Stability of Nonlinear Waves in Hamiltonian Dynamical Systems

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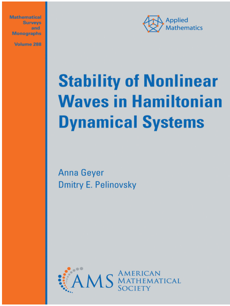
Outline of the book

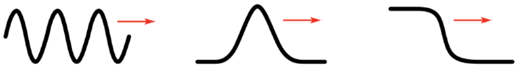


1. Stability in finite-dimensional systems
2. Stability of solitary waves
3. Stability of periodic waves
4. Stability in integrable Hamiltonian systems
5. Stability of peaked waves
6. Stability of domain walls

1. Introduction - nonlinear waves

- Nonlinear waves $\rightsquigarrow$ special solutions of some nonlinear PDE
- Steady traveling wave solutions

$$u(t,x) = U(x - ct)$$



periodic wave   solitary wave/pulse   front/kink

- Observed in nature, experiments, simulations

Introduction - dynamical systems

Consider an initial value problem

$$\begin{cases} u_t = \mathcal{F}(u, \partial_x), & t > 0, \\ u(0) = u_0, \end{cases}$$

- (IVP) is locally well-posed in a suitable function space.
- TW with profile $U(\xi)$, $\xi = x - ct$ exists from the steady equation

$$-c\partial_\xi U = \mathcal{F}(U, \partial_\xi)$$

- The profile U is an equilibrium point in the moving frame

$$u_t = c\partial_\xi u + \mathcal{F}(u, \partial_\xi)$$

What happens to the perturbation $w(t,\xi) := u(t,\xi) - U(\xi)$ as the time increases from $t = 0$ to $t \to +\infty$?

2. Stability in finite-dimensional systems

Consider the ODE

$$\begin{cases} \dfrac{du}{dt} = f(u), & t > 0, \\ u(0) = u_0, \end{cases}$$

where $u_0 \in \mathbb{R}^n$ and $f(u) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ for local well-posedness.

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Equilibrium point u_* is defined by $f(u_*) = 0$.

Expanding f at u_* with the perturbation $w(t) := u(t) - u_*$ as

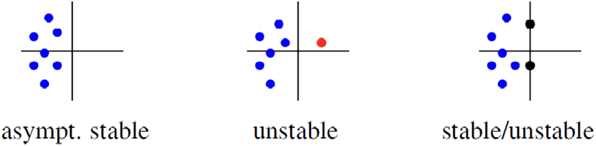
$$f(u) = f(u_* + w) = \underbrace{f(u_*)}_{=0} + \underbrace{D_u f(u_*)}_{=A} w + R(w),$$

where $A \in \mathbb{M}^{n\times n}$ is the linearized matrix and $R(w) = o_{\mathbb{R}^n}(w)$ is a remainder term, yields two systems:

$$\underbrace{\frac{dw}{dt} = Aw}_{\text{Linearized system}} \quad \text{and} \quad \underbrace{\frac{dw}{dt} = Aw + R(w)}_{\text{Nonlinear system}}.$$

Linear stability

Linear stability is determined by eigenvalues of $A = D_u f(u_*)$:



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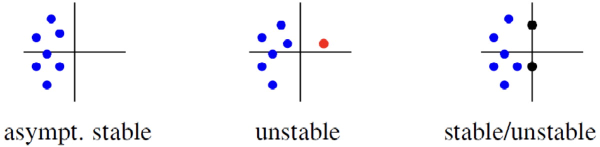


Nonlinear stability is defined as

- **Stable** if $\forall u_0$ close to u_*, $u(t)$ stays close to u_* for all $t \geq 0$.

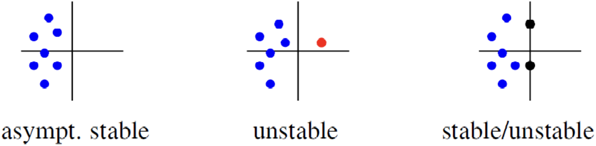
$$\forall \varepsilon > 0, \exists \delta > 0 : \quad \forall u_0 \in B_\delta(u_*) \quad \Rightarrow \quad u(t) \in B_\varepsilon(u_*), \quad t \geq 0.$$

- **Asymptotically stable** if it is stable and $u(t) \to u_*$ as $t \to +\infty$.
- **Unstable** if for at least one u_0 close to u_*, $u(t)$ leaves a ball at u_*.

$$\exists \varepsilon > 0 : \forall \delta > 0, \exists u_0 \in B_\delta(u_*) \Rightarrow u(t) \notin B_\varepsilon(u_*), \text{ for some } t > 0.$$

Linear stability

Linear stability is determined by eigenvalues of $A = D_u f(u_*)$:



Theorem 1

- *If* $\mathrm{Re}(\lambda_j) < 0$ *for every* $\lambda_j \in \sigma(A)$, *then* u_* *is nonlinearly asymptotically stable.*
- *If* $\mathrm{Re}(\lambda_j) > 0$ *for at least one* $\lambda_j \in \sigma(A)$, *then* u_* *is nonlinearly unstable.*

The case with $\mathrm{Re}(\lambda_j) \leq 0$ for every $\lambda_j \in \sigma(A)$ is inconclusive from the linearized system.

Finite-dimensional Hamiltonian systems

This includes Hamiltonian systems defined in the canonical form as

$$\begin{cases} \dfrac{du}{dt} = J\nabla H(u), \quad t > 0, \\ u(0) = u_0, \end{cases}$$

where $J = -J^*$ is invertible, $H \in C^2(\mathbb{R}^n, \mathbb{R})$, and n is even.

- Since $Jv \cdot v = 0$ for $v \in \mathbb{R}^n$, the energy is conserved:

$$H(u(t)) = H(u_0), \quad t \geq 0.$$

- Equilibrium point u_* is critical to the energy: $\nabla H(u_*) = 0$.
- The linearized system has a structure

$$\frac{dw}{dt} = J\mathcal{L}w,$$

where $\mathcal{L} := H''(u_*)$ is a self-adjoint (Hessian) matrix.

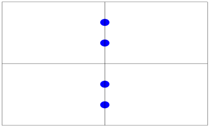
Linear stability in Hamiltonian systems

Eigenvalues of $J\mathcal{L}$ are symmetric

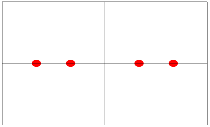
- about the real axis $\text{Im}(\lambda) = 0$: both λ, $\bar{\lambda}$ are eigenvalues
- about the imaginary axis $\text{Re}(\lambda) = 0$: both λ, $-\bar{\lambda}$ are eigenvalues.

$$(JL)^* = -LJ \quad \Rightarrow \quad (JL)^* = -J^{-1}(JL)J$$

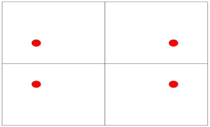
They appear in real or imaginary pairs or as complex quadruplets.



neutrally stable



unstable



unstable

Is u_* nonlinearly stable if $\lambda_j \in i\mathbb{R}$ for every $\lambda_j \in \sigma(J\mathcal{L})$?

Nonlinear stability in Hamiltonian systems

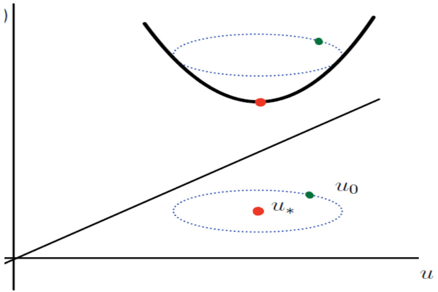
Theorem 2 (Lyapunov's Stability)

If there exists a Lyapunov function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ such that

(a) $V(u_*) = 0$ *and* $V(u) > 0$ *for all* $u \in B_\varepsilon(u_*) \backslash \{u_*\}$

(b) $\frac{d}{dt} V(u(t)) = \nabla V(u(t)) \cdot f(u(t)) \leq 0$ *for all* $u \in B_\varepsilon(u_*)$,

then u_* *is nonlinearly stable in* $\frac{du}{dt} = f(u)$.



Stability of isolated minima or maxima of $H(u)$

Theorem 3

If $H(u)$ has an isolated local minimum or maximum at u_*, then u_* is nonlinearly stable.

Stability of u_* relies only on $\mathcal{L} = H''(u_*)$, independently of J!
Morever, $\mathcal{L}$ may have zero eigenvalues, as long as u_* is isolated.

- If u_* is a minimum of $H(u)$, take $V(u) := H(u) - H(u_*) \geq 0$.
- If u_* is a maximum of $H(u)$, take $V(u) := H(u_*) - H(u) \geq 0$.

In either case,

$$\frac{d}{dt} H(u(t)) = 0$$

and Lyapunov's Stability Theorem applies.

What if u_* is a saddle point of $H(u)$? Is it unstable?

Nonlinear instability in Hamiltonian systems

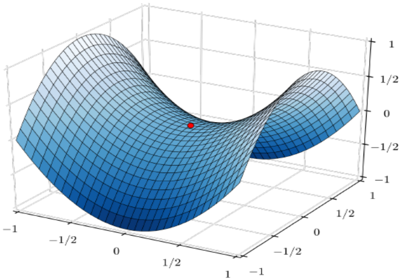
Theorem 4 (Lyapunov's Instability)

If there exists a Lyapunov function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ *such that*

(a) $V(u_*) = 0$ *and* $V(u) > 0$ *for all* $u \in \mathcal{B}_+ \subset B_\varepsilon(u_*) \backslash \{u_*\}$

(b) $\dfrac{d}{dt} V(u(t)) = \nabla V(u(t)) \cdot f(u(t)) > 0$ *for all* $u \in \mathcal{B}_+$,

then u_* *is nonlinearly unstable in* $\frac{du}{dt} = f(u)$.



