Domain walls in harmonic potentials

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Joint work with:
S. Alama, L. Bronsard, A. Contreras *(ARMA 2015)*
A. Contreras, M. Plum *(SIMA 2018)*
A. Contreras, V. Slastikov *(Calc Var PDE 2022)*

Classification of solitary waves

Bright soliton $\psi(t,x) = e^{it}\mathrm{sech}(x)$ of the focusing NLS equation

$$i\partial_t \psi + \partial_x^2 \psi + 2|\psi|^2\psi = 0$$

satisfying $|\psi(t,x)| \to 0$ as $|x| \to \infty$

Dark soliton $\psi(t,x) = e^{-2it}\tanh(x)$ of the defocusing NLS equation

$$i\partial_t \psi + \partial_x^2 \psi - 2|\psi|^2\psi = 0$$

satisfying $|\psi(t,x)| \to 1$ as $|x| \to \infty$

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Situation can be more interesting for the coupled NLS models

$$\begin{aligned} i\partial_t \psi_1 + \partial_x^2 \psi_1 + (|\psi_1|^2 + |\psi_2|^2)\psi_1 = 0, \\ i\partial_t \psi_2 + \partial_x^2 \psi_2 + (|\psi_1|^2 + |\psi_2|^2)\psi_2 = 0, \end{aligned}$$

with the bright-bright, bright-dark, and dark-dark solitons.

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with the bright-bright, bright-dark, and dark-dark solitons.

Domain walls satisfy

$$\begin{aligned} |\psi_1(t,x)| \to 0, \quad |\psi_2(t,x)| \to 1, \quad \text{as} \;\; x \to \mp\infty \\ |\psi_1(t,x)| \to 1, \quad |\psi_2(t,x)| \to 0, \quad \text{as} \;\; x \to \pm\infty \end{aligned}$$

B. Malomed, "Past and present trends in the development of the pattern-formation theory", arXiv:2110.14935 (2021)

Domain walls from the energetic point of view

- Bulk energy with stable states

$$W : \mathbb{R}^2 \to \mathbb{R}, \quad W(u) \geq 0, \quad W(p_+) = W(p_-) = 0$$

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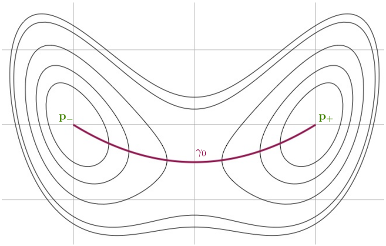
- Domain walls are stationary layers with profile U connecting $p_\pm$:

$$-U'' + DW(U) = 0, \qquad U \to p_\pm \text{ as } x \to \pm\infty$$



Example: Gross-Pitaevskii System

Motivated by two-component (repulsive) Bose-Einstein condensates,

$$
\begin{aligned}
i\partial_t \psi_1 &= -\partial_x^2 \psi_1 + (g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2)\psi_1, \\
i\partial_t \psi_2 &= -\partial_x^2 \psi_2 + (g_{12}|\psi_1|^2 + g_{22}|\psi_2|^2)\psi_2,
\end{aligned}
$$

with $g_{11} > 0$, $g_{22} > 0$, and $g_{12} > \sqrt{g_{11} g_{22}}$.

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with $g_{11} > 0$, $g_{22} > 0$, and $g_{12} > \sqrt{g_{11}g_{22}}$.

With normalization $g_{11} = g_{22} = 1$, $g_{12} = \gamma > 1$, the standing waves $\psi_j(t,x) = e^{-it}u_j(x)$ satisfy

$$-u_1'' + (u_1^2 + \gamma u_2^2 - 1)u_1 = 0,$$
$$-u_2'' + (\gamma u_1^2 + u_2^2 - 1)u_2 = 0,$$

with the bulk energy

$$W(u_1, u_2) = \frac{1}{2}\left(|u_1|^2 + |u_2|^2 - 1\right)^2 + (\gamma - 1)|u_1|^2|u_2|^2.$$

