phi} = L(\mathbf{u}_0; \lambda_0)\vec{\phi}$ such that $|\lambda_0 - e^{i\gamma/2}| \lesssim \epsilon$. Then, Bäcklund transformation

  $$\mathcal{B}_{-1} : (\mathbf{u}_0; \phi, \lambda_0) \mapsto \widetilde{\mathbf{u}}_0$$

  yields the estimate

  $$\|\widetilde{\mathbf{u}}_0\|_{L^2} \lesssim \|\mathbf{u}_0 - \mathbf{u}_\omega(0, \cdot)\|_{L^2}.$$

Steps in the proof of the stability result

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- Step 2: Time evolution of the small solution in $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$.

Steps in the proof of the stability result

- Step 3: From the small solution to the perturbed one-soliton:

  The Bäcklund transformation

  $$\mathbf{u}(t,\cdot) = \mathcal{B}_{+1}(\widetilde{\mathbf{u}}(t,\cdot)) \in H^2(\mathbb{R}), \quad \forall t \in \mathbb{R}$$

  yields the estimate

  $$\inf_{a,\theta\in\mathbb{R}} \|\mathbf{u}(t,\cdot) - e^{-i\theta}\mathbf{u}_\omega(t,\cdot+a)\|_{L^2_x} \lesssim \|\widetilde{\mathbf{u}}(t,\cdot)\|_{L^2} \quad \forall t \in \mathbb{R}.$$

Steps in the proof of the stability result

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  yields the estimate

  $$\inf_{a,\theta\in\mathbb{R}} \|\mathbf{u}(t,\cdot) - e^{-i\theta}\mathbf{u}_\omega(t,\cdot + a)\|_{L^2_x} \lesssim \|\widetilde{\mathbf{u}}(t,\cdot)\|_{L^2} \quad \forall t \in \mathbb{R}.$$

- Step 4: Approximation arguments in $H^2(\mathbb{R})$ as all three steps are performed in $L^2(\mathbb{R})$.

Sequences in $H^2(\mathbb{R})$ produce classical solutions (u,v) of the MTM, which are compatible with the Lax linear system for $\vec{\phi} \in C^2(\mathbb{R}\times\mathbb{R})$,

$$\vec{\phi}_x = L(u,v,\lambda)\vec{\phi} \quad \text{and} \quad \vec{\phi}_t = A(u,v,\lambda)\vec{\phi}.$$

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Thank you!!!