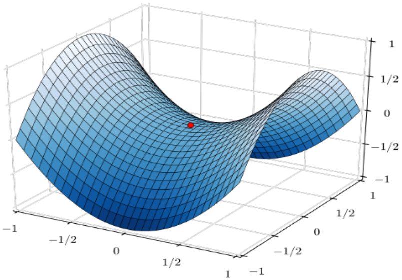
Theorem 5 (Lyapunov's Instability (reversed))

If there exists a Lyapunov function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ *such that*

(a) $V(u_*) = 0$ *and* $V(u) < 0$ *for all* $u \in \mathcal{B}_- \subset B_\varepsilon(u_*)\backslash\{u_*\}$

(b) $\displaystyle \frac{d}{dt}V(u(t)) = \nabla V(u(t)) \cdot f(u(t)) < 0$ *for all* $u \in \mathcal{B}_-$,

then u_* *is nonlinearly unstable in* $\frac{du}{dt} = f(u)$.



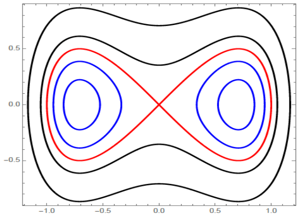
Instability of saddle points of $H(u)$?

Lyapunov's instability theorem is not applicable to saddle points since

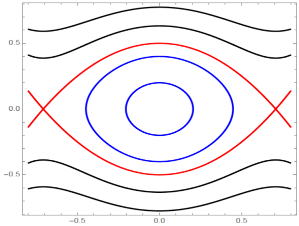
$$\frac{d}{dt}H(u(t)) = 0.$$

Saddle points in $\mathbb{R}^2$ are necessarily unstable, because the solution curves on the phase plane do not intersect.

$H(u,v) = \frac{1}{2}v^2 - \frac{1}{2}u^2 + \frac{1}{4}u^4$



$H(u,v) = \frac{1}{2}v^2 + \frac{1}{2}u^2 - \frac{1}{4}u^4$



Instability of saddle points of $H(u)$?

Lyapunov's instability theorem is not applicable to saddle points since

$$\frac{d}{dt}H(u(t)) = 0.$$

Saddle points in $\mathbb{R}^n$ with $n \geq 4$ do not have to be unstable, e.g.

$$H(u_1, v_1, u_2, v_2) = \frac{1}{2}(u_1^2 + v_1^2) - \frac{1}{2}(u_2^2 + v_2^2)$$

with

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial u_1} \\ \frac{\partial H}{\partial v_1} \\ \frac{\partial H}{\partial u_2} \\ \frac{\partial H}{\partial v_2} \end{bmatrix}$$

generates

$$\left\{ \begin{array}{l} \dot{u}_1 = v_1, \\ \dot{v}_1 = -u_1, \\ \dot{u}_2 = -v_2, \\ \dot{v}_2 = u_2, \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} \ddot{u}_1 + u_1 = 0, \\ \ddot{u}_2 + u_2 = 0. \end{array} \right.$$

Krein theory for saddle points

Let $\lambda_0 \in \sigma(J\mathcal{L})$ be an eigenvalue of algebraic multiplicity m and $\{u_j\}_{j=1}^m \subset \mathbb{R}^n$ be a basis of generalized eigenvectors. The matrix $K(\lambda_0)$ with elements

$$[K(\lambda_0)]_{ij} = \mathcal{L}u_i \cdot u_j, \quad 1 \leq i,j \leq m,$$

is called the Krein matrix associated with the eigenvalue λ_0.

- If $\mathrm{Re}(\lambda_0) \neq 0$, then $K(\lambda_0) = 0$.
- If $\mathrm{Re}(\lambda_0) = 0$ and $\mathrm{Im}(\lambda_0) \neq 0$, then $K(\lambda_0)$ is Hermitian and invertible with p_{λ_0} positive and n_{λ_0} negative eigenvalues such that $p_{\lambda_0} + n_{\lambda_0} = m$.

If $m = 1$ (simple eigenvalue), then $K(\lambda_0) = \mathcal{L}u_1 \cdot u_1$.

$$\mathcal{L}J\mathcal{L}u_1 = \lambda_0 \mathcal{L}u_1 \quad \Rightarrow \quad (\lambda_0 + \bar{\lambda}_0)\mathcal{L}u_1 \cdot u_1 = 0.$$

Hamiltonian–Krein theorem

Theorem 6 (Sylvester's inertial law)

Assume that $\mathcal{L} = H''(u_*)$ is invertible with $p(\mathcal{L})$ positive and $n(\mathcal{L})$ negative eigenvalues such that $p(\mathcal{L}) + n(\mathcal{L}) = n$. Then,

$$p(\mathcal{L}) = N_{\rm real}(J\mathcal{L}) + 2N_{\rm comp}(J\mathcal{L}) + 2 \sum_{\lambda_0 \in i\mathbb{R}_+} p_{\lambda_0},$$

$$n(\mathcal{L}) = N_{\rm real}(J\mathcal{L}) + 2N_{\rm comp}(J\mathcal{L}) + 2 \sum_{\lambda_0 \in i\mathbb{R}_+} n_{\lambda_0},$$

where $N_{\rm real}$ is the number of real positive eigenvalues and $2N_{\rm comp}$ is the number of complex eigenvalues with positive real part.

- If either $n(\mathcal{L}) = 0$ or $p(\mathcal{L}) = 0$, then $\lambda_j \in i\mathbb{R}$ for all $\lambda_j \in \sigma(J\mathcal{L})$.
- If $n(\mathcal{L})$ and $p(\mathcal{L})$ are odd, then $N_{\rm real}(J\mathcal{L}) \geq 1$ and the saddle point u_* is linearly unstable.

Example: a mechanical system

Consider a system of m particles with coordinates $\{q_k\}_{1\leq k\leq m}$ and momenta $\{p_k\}_{1\leq k\leq m}$ ($n = 2m$):

$$\frac{du}{dt} = J\nabla H(u), \quad J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad u = \begin{bmatrix} p \\ q \end{bmatrix},$$

where

$$H = \frac{1}{2} \underbrace{\mathcal{L}_+ p \cdot p}_{\text{kinetic energy}} + \frac{1}{2} \underbrace{\mathcal{L}_- q \cdot q}_{\text{potential energy}}$$

Equations of motion are

$$\frac{dq}{dt} = \mathcal{L}_+ p, \qquad \frac{dp}{dt} = -\mathcal{L}_- q$$

and the spectral problem is

$$\mathcal{L}_+ p = \lambda q, \qquad -\mathcal{L}_- q = \lambda p.$$

Stability theorems

Consider the spectral stability problem:

$$\mathcal{L}_+ p = \lambda q, \qquad -\mathcal{L}_- q = \lambda p, \quad p, q \in \mathbb{R}^m.$$

1. If either $\mathcal{L}_+$ or $\mathcal{L}_-$ is invertible, then the stability problem is equivalent to the generalized eigenvalue problems:

$$\mathcal{L}_+ p = (-\lambda^2)\mathcal{L}_-^{-1} p, \quad \text{or} \quad \mathcal{L}_- q = (-\lambda^2)\mathcal{L}_+^{-1} q.$$

2. If either $\mathcal{L}_+$ or $\mathcal{L}_-$ is positive, then $z := -\lambda^2$ is real and the number of negative eigenvalues of z (real positive eigenvalues of λ) is uniquely determined by the number of negative eigenvalues of the other operator.

Sharp Hamiltonian–Krein theorem

Assume invertibility of $\mathcal{L}_+$ and $\mathcal{L}_-$ in

$$\mathcal{L}_+ p = (-\lambda^2)\mathcal{L}_-^{-1} p, \quad \text{and} \quad \mathcal{L}_- q = (-\lambda^2)\mathcal{L}_+^{-1} q.$$

3 Hamiltonian–Krein theorem follows by Sylvester inertial law:

$$n(\mathcal{L}_+) = N_{\rm real}^-(J\mathcal{L}) + N_{\rm comp}(J\mathcal{L}) + \sum_{\lambda_0 \in i\mathbb{R}_+} n_{\lambda_0},$$

$$n(\mathcal{L}_-) = N_{\rm real}^+(J\mathcal{L}) + N_{\rm comp}(J\mathcal{L}) + \sum_{\lambda_0 \in i\mathbb{R}_+} n_{\lambda_0},$$

where $N_{\rm real}^{\pm}$ is related to p_{λ_0} positive and n_{λ_0} negative eigenvalues of Krein matrix $K(\lambda_0)$ associated with $\mathcal{L}_+$ for $\lambda_0 \in \mathbb{R}_+$.

4 The theorem recovers the general result

$$n(\mathcal{L}_+) + n(\mathcal{L}_-) = N_{\rm real}(J\mathcal{L}) + 2N_{\rm comp}(J\mathcal{L}) + 2\sum_{\lambda_0 \in i\mathbb{R}_+} n_{\lambda_0},$$

but also gives a sharper count of unstable eigenvalues.

More examples

Here we illustrate all three possible cases for

$$1 = n(\mathcal{L}_+) = N^-_{\rm real}(J\mathcal{L}) + N_{\rm comp}(J\mathcal{L}) + \sum_{\lambda_0 \in i\mathbb{R}_+} n_{\lambda_0},$$

$$1 = n(\mathcal{L}_-) = N^+_{\rm real}(J\mathcal{L}) + N_{\rm comp}(J\mathcal{L}) + \sum_{\lambda_0 \in i\mathbb{R}_+} n_{\lambda_0}.$$

- For the previous example of $H(p,q) = \frac{1}{2}(p_1^2 - p_2^2) + \frac{1}{2}(q_1^2 - q_2^2)$ we have $\ddot{q}_1 + q_1 = 0$ and $\ddot{q}_2 + q_2 = 0$ with $n_{\lambda_0} = 1$.
- For another example of $H(p,q) = \frac{1}{2}(p_1^2 - p_2^2) - \frac{1}{2}(q_1^2 - q_2^2)$ we have $\ddot{q}_1 - q_1 = 0$ and $\ddot{q}_2 - q_2 = 0$ with $N^+_{\rm real} = N^-_{\rm real} = 1$.
- For another example of $H(p,q) = p_1 p_2 + \frac{1}{2}(q_1^2 - q_2^2)$ we have

  $$\ddot{q}_1 + q_2 = 0, \quad \ddot{q}_2 - q_1 = 0 \quad \Rightarrow \quad \ddddot{q}_1 + q_1 = 0,$$

  with $N_{\rm comp} = 1$.