Barankov (2002), Dror-Malomed-Zeng (2011), Filatrella–Malomed (2014)

Domain wall solutions

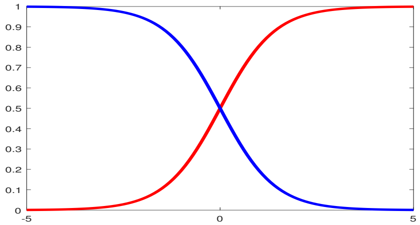
Domain walls satisfy the boundary-value problem:

$$-u_1'' + (u_1^2 + \gamma u_2^2 - 1)u_1 = 0,$$
$$-u_2'' + (\gamma u_1^2 + u_2^2 - 1)u_2 = 0,$$

with $(u_1, u_2) \to (0, 1)$ as $x \to \mp\infty$, and $(u_1, u_2) \to (1, 0)$ as $x \to \pm\infty$.

Example: exact solution for $\gamma = 3$:

$$u_1(x) = \frac{1}{2}\left[1 + \tanh\left(\frac{x}{\sqrt{2}}\right)\right], \quad u_2(x) = \frac{1}{2}\left[1 - \tanh\left(\frac{x}{\sqrt{2}}\right)\right].$$



Existence Theorem

Recall the energy $E(U) = \int_{\mathbb{R}} [\frac{1}{2}|U'|^2 + W(U)]dx$ with $U = (u_1, u_2)$ and

$$W(U) = \frac{1}{2}\left(|u_1|^2 + |u_2|^2 - 1\right)^2 + (\gamma - 1)|u_1|^2|u_2|^2.$$

Theorem (Alama–Bronsard–Contreras–P., 2015)

For $\gamma > 1$,

- *The infimum of $E(U)$ is attained among the solutions with $U(x) \to e_\pm$ as $x \to \pm\infty$, where $e_+ = (1, 0)$ and $e_- = (0, 1)$.*
- *Every minimizer $U = (u_1, u_2)$ satisfies*
  - (a) *$u_1(x) = u_2(-x)$ for all $x \in \mathbb{R}$.*
  - (b) *$u_1^2(x) + u_2^2(x) \leq 1$ for all $x \in \mathbb{R}$.*
  - (c) *$u_1'(x) > 0$ and $u_2'(x) < 0$ for all $x \in \mathbb{R}$.*
  - (d) *$0 < u_{1,2}(x) < 1$ with exponential convergence to constant states.*

Uniqueness in Aftalion-Sourdis (2016); Farina-Sciunzi-Soave (2017).

Domain walls among other stationary states

Other positive solutions exist in the coupled system:

$$-u_1'' + (u_1^2 + \gamma u_2^2 - 1)u_1 = 0,$$
$$-u_2'' + (\gamma u_1^2 + u_2^2 - 1)u_2 = 0,$$

such as the uncoupled states $(u_1, u_2) = (1, 0)$ and $(u_1, u_2) = (0, 1)$ or the symmetric state $(u_1, u_2) = (1 + \gamma)^{-1/2}(1, 1)$.

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Recall

$$W(u_1, u_2) = \frac{1}{2}\left(u_1^2 + u_2^2 - 1\right)^2 + (\gamma - 1)u_1^2 u_2^2.$$

For $\gamma \in (0, 1)$,

$$W(u_1, u_2) \geq -\frac{\gamma(1-\gamma)}{2(1+\gamma)^2} = W((1+\gamma)^{-1/2}, (1+\gamma)^{-1/2})$$

hence the symmetric state is the minimizer of $W(u_1, u_2)$ for $\gamma \in (0, 1)$.

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W(u_1, u_2) = \frac{1}{2}\left(u_1^2 + u_2^2 - 1\right)^2 + (\gamma - 1)u_1^2 u_2^2.
$$

For $\gamma > 1$,

$$
W(u_1, u_2) \geq 0 = W(1, 0) = W(0, 1)
$$

hence the uncoupled states are the minimizers of $W(u_1, u_2)$ for $\gamma > 1$.