3. Towards the infinite-dimensional Hamiltonian systems

Consider a Hamiltonian dynamical system

$$\frac{du}{dt} = J\nabla H(u),$$

where

- $H : X \subset \mathcal{H} \to \mathbb{R}$ is a C^2 functional on a subset X of a Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$
- $J : \mathcal{H} \to \mathcal{H}$ is a bounded invertible operator with a bounded inverse which is skew-adjoint:

$$\langle Ju, w \rangle = -\langle u, Jw \rangle \quad \text{for } u, w \in \mathcal{H}$$

The first variation of H is defined as

$$\langle \nabla H(u), w \rangle = \lim_{\epsilon \to 0} \frac{H(u + \epsilon w) - H(u)}{\epsilon} = \frac{d}{d\epsilon} H(u+\epsilon w)\Big|_{\epsilon=0} \quad \text{for } u, w \in X,$$

so that $\nabla H(u) \in X^*$, where X^* is dual to X with respect to $\mathcal{H}$.

Example: nonlinear wave equation

Consider the nonlinear wave equation on the infinite line $x \in \mathbb{R}$:

$$u_{tt} - u_{xx} + V'(u) = 0 \quad \Rightarrow \frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \nabla_u H(u,v) \\ \nabla_v H(u,v) \end{bmatrix},$$

where

$$H(u,v) = \int_{\mathbb{R}} \left(\frac{1}{2}(u_x)^2 + V(u) + \frac{1}{2}v^2 \right) dx$$

with $V(0) = V'(0) = 0$, and $V''(0) \neq 0$.

- Hilbert space for (u,v): $\mathcal{H} = L^2(\mathbb{R}) \times L^2(\mathbb{R})$
- Energy space for (u,v): $X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$
- Dual space of X w.r.t. $\mathcal{H}$: $X^* = H^{-1}(\mathbb{R}) \times L^2(\mathbb{R})$.

Initial-value problem

Consider the initial-value problem:

$$\begin{cases} \dfrac{du}{dt} = J\nabla H(u), & t > 0, \\ u(0) = u_0 \in X, \end{cases}$$

We assume that the local well-posedness holds in the energy space with the local solution

$$u \in C^0([0,\tau_0), X) \cap C^1((0,\tau_0), X^*)$$

for every $u_0 \in X$ and some $\tau_0 > 0$.

The Hamiltonian is conserved in time: $H(u(t,\cdot)) = H(u_0)$, $t \in [0,\tau_0)$. This follows formally (for sufficiently smooth solutions):

$$\frac{d}{dt}H(u(t,\cdot)) = \langle \nabla H(u), \frac{du}{dt} \rangle = \langle \nabla H(u), J\nabla H(u) \rangle = 0.$$

Critical points of $H(u)$

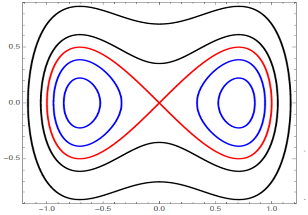
$u_* \in X$ is a critical point of $H : X \to \mathbb{R}$ if $\nabla H(u_*) = 0$.

For the nonlinear wave equation

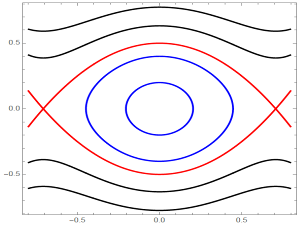
$$u_{tt} - u_{xx} + V'(u) = 0, \quad H(u,v) = \int_{\mathbb{R}} \left(\frac{1}{2}(u_x)^2 + V(u) + \frac{1}{2}v^2 \right) dx,$$

$(u_*, 0)$ is a critical point of $H(u,v)$ if $-u_*'' + V'(u_*) = 0$.

If $V(u) = \frac{1}{2}u^2 - \frac{1}{4}u^4$, then



If $V(u) = -\frac{1}{2}u^2 + \frac{1}{4}u^4$, then



Stability of critical points

If $H \in C^2(X, \mathbb{R})$, expansion of H about $u_* \in X$ with the perturbation $w = u - u_* \in X$ yields

$$H(u_* + w) = H(u_*) + \underbrace{\langle \nabla H(u_*), w \rangle}_{=0} + \frac{1}{2} \langle \mathcal{L} w, w \rangle + R(w),$$

where $\mathcal{L} := H''(u_*) : X \to X^*$ is a self-adjoint linear operator in $\mathcal{H}$ and $R(w) = o_X(w^2)$ is the remainder term.

We assume that $\mathcal{L} : \mathrm{dom}(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$ is bounded from below.

- If u_* is a minimizer of $H(u)$, is it nonlinearly stable?
- If u_* is a saddle point of $H(u)$, is it nonlinearly unstable?

Difference from the finite-dimensional case in $\mathbb{R}^n$:

- $\mathcal{L}$ may not be bounded from above.
- $\ker(\mathcal{L}) \neq \{0\}$ due to translational (and other) symmetries.

Strict minimizers of energy $H(u)$

Theorem 7

Let $u_* \in X$ be a critical point of $H(u)$ and there exists $C > 0$ such that $\mathcal{L} := H''(u_*)$ satisfies

$$\langle \mathcal{L}w, w \rangle \geq C\|w\|_X^2, \quad w \in X.$$

then u_* is nonlinearly stable:

$$\forall \varepsilon > 0, \exists \delta > 0 : \quad \forall u_0 \in B_\delta(u_*) \quad \Rightarrow \quad u(t) \in B_\varepsilon(u_*), \quad t \geq 0.$$

The proof follows from:

$$H(u_0) - H(u_*) = H(u(t,\cdot)) - H(u_*) = H(u_* + w(t,\cdot)) - H(u_*)$$
$$= \frac{1}{2}\langle \mathcal{L}w(t,\cdot), w(t,\cdot)\rangle + R(w(t,\cdot)) \geq \frac{1}{2}C\|w(t,\cdot)\|_X^2 - C_R(\varepsilon)\|w(t,\cdot)\|_X^2.$$

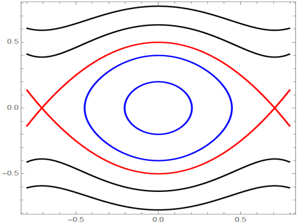
$$H(u_0) - H(u_*) \leq \tilde{C}\|u_0 - u_*\|_X^2 \Rightarrow \|u(t,\cdot) - u_*\|_X^2 \leq \frac{4\tilde{C}}{C}\|u_0 - u_*\|_X^2.$$

Example: kinks in the nonlinear wave equation

Consider the Hamiltonian

$$H(u,v) = \int_{\mathbb{R}} \left(\frac{1}{2}(u_x)^2 + \frac{1}{4}(1-u^2)^2 + \frac{1}{2}v^2 \right) dx$$

subject to $(u,v) \to (\pm 1, 0)$ as $x \to \pm\infty$, e.g. $u_*(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)$.



Perturbation $(u_* + w, v)$ is in $(w,v) \in X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with

$$\mathcal{L} = H''(u,v) = \begin{bmatrix} -\partial_x^2 + 3u_*^2(x) - 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} w \\ v \end{bmatrix}.$$

Sturm's theory for the Schrödinger operator

We have

$$\mathcal{L}_0 : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \text{with} \quad \mathcal{L}_0 = -\partial_x^2 + 3u_*^2(x) - 1.$$

(a) $\sigma(\mathcal{L}_0) = \sigma_p(\mathcal{L}_0) \cup \sigma_c(\mathcal{L}_0) \subset \mathbb{R}$ since $\mathcal{L}_0$ is self-adjoint in $L^2(\mathbb{R})$.
(b) $\sigma_c(\mathcal{L}_0) = [2, \infty)$ since $u_*(x) \to \pm 1$ at $\pm\infty$ exponentially fast.
(c) Eigenvalues in $\sigma_p(L) = \{\lambda_1, \lambda_2, \dots\} \in (-\infty, 2)$ are simple
(d) Eigenfunction for eigenvalue λ_n has $n - 1$ simple zeros on $\mathbb{R}$.
(e) $\lambda_1 = 0$ since $\mathcal{L}_0 \partial_x u_* = 0$ and $\partial_x u_*(x) > 0$ for all $x \in \mathbb{R}$.

Hence $\mathcal{L}_0 : H^2_{\text{odd}}(\mathbb{R}) \subset L^2_{\text{odd}}(\mathbb{R}) \to L^2_{\text{odd}}(\mathbb{R})$ is strictly positive and $\exists C > 0$ such that $\langle \mathcal{L}_0 w, w \rangle \geq C \|w\|^2_{L^2}$, $\forall w \in H^2_{\text{odd}}$. This implies via Gärdiner's inequality that

$$\langle \mathcal{L}_0 w, w \rangle \geq C \|w\|^2_{H^1}, \quad \forall w \in H^1_{\text{odd}}(\mathbb{R}).$$

The kink is nonlinearly stable for $(w, v) \in X_{\text{odd}} = H^1_{\text{odd}}(\mathbb{R}) \times L^2_{\text{odd}}(\mathbb{R})$.

Saddle points of energy $H(u)$

Theorem 8

Let $u_* \in X$ be a critical point of $H(u)$ and there exists $w_+, w_- \in X$ such that $\mathcal{L} := H''(u_*)$ satisfies

$$\langle \mathcal{L} w_+, w_+ \rangle > 0 \quad \text{and} \quad \langle \mathcal{L} w_-, w_- \rangle < 0.$$

If u_* is linearly unstable and $\nabla H(u_* + w) = \mathcal{L} w + \mathrm{o}_X(w)$, then u_* is nonlinearly unstable:

$$\exists \varepsilon > 0 : \quad \forall \delta > 0, \quad \exists u_0 \in B_\delta(u_*) : \quad u(t_0) \notin B_\varepsilon(u_*), \quad t_0 > 0.$$

Saddle points of energy $H(u)$

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$$\exists \varepsilon > 0 : \quad \forall \delta > 0, \quad \exists u_0 \in B_\delta(u_*) : \quad u(t_0) \notin B_\varepsilon(u_*), \quad t_0 > 0.$$

It follows from the Hamiltonian–Krein theorem,

$$n(\mathcal{L}) = N_{\text{real}}(J\mathcal{L}) + 2N_{\text{comp}}(J\mathcal{L}) + 2 \sum_{\lambda_0 \in i\mathbb{R}_+} n_{\lambda_0},$$

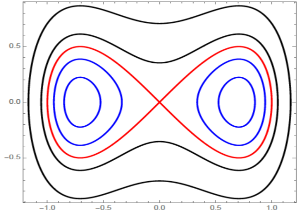
that u_* is linearly unstable if $n(\mathcal{L})$ is odd.