Spaces for Minimization

Recall the energy $E(U) = \int_{\mathbb{R}} [\frac{1}{2}|U'|^2 + W(U)]dx$ with $U = (u_1, u_2)$ and

$$W(U) = \frac{1}{2}\left(|u_1|^2 + |u_2|^2 - 1\right)^2 + (\gamma - 1)|u_1|^2|u_2|^2.$$

A minimizing sequence belongs to the energy space

$$\mathcal{D} = \left\{ U \in H^1_{loc}(\mathbb{R}) : \quad |U(x)| \to e_\pm \quad \text{as} \quad x \to \pm\infty \right\}.$$

Decomposition of energy for stability argument

Let us equip the energy space with the family of distances for $A > 0$:

$$\rho_A(\Psi, \Phi) := \sum_{j=1,2} \left[\|\psi_j' - \varphi_j'\|_{L^2(\mathbb{R})} + \||\psi_j| - |\varphi_j|\|_{L^2(\mathbb{R})} + \|\psi_j - \varphi_j\|_{L^\infty(-A,A)} \right]$$

F. Bethuel, P. Gravejat, J.C. Saut, D. Smets (2008)

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Energy can be decomposed in the form:

$$E(U+V+iW) - E(U) = (L_+ V, V)_{L^2} + (L_- W, W)_{L^2} + \mathcal{O}(\|V+iW\|^3_{H^1(\mathbb{R})}).$$

Cubic terms cannot be controlled in ρ_A because of phase modulations.

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The second variation satisfies the following properties:

- Self-adjoint operator L_+ and L_- are positive semi-definite in $H^1(\mathbb{R})$.
- $\exists\ \Sigma_0 > 0 : \sigma_{ess}(L_+) = [\Sigma_0, \infty)$. $\sigma_{ess}(L_-) = [0, \infty)$
- Zero is a simple eigenvalue of L_+, with eigenfunction $\partial_x U > 0$.
- $L_- U_1 = L_- U_2 = 0$ with $U_1 = (u_1, 0)$ and $U_2 = (0, u_2)$.

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As a result, we have

$$(L_+ V, V)_{L^2} \geq C_0 \|V\|_{H^1}^2 \quad \text{for every} \quad V \in H^1(\mathbb{R}) : \quad (V, \partial_x U)_{L^2} = 0$$

but

$$(L_- W, W)_{L^2} \geq 0, \quad \text{with} \quad L_- U_1 = L_- U_2 = 0.$$

Complex phases can not be controlled far away from the domain walls.

Alternative decomposition of energy

Energy can be decomposed in the equivalent way:

$$E(U+V+iW) - E(U) = (L_- V, V)_{L^2} + (L_- W, W)_{L^2} + \frac{1}{2}(M\Upsilon, \Upsilon)_{L^2},$$

where $\Upsilon = (\eta_1, \eta_2)$ with $\eta_j := |u_j + v_j + iw_j|^2 - u_j^2 = 2u_j v_j + v_j^2 + w_j^2$ and

$$M = \begin{bmatrix} 1 & \gamma \\ \gamma & 1 \end{bmatrix} : \quad \det(M) = 1 - \gamma^2 < 0.$$

Gravejat–Smets (2015)

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One can introduce weighted H^1 space:

$$\langle \Psi, \Phi \rangle_{\mathcal{H}} := \sum_{j=1}^{2} \int_{\mathbb{R}} \left[\frac{d\psi_j}{dx} \frac{d\bar{\varphi}_j}{dx} + (\gamma - 1)(1 - u_j^2)\psi_j \bar{\varphi}_j \right] dx$$

and write

$$(L_- W, W)_{L^2} = \|W\|_{\mathcal{H}}^2 - \gamma \langle TW, W \rangle_{\mathcal{H}},$$

where $T : \mathcal{H} \to \mathcal{H}$ is the compact positive operator defined by

$$\langle T\Psi, \Phi \rangle_{\mathcal{H}} := \int_{\mathbb{R}} \left(1 - u_1^2 - u_2^2\right) \left(\psi_1 \bar{\varphi}_1 + \psi_2 \bar{\varphi}_2\right) dx.$$