Example: pulses in the nonlinear wave equation

Consider the Hamiltonian

$$H(u,v) = \int_{\mathbb{R}} \left(\frac{1}{2}(u_x)^2 - \frac{1}{2}u^2 + \frac{1}{4}u^4 + \frac{1}{2}v^2 \right) dx$$

subject to $(u,v) \to (0,0)$ as $x \to \pm\infty$, e.g. $u_*(x) = \sqrt{2}\,\mathrm{sech}(x)$.



Perturbation $(u_* + w, v)$ is in $(w,v) \in X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with

$$\mathcal{L} = H''(u,v) = \begin{bmatrix} -\partial_x^2 + 1 - 3u_*^2(x) & 0 \\ 0 & 1 \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} w \\ v \end{bmatrix}.$$

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We have

$$\mathcal{L}_0 : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \text{with} \quad \mathcal{L}_0 = -\partial_x^2 + 1 - 3u_*^2(x).$$

(a) $\sigma(\mathcal{L}_0) = \sigma_p(\mathcal{L}_0) \cup \sigma_c(\mathcal{L}_0) \subset \mathbb{R}$ since $\mathcal{L}_0$ is self-adjoint in $L^2(\mathbb{R})$.
(b) $\sigma_c(\mathcal{L}_0) = [1, \infty)$ since $u_*(x) \to 0$ at $\pm\infty$ exponentially fast.
(c) Eigenvalues in $\sigma_p(L) = \{\lambda_1, \lambda_2, \dots\} \in (-\infty, 1)$ are simple
(d) Eigenfunction for eigenvalue λ_n has $n-1$ simple zeros on $\mathbb{R}$.
(e) $\lambda_1 < 0$, $\lambda_2 = 0$ since $\mathcal{L}_0 \partial_x u_* = 0$ and $\partial_x u_*(x)$ has one zero on $\mathbb{R}$.

The spectral stability problem

$$\lambda \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -\partial_x^2 + 1 - 3u_*^2(x) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}$$

yields $\lambda w = v$ and $-\lambda^2 w = \mathcal{L}_0 w$ with $\lambda^2 = -\lambda_1 > 0$.
The pulse is nonlinearly unstable for $(w, v) \in X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$.

4. Towards to Hamiltonian systems with symmetries

- Hamiltonian systems are invariant under translations in time. This symmetry is associated to the conservation of $H(u)$.

- Other symmetries are common in physical systems such as spatial translations and rotations, phase rotations, ... These symmetries lead to the degeneracy of the kernel of $\mathcal{L} = H''(u_*)$.

- Additional symmetries lead to additional conserved quantities.

- Additional conservative quantities may affect linear and nonlinear stability of saddle points of $H(u)$.

Definition of a symmetry

Consider the symmetry transformation $T(\theta) : \mathcal{H} \to \mathcal{H}$ for $\theta \in \mathbb{R}$ with the infinitesimal generator $T' : \mathrm{dom}(T') \subset \mathcal{H} \to \mathcal{H}$ defined by

$$T'u := \lim_{\theta \to 0} \frac{T(\theta)u - u}{\theta},$$

for every $u \in \mathrm{dom}(T')$ for which the limit exists. Assume properties:

- group:
  $T(0) = \mathrm{Id}$, $T(\theta_1 + \theta_2) = T(\theta_1)T(\theta_2) = T(\theta_2)T(\theta_1) \quad \forall \theta_1, \theta_2.$
- isometry:
  $\langle T(\theta)f, T(\theta)g \rangle = \langle f, g \rangle, \forall f, g \in \mathcal{H}.$
- invariance: $H(T(\theta)u) = H(u), \forall u \in X.$
- commutativity: $JT(\theta) = T(\theta)J.$
- skew-adjointness: $\langle T'f, g \rangle = -\langle f, T'g \rangle, \forall f, g \in \mathrm{dom}(T') \subset \mathcal{H}.$

An associated conserved quantity

Let $T(\theta) : \mathcal{H} \to \mathcal{H}$, $\theta \in \mathbb{R}$ be a symmetry of

$$\frac{du}{dt} = J\nabla H(u),$$

such that if $u(t)$ is a solution, so is $T(\theta)u(t)$ for every $\theta \in \mathbb{R}$.

If $u \in X$ and $J^{-1}T'u \in X^*$, then the quadratic functional $Q(u) : X \to \mathbb{R}$ is well-defined by

$$Q(u) := \frac{1}{2}\langle J^{-1}T'u, u\rangle.$$

For every local solution $u \in C^0([0,\tau_0), X)$, the value of $Q(u)$ is conserved in time:

$$Q(u(t,\cdot)) = Q(u_0), \quad t \in [0,\tau_0).$$

Furthermore we have $Q(T(\theta)u) = Q(u)$, $T'u = J\nabla Q(u)$, and

$$\langle \nabla Q(u), J\nabla H(u)\rangle = 0.$$

Stationary states (traveling waves)

In application to the Hamiltonian system

$$\frac{du}{dt} = J\nabla H(u),$$

the symmetry $T(\theta) : \mathcal{H} \to \mathcal{H}$, $\theta \in \mathbb{R}$ allows us to consider stationary states (traveling waves) in the form:

$$u(t,\cdot) = T(\omega t + \theta)u_\omega, \quad \omega \in \Omega \subset \mathbb{R}, \quad \theta \in \mathbb{R},$$

where the profile u_ω is obtained from

$$\omega T' u_\omega = J\nabla H(u_\omega)$$

Since $T'u = J\nabla Q(u)$ and J is invertible, we characterize the profile u_ω as a critical point of the *augmented energy*

$$\Lambda_\omega(u) := H(u) - \omega Q(u).$$

Example: nonlinear Schrödinger equation

Consider the NLS equation on the infinite line $x \in \mathbb{R}$:

$$i\psi_t = -\psi_{xx} + W'(|\psi|^2)\psi \Rightarrow \frac{d}{dt}\begin{bmatrix}\psi \\ \bar{\psi}\end{bmatrix} = \begin{bmatrix}0 & -i \\ i & 0\end{bmatrix}\begin{bmatrix}\nabla_{\psi}H(\psi,\bar{\psi}) \\ \nabla_{\bar{\psi}}H(\psi,\bar{\psi})\end{bmatrix},$$

where

$$H(\psi,\bar{\psi}) = \int_{\mathbb{R}} \left(|\psi_x|^2 + W(|\psi|^2)\right) dx,$$

with $\mathcal{H} = L^2(\mathbb{R},\mathbb{C})$, $X = H^1(\mathbb{R},\mathbb{C})$, and $X^* = H^{-1}(\mathbb{R},\mathbb{C})$.

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with $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$, $X = H^1(\mathbb{R}, \mathbb{C})$, and $X^* = H^{-1}(\mathbb{R}, \mathbb{C})$.

Phase rotation symmetry

$$T_1(\theta)\begin{bmatrix}\psi \\ \bar{\psi}\end{bmatrix} = \begin{bmatrix}e^{-i\theta}\psi \\ e^{i\theta}\bar{\psi}\end{bmatrix} \Rightarrow T_1'\begin{bmatrix}\psi \\ \bar{\psi}\end{bmatrix} = \begin{bmatrix}-i\psi \\ i\bar{\psi}\end{bmatrix} \Rightarrow Q_1(\psi) = \int_{\mathbb{R}} |\psi|^2 dx.$$

Spatial translation symmetry

$$T_2(\theta)\begin{bmatrix}\psi \\ \bar{\psi}\end{bmatrix} = \begin{bmatrix}\psi(\cdot + \theta) \\ \bar{\psi}(\cdot + \theta)\end{bmatrix} \Rightarrow T_2'\begin{bmatrix}\psi \\ \bar{\psi}\end{bmatrix} = \begin{bmatrix}\partial_x\psi \\ \partial_x\bar{\psi}\end{bmatrix} \Rightarrow Q_2(\psi) = \mathrm{Im}\int_{\mathbb{R}} \psi\partial_x\bar{\psi} dx.$$

Stationary (TW) states in the NLS equation

Critical points of $\Lambda_{\omega_1,\omega_2} := H - \omega_1 Q_1 - \omega_2 Q_2$ are found from Euler–Lagrange equations

$$-\Psi'' + W'(|\Psi|^2)\Psi - \omega_1 \Psi - i\omega_2 \Psi' = 0,$$

which coincides with the stationary (TW) states of the NLS equation

$$i\psi_t = -\psi_{xx} + W'(|\psi|^2)\psi \qquad \psi(t,x) = e^{-i\omega_1 t}\Psi(x + \omega_2 t).$$

Due to the Galilean transformation

$$\Psi(x) = e^{-\frac{i}{2}\omega_2 x}\Phi(x) \quad \Rightarrow \quad -\Phi'' + W'(|\Phi|^2)\Phi - \omega\Phi = 0, \;\; \omega = \omega_1 + \frac{\omega_2^2}{4},$$

we can obtain stationary states from a single symmetry with the augmented energy $\Lambda_\omega = H - \omega Q$ with a single conservation $Q \equiv Q_1$.

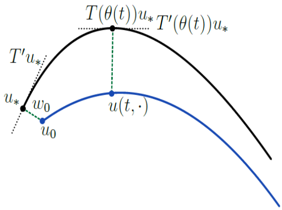
Consequencies of the symmetry I

If u_ω is a critical point of $\Lambda_\omega(u) = H(u) - \omega Q(u)$, then the kernel of $\mathcal{L}_\omega := H''(u_\omega) - \omega Q''(u_\omega)$ contains $T'u_\omega = J\nabla Q(u_\omega)$ (degeneracy).

To deal with the degeneracy due to symmetry, we can introduce the concept of orbital stability:

We say that u_ω is *orbitally stable in X* if $\forall \epsilon > 0$, $\exists \delta > 0$: $\|u_0 - u_\omega\|_X < \delta$ implies

$$\inf_{\theta \in \mathbb{R}} \|u(t,\cdot) - T(\theta)u_\omega\|_X < \epsilon, \quad \text{for } t > 0.$$

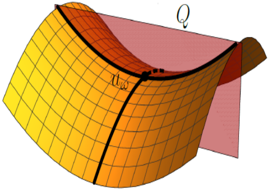


Since $X \subset \mathcal{H}$ with the inner product $\langle \cdot, \cdot \rangle$, we can look for $\theta(t)$ such that

$$\langle u(t,\cdot) - T(\theta(t))u_\omega, T'(\theta(t))u_\omega \rangle = 0$$

Consequencies of the symmetry II

If u_ω is a saddle point of $\Lambda_\omega(u) = H(u) - \omega Q(u)$, then it may still be a minimizer of energy $H(u)$ subject to fixed mass $Q(u)$.



Since $Q(u_\omega + w) =$
$Q(u_\omega) + \langle \nabla Q(u_\omega), w \rangle + \mathcal{O}(\|w\|_X^2)$,
we can use

$$\langle u(t,\cdot) - T(\theta(t))u_\omega, T(\theta(t))\nabla Q(u_\omega)\rangle = 0$$

We say that u_ω is *a strict constrained minimizer of energy* if

$$\exists C_\omega > 0 : \quad \langle \mathcal{L}_\omega w, w \rangle \geq C_\omega \|w\|_X^2, \quad \forall w \in X_{\{J\nabla Q(u_\omega), \nabla Q(u_\omega)\}^\perp}.$$

Strict constrained minimizers of energy

Theorem 9

Let $u_\omega \in X$ be a critical point of $\Lambda_\omega(u) = H(u) - \omega Q(u)$. Assume

- *the spectral gap condition*

$$\exists \delta > 0 : \quad (-\delta, \delta) \cap \sigma(\mathcal{L}_\omega) = \{0\}.$$

- *the non-degeneracy condition*
  $\ker(\mathcal{L}_\omega) = \mathrm{span}(J\nabla Q(u_\omega))$ *with* $J\nabla Q(u_\omega) \in X \subset \mathcal{H}$.

- *the robust norm condition*
  If $\|u - u_\omega\|_X < \epsilon_0 \ll 1$, *then*

$$\exists \theta_0 \in \mathbb{R} : \quad \|u - T(\theta_0)u_\omega\|_X = \inf_{\theta \in \mathbb{R}} \|u - T(\theta)u_\omega\|_X.$$

If u_ω is a constrained minimizer of $H(u)$, then u_ω is orbitally stable.

Constrained saddle points of energy

Theorem 10

Let $u_\omega \in X$ be a critical point of $\Lambda_\omega(u)$ such that there exist $w_+, w_- \in X_{\{J\nabla Q(u_\omega), \nabla Q(u_\omega)\}^\perp}$ such that $\mathcal{L}_\omega$ satisfies

$$\langle \mathcal{L}_\omega w_+, w_+ \rangle > 0 \quad \text{and} \quad \langle \mathcal{L}_\omega w_-, w_- \rangle < 0.$$

If u_ω is linearly unstable and $\nabla \Lambda_\omega(u_\omega + w) = \mathcal{L}_\omega w + \mathrm{o}_X(w)$, then u_ω is nonlinearly orbitally unstable:

$$\exists \varepsilon > 0 : \; \forall \delta > 0, \; \exists u_0 \in B_\delta(u_\omega) : u(t_0) \notin B_\varepsilon(T(\theta) u_\omega), \forall \theta \in \mathbb{R}, \; t_0 > 0.$$

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If u_ω is linearly unstable and $\nabla \Lambda_\omega(u_\omega + w) = \mathcal{L}_\omega w + o_X(w)$, then u_ω is nonlinearly orbitally unstable:

$$\exists \varepsilon > 0 : \ \forall \delta > 0, \ \exists u_0 \in B_\delta(u_\omega) : u(t_0) \notin B_\varepsilon(T(\theta) u_\omega), \forall \theta \in \mathbb{R}, \ t_0 > 0.$$

Linear instability implies nonlinear instability if the nonlinearity is bounded in X like in $\mathbb{R}^n$ and the (IVP) is well-posed in X. Constraint of fixed mass $Q(u)$ does not help to stabilize the saddle point.

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It follows from the Hamiltonian–Krein theorem,

$$n(\mathcal{L}_\omega|_{\{J\nabla Q(u_\omega), \nabla Q(u_\omega)\}^\perp}) = N_{\rm real}(J\mathcal{L}) + 2N_{\rm comp}(J\mathcal{L}) + 2 \sum_{\lambda_0 \in i\mathbb{R}_+} n_{\lambda_0},$$

that u_ω is linearly unstable if $n(\mathcal{L}_\omega|_{\{J\nabla Q(u_\omega), \nabla Q(u_\omega)\}^\perp})$ is odd.

Constrained minimizers of energy

The goal is to show that

$$\exists C_\omega > 0 : \quad \langle \mathcal{L}_\omega w, w \rangle \geq C_\omega \|w\|_X^2, \quad \forall w \in X_{\{J\nabla Q(u_\omega), \nabla Q(u_\omega)\}^\perp},$$

even if $\mathcal{L}_\omega$ has negative eigenvalues in $\mathcal{H}$.

Constrained minimizers of energy

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$$\exists C_\omega > 0 : \quad \langle \mathcal{L}_\omega w, w \rangle \geq C_\omega \|w\|_X^2, \quad \forall w \in X_{\{J\nabla Q(u_\omega), \nabla Q(u_\omega)\}^\perp},$$

even if $\mathcal{L}_\omega$ has negative eigenvalues in $\mathcal{H}$.

Theorem 11

Assume the spectral gap and non-degeneracy conditions. Then, u_ω is a constrained minimizer of $H(u)$ if either

- $\sigma(\mathcal{L}_\omega) \geq 0$ *or*
- $\mathcal{L}_\omega$ *has one simple negative eigenvalue and* $\frac{d}{d\omega} Q(u_\omega) < 0$*.*

u_ω is a constrained saddle point of $H(u)$ if either

- $\mathcal{L}_\omega$ *has one simple negative eigenvalue and* $\frac{d}{d\omega} Q(u_\omega) > 0$ *or*
- $\mathcal{L}_\omega$ *has two or more negative eigenvalues.*

[Vakhitov-Kolokolov, Bona-Souganidis-Strauss, Weinstein, Shatah-Strauss, ...]

Spectrum of $\mathcal{L}_\omega$ under constrains

We are looking for

$$\begin{aligned}
\lambda_0 &= \inf_{w \in X_{\{J\nabla Q(u_\omega), \nabla Q(u_\omega)\}^\perp}} \frac{\langle \mathcal{L}_\omega w, w \rangle}{\|w\|^2} \\
&= \inf_{w \in X, (\nu,\mu) \in \mathbb{R}^2} \frac{\langle \mathcal{L}_\omega w, w \rangle - 2\nu \langle w, \nabla Q(u_\omega) \rangle - 2\mu \langle w, J\nabla Q(u_\omega) \rangle}{\|w\|^2},
\end{aligned}$$

or, equivalently, at the lowest eigenvalue λ of

$$(\mathcal{L}_\omega - \lambda I)w = \nu \nabla Q(u_\omega) + \mu J \nabla Q(u_\omega), \quad w \in X_{\{J\nabla Q(u_\omega), \nabla Q(u_\omega)\}^\perp},$$

where $(\nu, \mu) \in \mathbb{R}^2$ are Lagrange multipliers.

Spectrum of $\mathcal{L}_\omega$ under constrains

We are looking for

$$\lambda_0 = \inf_{w \in X_{\{J\nabla Q(u_\omega), \nabla Q(u_\omega)\}^\perp}} \frac{\langle \mathcal{L}_\omega w, w\rangle}{\|w\|^2}$$
$$= \inf_{w \in X, (\nu,\mu) \in \mathbb{R}^2} \frac{\langle \mathcal{L}_\omega w, w\rangle - 2\nu \langle w, \nabla Q(u_\omega)\rangle - 2\mu \langle w, J\nabla Q(u_\omega)\rangle}{\|w\|^2},$$

or, equivalently, at the lowest eigenvalue λ of

$$(\mathcal{L}_\omega - \lambda I) w = \nu \nabla Q(u_\omega) + \mu J \nabla Q(u_\omega), \quad w \in X_{\{J\nabla Q(u_\omega), \nabla Q(u_\omega)\}^\perp},$$

where $(\nu, \mu) \in \mathbb{R}^2$ are Lagrange multipliers.
Since

$$\langle \nabla Q(u_\omega), J\nabla Q(u_\omega)\rangle = 0,$$

we have $\mu \|J\nabla Q(u_\omega)\|^2 = 0$, so that $\mu = 0$. Furthermore, the constraint $\langle w, J\nabla Q(u_\omega)\rangle = 0$ is satisfied for every $\lambda \neq 0$.

Spectrum of $\mathcal{L}_\omega$ under constrains

Hence, we are looking at the lowest eigenvalue λ in $(-\infty, 0)$ of

$$(\mathcal{L}_\omega - \lambda I) w = \nu \nabla Q(u_\omega), \quad w \in X_{\{\nabla Q(u_\omega)\}^\perp}.$$

- Either $\lambda \in \sigma(\mathcal{L}_\omega)$ with $w \in X_{\{\nabla Q(u_\omega)\}^\perp}$ and $\nu = 0$ or
- $\lambda \notin \sigma(\mathcal{L}_\omega)$ and

$$F(\lambda) = \langle (\mathcal{L}_\omega - \lambda I)^{-1} \nabla Q(u_\omega), \nabla Q(u_\omega) \rangle = 0,$$

  with $\nu \neq 0$.

The first case is included in the study of properties of $F(\lambda)$ for $\lambda \in (-\infty, 0)$.

Spectrum of $\mathcal{L}_\omega$ under constrains

Properties of $F(\lambda)$ given by $F(\lambda) = \langle (\mathcal{L}_\omega - \lambda I)^{-1} \nabla Q(u_\omega), \nabla Q(u_\omega) \rangle$.

1. F is analytic for $\lambda \notin \sigma(\mathcal{L}_\omega)$ and may have infinite jump discontinuity at $\lambda \in \sigma(\mathcal{L}_\omega)$.
2. $F(\lambda) \to 0^+$ as $\lambda \to -\infty$ since
$$|F(\lambda)| \leq |\lambda - \lambda_{\min}|^{-1} \|\nabla Q(u_\omega)\|^2, \quad \lambda < \lambda_{\min} = \inf \sigma(\mathcal{L}_\omega).$$
3. $F'(\lambda) > 0$ since $F'(\lambda) = \|(\mathcal{L}_\omega - \lambda I)^{-1} \nabla Q(u_\omega)\|^2$.
4. If $\lambda_0 \in \sigma(\mathcal{L}_\omega)$ (simple) with $w_0 \in X$, then
$$F(\lambda) = \frac{|\langle w_0, \nabla Q(u_\omega) \rangle|^2}{\lambda_0 - \lambda} + \tilde{F}(\lambda),$$
with analytic $\tilde{F}$ across $\lambda = \lambda_0$.
5. F is analytic at $\lambda = 0$ as $w_0 = J\nabla Q(u_\omega)$: $\langle w_0, \nabla Q(u_\omega) \rangle = 0$.

Spectrum of $\mathcal{L}_\omega$ under constrains

Finally, we compute the slope criterion

$$\lim_{\lambda \to 0} F(\lambda) = \langle \mathcal{L}_\omega^{-1} \nabla Q(u_\omega), \nabla Q(u_\omega) \rangle = \langle \partial_\omega u_\omega, \nabla Q(u_\omega) \rangle = \frac{d}{d\omega} Q(u_\omega).$$

Here we have used C^1 smoothness of $\Omega \ni \omega \to u_\omega \in X$ and

$$\mathcal{L}_\omega \partial_\omega u_\omega = \nabla Q(u_\omega),$$

obtained from $\nabla \Lambda_\omega(u_\omega) = \nabla H(u_\omega) - \omega \nabla Q(u_\omega) = 0$.

C^1-smoothness $\Omega \ni \omega \to u_\omega \in X$ holds under the spectral gap and non-degeneracy conditions.

Spectrum of $\mathcal{L}_\omega$ under constrains

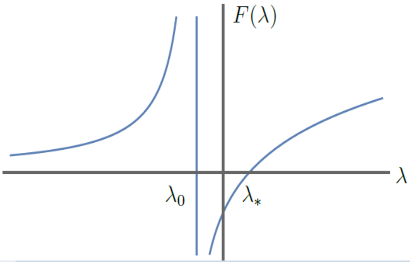
Finally, we compute the slope criterion

$$\lim_{\lambda \to 0} F(\lambda) = \langle \mathcal{L}_\omega^{-1} \nabla Q(u_\omega), \nabla Q(u_\omega) \rangle = \langle \partial_\omega u_\omega, \nabla Q(u_\omega) \rangle = \frac{d}{d\omega} Q(u_\omega).$$

If $\mathcal{L}_\omega$ has one simple negative eigenvalue and

$$\frac{d}{d\omega} Q(u_\omega) < 0$$

then



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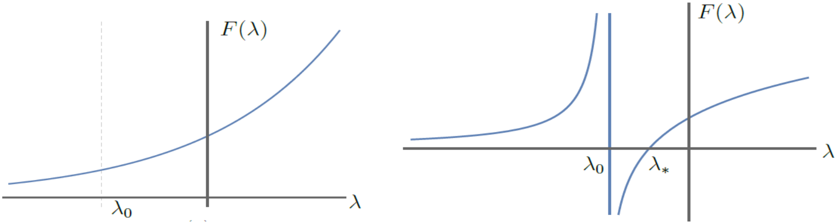
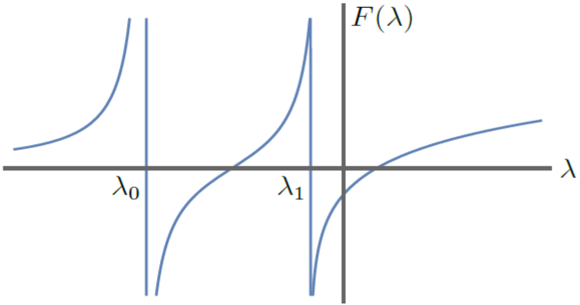


Spectrum of $\mathcal{L}_\omega$ under constrains

Finally, we compute the slope criterion

$$\lim_{\lambda\to 0} F(\lambda) = \langle \mathcal{L}_\omega^{-1} \nabla Q(u_\omega), \nabla Q(u_\omega)\rangle = \langle \partial_\omega u_\omega, \nabla Q(u_\omega)\rangle = \frac{d}{d\omega} Q(u_\omega).$$

If $\mathcal{L}_\omega$ has two (or more) negative eigenvalues, then



Example: solitons in the nonlinear Schrödinger equation

Recall the NLS equation

$$i\psi_t = -\psi_{xx} + W'(|\psi|^2)\psi, \quad W'(0) = 0,$$

with $\psi(t,x) = e^{-i\omega t}\Phi(x)$ from

$$-\Phi'' + W'(|\Phi|^2)\Phi - \omega\Phi = 0.$$

If $\Phi(x) \to 0$ as $|x| \to \infty$ (only if $\omega < 0$), then $\Phi \in \mathbb{R}$ up to the phase rotation. Perturbation $\psi - \Phi = u + iv$ is defined for $(u,v) \in X = H^1(\mathbb{R}) \times H^1(\mathbb{R})$ with

$$\mathcal{L}_\omega = \begin{bmatrix} -\partial_x^2 + W'(\Phi^2) + 2\Phi^2 W''(\Phi^2) - \omega & 0 \\ 0 & -\partial_x^2 + W'(\Phi^2) - \omega \end{bmatrix}$$

acting on (u,v).

Sturm's theory for the Schrödinger operators

We have

$$\mathcal{L}_+ : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

with

$$\mathcal{L}_+ = -\partial_x^2 + W'(\Phi^2) + 2\Phi^2 W''(\Phi^2) - \omega, \quad \omega < 0.$$

(a) $\sigma(\mathcal{L}_+) = \sigma_p(\mathcal{L}_+) \cup \sigma_c(\mathcal{L}_+) \subset \mathbb{R}$ since $\mathcal{L}_+$ is self-adjoint in $L^2(\mathbb{R})$.

(b) $\sigma_c(\mathcal{L}_+) = [|\omega|, \infty)$ since $\Phi(x) \to 0$ at $\pm\infty$ exponentially fast.

(c) Eigenvalues in $\sigma_p(L_+) = \{\lambda_1, \lambda_2, \dots\} \in (-\infty, |\omega|)$ are simple

(d) Eigenfunction for eigenvalue λ_n has $n-1$ simple zeros on $\mathbb{R}$.

(e) $\lambda_1 < 0$, $\lambda_2 = 0$ since $\mathcal{L}_+ \partial_x \Phi = 0$ and $\partial_x \Phi$ has one zero on $\mathbb{R}$.

Sturm's theory for the Schrödinger operators

We have

$$\mathcal{L}_- : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

with

$$\mathcal{L}_- = -\partial_x^2 + W'(\Phi^2) - \omega, \quad \omega < 0$$

(a) $\sigma(\mathcal{L}_-) = \sigma_p(\mathcal{L}_-) \cup \sigma_c(\mathcal{L}_-) \subset \mathbb{R}$ since $\mathcal{L}_-$ is self-adjoint in $L^2(\mathbb{R})$.

(b) $\sigma_c(\mathcal{L}_-) = [|\omega|, \infty)$ since $\Phi(x) \to 0$ at $\pm\infty$ exponentially fast.

(c) Eigenvalues in $\sigma_p(L_-) = \{\lambda_1, \lambda_2, \dots\} \in (-\infty, |\omega|)$ are simple

(d) Eigenfunction for eigenvalue λ_n has $n-1$ simple zeros on $\mathbb{R}$.

(e) $\lambda_1 = 0$ since $\mathcal{L}_-\Phi = 0$ and $\Phi(x) > 0$ for all $x \in \mathbb{R}$.

Sturm's theory for the Schrödinger operators

Hence Φ is a saddle point of energy

$$H(\psi) = \int_{\mathbb{R}} \left(|\psi_x|^2 + W(|\psi|^2) \right) dx.$$

The phase rotation symmetry gives conserved mass

$$Q(\psi) = \int_{\mathbb{R}} |\psi|^2 dx.$$

If $\frac{d}{d\omega}\|\Phi\|^2 < 0$, then Φ is orbitally stable with respect to

$$\inf_{(\theta,\xi)\in\mathbb{R}^2} \|\psi(t,\cdot) - e^{-i\theta}\Phi(\cdot+\xi)\|_{H^1(\mathbb{R},\mathbb{C})}.$$

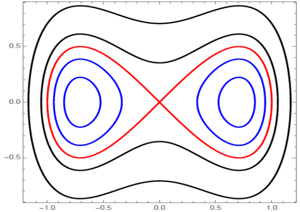
If $\frac{d}{d\omega}\|\Phi\|^2 > 0$, then Φ is orbitally unstable.

Example: pulses in the nonlinear wave equation

Consider the Hamiltonian

$$H(u,v) = \int_{\mathbb{R}} \left(\frac{1}{2}(u_x)^2 - \frac{1}{2}u^2 + \frac{1}{4}u^4 + \frac{1}{2}v^2 \right) dx$$

subject to $(u,v) \to (0,0)$ as $x \to \pm\infty$, e.g. $u_*(x) = \sqrt{2}\mathrm{sech}(x)$.



Perturbation $(u_* + w, v)$ is in $(w,v) \in X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with

$$\mathcal{L} = H''(u,v) = \begin{bmatrix} -\partial_x^2 + 1 - 3u_*^2(x) & 0 \\ 0 & 1 \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} w \\ v \end{bmatrix}.$$

Example: pulses in the nonlinear wave equation

Recall that $\mathcal{L}_0 = -\partial_x^2 + 1 - 3u_*^2$ has a simple negative eigenvalue, and hence $(u_*, 0)$ is a saddle point of energy.

The spatial translation symmetry gives the conserved momentum

$$Q(u, v) = \frac{1}{2} \int_{\mathbb{R}} (v u_x - v_x u) dx$$

Q: Is it possible for $(u_*, 0)$ to be a constrained minimizer of energy?

Example: pulses in the nonlinear wave equation

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The spatial translation symmetry gives the conserved momentum

$$Q(u, v) = \frac{1}{2} \int_{\mathbb{R}} (v u_x - v_x u) dx$$

Q: Is it possible for $(u_*, 0)$ to be a constrained minimizer of energy?

Let us check: $(u_\omega, \omega \partial_x u_\omega)$ is a critical point of $\Lambda_\omega = H - \omega Q$, which is a traveling wave solution $u(t, x) = u_\omega(x + \omega t)$. Since it satisfies $-(1 - \omega^2) u_\omega'' - u_\omega + u_\omega^3 = 0$, it is given by

$$u_\omega(x) = u_* \left(\frac{x}{\sqrt{1 - \omega^2}} \right), \quad \omega \in (-1, 1).$$

Then $Q(u_\omega, \omega \partial_x u_\omega) = \frac{\omega}{\sqrt{1-\omega^2}} \|\partial_x u_*\|^2$ and $\frac{d}{d\omega} Q(u_\omega, \omega \partial_x u_\omega) > 0$.

A: $(u_*, 0)$ is a constrained saddle point of energy.

Sharp Hamiltonian–Krein theorem for the NLS equation

Recall the pair of Schrödinger operators for $\omega < 0$:

$$\mathcal{L}_+ = -\partial_x^2 + W'(\Phi^2) + 2\Phi^2 W''(\Phi^2) + |\omega|,$$
$$\mathcal{L}_- = -\partial_x^2 + W'(\Phi^2) + |\omega|.$$

Due to the Hamiltonian structure, the spectral stability problem is

$$\mathcal{L}_+ p = \lambda q, \qquad -\mathcal{L}_- q = \lambda p, \qquad p, q \in H^2(\mathbb{R})$$

similar to the example of a mechanical system.

If $\ker(\mathcal{L}_+) = \text{span}(\partial_x \Phi)$ and $\ker(\mathcal{L}_-) = \text{span}(\Phi)$, then the eigenfunction $(p, q) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ for $\lambda \neq 0$ satisfies

$$\langle \Phi, p \rangle = 0, \qquad \langle \partial_x \Phi, q \rangle = 0.$$

This yields the generalized eigenvalue problem

$$q = -\lambda (\mathcal{L}_-|_{\{\Phi\}^\perp})^{-1} p \quad \Rightarrow \mathcal{L}_+|_{\{\Phi\}^\perp} p = -\lambda^2 (\mathcal{L}_-|_{\{\Phi\}^\perp})^{-1} p.$$

Sharp Hamiltonian–Krein theorem for the NLS equation

Hamiltonian–Krein theorem is now read as

$$n(\mathcal{L}_+|_{\{\Phi\}^\perp}) = N^-_{\rm real}(J\mathcal{L}) + N_{\rm comp}(J\mathcal{L}) + \sum_{\lambda_0 \in i\mathbb{R}_+} n_{\lambda_0},$$

$$n(\mathcal{L}_-|_{\{\Phi\}^\perp}) = N^+_{\rm real}(J\mathcal{L}) + N_{\rm comp}(J\mathcal{L}) + \sum_{\lambda_0 \in i\mathbb{R}_+} n_{\lambda_0},$$

where $N^{\pm}_{\rm real}$ is related to p_{λ_0} positive and n_{λ_0} negative eigenvalues of Krein matrix $K(\lambda_0)$ associated with $\mathcal{L}_+|_{\{\Phi\}^\perp}$ for $\lambda_0 \in \mathbb{R}_+$.

- The sum yields Grillakis–Shatah–Strauss' 90 theorem:

$$n(\mathcal{L}_+|_{\{\Phi\}^\perp}) + n(\mathcal{L}_-) = N_{\rm real}(J\mathcal{L}) + 2N_{\rm comp}(J\mathcal{L}) + 2\sum_{\lambda_0 \in i\mathbb{R}_+} n_{\lambda_0}.$$

- The difference yields Grillakis'90 or Jones'88 theorems:

$$n(\mathcal{L}_+|_{\{\Phi\}^\perp}) - n(\mathcal{L}_-) = N^-_{\rm real}(J\mathcal{L}) - N^-_{\rm real}(J\mathcal{L}).$$

Strict constrained minimizers of energy

Let us give the proof of the orbital stability result for a strict constrained minimizer of energy.

Theorem 12

Let $u_\omega \in X$ be a critical point of $\Lambda_\omega(u) = H(u) - \omega Q(u)$. Assume

- *the spectral gap condition*

$$\exists \delta > 0 : \quad (-\delta, \delta) \cap \sigma(\mathcal{L}_\omega) = \{0\}.$$

- *the non-degeneracy condition*
  $\ker(\mathcal{L}_\omega) = \mathrm{span}(J\nabla Q(u_\omega))$ *with* $J\nabla Q(u_\omega) \in X \subset \mathcal{H}$.
- *the robust norm condition*
  If $\|u - u_\omega\|_X < \epsilon_0 \ll 1$*, then*

$$\exists \theta_0 \in \mathbb{R} : \quad \|u - T(\theta_0)u_\omega\|_X = \inf_{\theta \in \mathbb{R}} \|u - T(\theta)u_\omega\|_X.$$

If u_ω is a constrained minimizer of $H(u)$, then u_ω is orbitally stable.

Proof of orbital stability of constrained minimizers I

Decomposition:

$\exists \epsilon_0 > 0$ and unique functions $(\theta, \omega, w) \in \mathbb{R} \times \Omega \times X$ near $(\theta_0, \omega_0, 0)$ such that all $u \in X$ satisfying

$$\epsilon := \|u - T(\theta_0) u_{\omega_0}\|_X = \inf_{\theta \in \mathbb{R}} \|u - T(\theta) u_{\omega_0}\|_X < \epsilon_0$$

can be written as

$$\boxed{u = T(\theta)(u_\omega + w) \quad \text{with} \quad \langle w, J\nabla Q(u_\omega)\rangle = 0, \langle w, \nabla Q(u_\omega)\rangle = 0}$$

Proof: Define the function $f : \mathbb{R} \times \Omega \times X \to \mathbb{R}^2$,

$$f(\theta, \omega, u) := \begin{bmatrix} \langle u - T(\theta) u_\omega, T'(\theta) u_\omega \rangle \\ \langle u - T(\theta) u_\omega, T(\theta) \nabla Q(u_\omega)\rangle \end{bmatrix}, \quad \theta \in \mathbb{R}, \quad \omega \in \Omega \subset \mathbb{R},$$

so that $f(\theta, \omega, u) = 0$ if and only if u can be represented as in the box.

Proof of orbital stability of constrained minimizers I

Let $\theta_0 \in \mathbb{R}$ be the argument in the infimum. By Cauchy–Schwarz inequality, we have $|f(\theta_0, \omega_0, u)| \le C_0 \epsilon$ and the Jacobian of f is

$$D_{(\theta,\omega)} f(\theta, \omega, u) = -\left[\begin{array}{cc} \|T' u_\omega\|^2 & \langle \partial_\omega u_\omega, T' u_\omega \rangle \\ \langle T' u_\omega, \nabla Q(u_\omega) \rangle & \langle \partial_\omega u_\omega, \nabla Q(u_\omega) \rangle \end{array}\right]$$

$$+ \left[\begin{array}{cc} \langle u - T(\theta) u_\omega, T''(\theta) u_\omega \rangle & \langle u - T(\theta) u_\omega, T'(\theta) \partial_\omega u_\omega \rangle \\ \langle u - T(\theta) u_\omega, T'(\theta) \delta Q(u_\omega) \rangle & \langle u - T(\theta) u_\omega, T(\theta) \partial_\omega \delta Q(u_\omega) \rangle \end{array}\right],$$

Since $\langle T' u_\omega, \nabla Q(u_\omega) \rangle = \langle J \nabla Q(u_\omega), \nabla Q(u_\omega) \rangle = 0$ and $\langle \partial_\omega u_\omega, \nabla Q(u_\omega) \rangle = \frac{d}{d\omega} Q(u_\omega) \neq 0$, the Jacobian is invertible if ϵ_0 is small enough.

By the inverse function theorem, $f(\theta, \omega, u) = 0$ has a unique solution for $(\theta, \omega) \in \mathbb{R} \times \Omega$ satisfying

$$|\theta - \theta_0| + |\omega - \omega_0| \le C_0 \epsilon$$

and the decomposition is justified.

Proof of orbital stability of constrained minimizers II

Control of the error in time:

By LWP: $\|u_0 - u_\omega\| < \delta \Rightarrow \inf_{\theta \in \mathbb{R}} \|u(t,\cdot) - T(\theta)u_\omega\|_X < \epsilon$ for $t \in [0, \tau_0)$ for some $\tau_0 > 0$. So $u(t,\cdot)$ an be decomposed as in the box.

Define

$$V(t) := H(u(t,\cdot)) - H(u_{\omega_0}) - \omega(t)\left[Q(u(t,\cdot)) - Q(u_{\omega_0})\right]$$

and

$$\Delta(\omega(t)) := H(u_{\omega(t)}) - H(u_{\omega_0}) - \omega(t)\left[Q(u_{\omega(t)}) - Q(u_{\omega_0})\right].$$

Expand $\Lambda_{\omega(t)}(u)$ near $u_{\omega(t)}$ and subtract $H(u_{\omega_0}) - \omega(t)Q(u_{\omega_0})$ from both sides:

$$V(t) = \Delta(\omega(t)) + \frac{1}{2}\langle \mathcal{L}_{\omega(t)} w(t,\cdot), w(t,\cdot)\rangle + R_\Lambda(w(t,\cdot)).$$

Proof of orbital stability of constrained minimizers II

The mapping $\omega \mapsto \Delta(\omega)$ is C^1, so we differentiate it in ω and obtain

$$\Delta'(\omega) = Q(u_{\omega_0}) - Q(u_\omega).$$

Hence, $\Delta(\omega_0) = \Delta'(\omega_0) = 0$ and $\Delta''(\omega_0) = -\frac{d}{d\omega} Q(u_\omega)|_{\omega=\omega_0} > 0$, so $\Delta(\omega)$ is quadratic near ω_0.

Together with the lower bound on $\mathcal{L}_\omega$ we obtain that $\exists C_- > 0$

$$V(t) \geq C_- |\omega(t) - \omega_0|^2 + C_- \|w(t,\cdot)\|_X^2$$

as long as $|\omega(t) - \omega_0|$ and $\|w(t,\cdot)\|_X$ are small.

Due to conservation of $H(u)$ and $Q(u)$, we have

$$\begin{aligned} V(t) &= H(u_0) - H(u_{\omega_0}) - \omega(t) \left[Q(u_0) - Q(u_{\omega_0}) \right] \\ &= V(0) - \left[\omega(t) - \omega_0 \right] \left[Q(u_0) - Q(u_{\omega_0}) \right], \end{aligned}$$

where

$$|V(0)| \leq C_0 \delta^2, \quad |Q(u_0) - Q(u_{\omega_0})| \leq 2C_0 \delta.$$

Proof of orbital stability of constrained minimizers II

The two bounds give with the Young's inequality

$$C_+\delta^2 \geq C_-|\omega(t) - \omega_0 - C_0\delta|^2 + C_-\|w(t,\cdot)\|_X^2$$

which yields

$$|\omega(t) - \omega_0| + \|w(t,\cdot)\|_X \leq C\delta, \quad t \in [0, \tau_0).$$

The triangle inequality and C^1 smoothness of $\omega \mapsto u_\omega$ yields for $\delta < \epsilon/C$:

$$\begin{aligned}
\inf_{\theta\in\mathbb{R}} \|u(t,\cdot) - T(\theta)u_{\omega_0}\|_X &\leq \|u(t,\cdot) - T(\theta(t))u_{\omega_0}\|_X \\
&\leq \|u_{\omega(t)} - u_{\omega_0}\|_X + \|u(t,\cdot) - T(\theta(t))u_{\omega(t)}\|_X \\
&\leq C|\omega(t) - \omega_0| + \|w(t,\cdot)\| \leq C\delta < \epsilon,
\end{aligned}$$

which is the orbital stability of u_{ω_0}.

5. Concluding remarks

The method of stability can be extended to constrained minimizers of energy with several constraints. This includes

- traveling solitary waves with several parameters:

$$i\psi_t = -\psi_{xx} + W'(|\psi|^2)\psi \qquad \psi(t,x) = e^{-i\omega t}\Psi(x+ct).$$

- traveling periodic waves with several parameters:

$$u_t + 2uu_x + u_{xxx} = 0, \quad u(t,x) = U(x+ct), \quad U'' + U^2 + cU = b.$$

- integrable Hamiltonian systems with commuting flows.

Integrable Hamiltonian systems

We say that the Hamiltonian system

$$\frac{du}{dt} = J\nabla H(u), \quad H(u) : X \subset \mathcal{H} \to \mathbb{R}, \quad J : \mathcal{H} \to \mathcal{H}$$

is integrable if there exists a recursion operator $R : \mathcal{H} \to \mathcal{H}$ and another energy $\tilde{H}(u) : \tilde{X} \subset \mathcal{H}$ such that

$$JR = R^* J, \quad \nabla H(u) = R\nabla \tilde{H}(u).$$

This implies that

$$\frac{du}{dt} = \tilde{J}\nabla \tilde{H}(u), \quad \tilde{H}(u) : \tilde{X} \subset \mathcal{H} \to \mathbb{R}, \quad \tilde{J} : \mathcal{H} \to \mathcal{H}$$

with $\tilde{J} = JR$. If $J^* = -J$, then $\tilde{J}^* = (JR)^* = -R^* J = -JR = -\tilde{J}$.

If $H(u)$ is conserved in t, so is $\tilde{H}(u)$ since

$$\langle J\nabla H(u), \nabla \tilde{H}(u)\rangle = \langle \tilde{J}\nabla \tilde{H}(u), \nabla \tilde{H}(u)\rangle = 0.$$

A hierarchy of integrable Hamiltonian systems

We can then introduce a sequence of Hamiltonians $\{H_k(u)\}_{k\in\mathbb{N}}$ with $H_k(u) : X_k \subset \mathcal{H} \to \mathbb{R}$ such that

$$R\nabla H_k(u) = \nabla H_{k+1}(u),$$

and

$$\frac{du}{dt_k} = J\nabla H_k(u),$$

such that

$$\langle J\nabla H_k(u), \nabla H_m(u)\rangle = 0, \quad \forall k, m \in \mathbb{N}.$$

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such that

$$\langle J\nabla H_k(u), \nabla H_m(u)\rangle = 0, \quad \forall k, m \in \mathbb{N}.$$

- For a particular PDE with fixed $k \in \mathbb{N}$, all Hamiltonians $\{H_m(u)\}_{m\in\mathbb{N}}$ are formally conserved.
- Traveling waves (multi-solitons, breathers) are critical points of

$$\Lambda_{\omega_1,\dots,\omega_k}(u) = H_{k+1}(u) - \sum_{m=1}^{k} \omega_m H_m(u).$$

Example: the NLS hierarchy

$$J = i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad R = i\begin{bmatrix} \partial_x + 2\bar{u}\partial_x^{-1}u & -2\bar{u}\partial_x^{-1}\bar{u} \\ 2u\partial_x^{-1}u & -\partial_x - 2u\partial_x^{-1}\bar{u} \end{bmatrix}$$

satisfying $JR = R^*J$. Then, $R\nabla H_k(u) = \nabla H_{k+1}(u)$ generates

$$\begin{aligned}
H_0(u,\bar{u}) &= \int |u|^2 dx, \\
H_1(u,\bar{u}) &= \frac{i}{2}\int (u\bar{u}_x - \bar{u}u_x)dx, \\
H_2(u,\bar{u}) &= \int (|u_x|^2 - |u|^4)dx, \\
H_3(u,\bar{u}) &= \frac{i}{2}\int (u_x\bar{u}_{xx} - \bar{u}_x u_{xx} - 3|u|^2(u\bar{u}_x - \bar{u}u_x))dx, \\
H_4(u,\bar{u}) &= \int (|u_{xx}|^2 - 6|u|^2|u_x|^2 - (u\bar{u}_x + \bar{u}u_x)^2 + 2|u|^6)dx.
\end{aligned}$$

Note that H_0 is defined in $\mathcal{H} = L^2(\mathbb{R},\mathbb{C})$, H_2 is defined in $X = H^1(\mathbb{R},\mathbb{C})$, and H_4 is defined in $\tilde{X} = H^2(\mathbb{R},\mathbb{C})$.

Example: traveling solitary wave in the NLS equation

The NLS equation appears for $t = t_2$:

$$\frac{d}{dt}\begin{bmatrix} u \\ \bar{u} \end{bmatrix} = i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \nabla_u H_2(u,\bar{u}) \\ \nabla_{\bar{u}} H_2(u,\bar{u}) \end{bmatrix}$$

Traveling solitary waves with two parameters:

$$iu_t + u_{xx} + 2|u|^2 u = 0, \qquad u(t,x) = e^{-i\omega t} U(x - ct)$$

satisfy

$$U'' + 2|U|^2 U - icU' + \omega U = 0, \quad \Lambda_{\omega,c} = H_2 - cH_1 - \omega H_0$$

U is a constrained minimizer of energy H_2 in $X = H^1(\mathbb{R}, \mathbb{C})$ for fixed H_1, H_0. Simultaneously, U is a constrained minimizer of H_4 in $\tilde{X} = H^2(\mathbb{R}, \mathbb{C})$ for fixed H_3, H_2 from $\tilde{\Lambda}_{\omega,c} = H_4 - cH_3 - \omega H_2$:

$$\nabla \tilde{\Lambda}_{\omega,c}(u,\bar{u}) = R^2 \nabla \Lambda_{\omega,c}(u,\bar{u}) = 0.$$

Striking example: the nonlinear Dirac equation

The massive Thirring model in laboratory coordinates

$$\left\{ \begin{array}{l} \mathrm{i}(u_t + u_x) + v = |v|^2 u, \\ \mathrm{i}(v_t - v_x) + u = |u|^2 v, \end{array} \right.$$

Conservation of mass, momentum and energy:

$$Q = \int_{\mathbb{R}} \left(|u|^2 + |v|^2\right) dx,$$

$$P = \frac{i}{2} \int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u} + v\bar{v}_x - v_x\bar{v})\, dx,$$

$$H = \frac{i}{2} \int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u} - v\bar{v}_x + v_x\bar{v})\, dx + \int_{\mathbb{R}} \left(-v\bar{u} - u\bar{v} + 2|u|^2|v|^2\right) dx,$$

and the higher-order energy (due to integrability):

$$R = \int_{\mathbb{R}} \Big[|u_x|^2 + |v_x|^2 - \frac{i}{2}(u_x\bar{u} - \bar{u}_x u)(|u|^2 + 2|v|^2) \\ -(u\bar{v} + \bar{u}v)(|u|^2 + |v|^2) + 2|u|^2|v|^2(|u|^2 + |v|^2) \Big]\, dx$$

Example: solitary waves in the nonlinear Dirac equation

Solitary waves are given by

$$\mathbf{u}(x,t) = \mathbf{U}_\omega(x) e^{i\omega t}, \quad \mathbf{U}_\omega(x) = \sin\gamma \begin{bmatrix} \operatorname{sech}\left(x\ \sin\gamma + \frac{\mathrm{i}\gamma}{2}\right) \\ \operatorname{sech}\left(x\ \sin\gamma - \frac{\mathrm{i}\gamma}{2}\right) \end{bmatrix}.$$

where $\omega := \cos\gamma \in (-1,1)$.

First derivative test: $\mathbf{U}_\omega$ is a critical point of $H + \omega Q$ and a critical point of $\Lambda_\omega := R + (1-\omega^2)Q$.

Second derivative test: $\mathbf{U}_\omega$ is a strict minimizer of R in $H^1(\mathbb{R},\mathbb{C}^2)$ for fixed Q and P. Hence, it is orbitally stable in $H^1(\mathbb{R},\mathbb{C}^2)$

[D.P.–Y. Shimabukuro, 2014]

Further directions

- Orbital and asymptotic stability of multi-solitons and breathers in integrable systems
- Spectral and orbital stability of periodic waves in integrable systems
- Casimir integrals for non-invertible J and their roles in the orbital stability analysis
- Waves of the peaked (singular) profiles and failure of "conditional" orbital stability
- Solitary waves with nonzero boundary conditions and failure of coercivity.
- Degeneracy of the second variation and the role of bifurcations.