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$$E(U + V + iW) - E(U) = (L_- V, V)_{L^2} + (L_- W, W)_{L^2} + \frac{1}{2}(M\Upsilon, \Upsilon)_{L^2},$$

Then,

- The spectrum of L_- in $\mathcal{H}$ consists of isolated eigenvalues accumulating to 1.
- The smallest eigenvalue of L_- is a double zero with $U_1 = (u_1, 0) \in \mathcal{H}$ and $U_2 = (0, u_2) \in \mathcal{H}$.

As a result, the quadratic form is coercive under the two constraints

$$(L_- W, W)_{L^2} \geq C\|W\|_{\mathcal{H}}^2 \quad \forall W \in \mathcal{H}: \quad \langle W, U_1 \rangle_{\mathcal{H}} = \langle W, U_2 \rangle_{\mathcal{H}} = 0.$$

Alternative decomposition of energy

Energy can be decomposed in the equivalent way:

$$E(U+V+iW)-E(U)=(L_-V,V)_{L^2}+(L_-W,W)_{L^2}+\frac{1}{2}(M\Upsilon,\Upsilon)_{L^2},$$

However,

- Only one constraint can be set on V in $(L_-V,V)_{L^2}$.
- The nonlinear part $(M\Upsilon,\Upsilon)_{L^2}$ is sign-indefinite.

In order to control these two terms, we introduce the family of distances parameterized by $R>0$:

$$\rho_R(\Psi,\Phi) := \|\Psi-\Phi\|_{\mathcal{H}} + \sum_{j=1,2} \big\| |\psi_j| - |\varphi_j| \big\|_{L^2(|x|\geq R)}.$$

Alternative decomposition of energy (revised)

The revised alternative decomposition can be controlled in ρ_R:

$$E(U + V + iW) - E(U) = (L_R V, V)_{L^2} + (L_- W, W)_{L^2}$$
$$+ \int_{-R}^{R} [N_3(V, W) + N_4(V, W)]\, dx + \frac{1}{2}\left(\int_{-\infty}^{-R} + \int_{R}^{\infty}\right)(\eta_1^2 + \eta_2^2) dx$$
$$+ \gamma \int_{-\infty}^{-R} \eta_2 (2u_1 v_1 + v_1^2 + w_1^2) dx + \gamma \int_{R}^{\infty} \eta_1 (2u_2 v_2 + v_2^2 + w_2^2) dx,$$

where

$$\begin{aligned} L_R &= L_- + 2 \begin{bmatrix} u_1^2 & \gamma u_1 u_2 \\ \gamma u_1 u_2 & u_2^2 \end{bmatrix} \chi_{[-R,R]} \\ &= L_+ - 2 \begin{bmatrix} u_1^2 & \gamma u_1 u_2 \\ \gamma u_1 u_2 & u_2^2 \end{bmatrix} \chi_{(-\infty,-R)\cup(R,\infty)}. \end{aligned}$$

As $R \to \infty$, $L_R \to L_+$ and $L_+ \partial_x U = 0$ with $\partial_x U \in \mathcal{H}$.

Alternative decomposition of energy (revised)

- The spectrum of L_R in $\mathcal{H}$ consists of isolated eigenvalues accumulating to 1.
- The zero eigenvalue is shifted for $R < \infty$ but is near 0 if R is large.

As a result, the quadratic form is coercive under one constraint

$$(L_R V, V)_{L^2} \geq C\|V\|^2_{\mathcal{H}} \quad \forall V \in \mathcal{H}: \quad \langle V, \partial_x U\rangle_{\mathcal{H}} = 0.$$

The nonlinear terms can be controlled inside and outside of $[-R, R]$, e.g.

$$\|V + iW\|_{H^1(-R,R)} \leq Ce^{\kappa R}\|V + iW\|_{\mathcal{H}}$$

and

$$\left|\int_R^\infty \eta_1(2u_2 v_2 + v_2^2 + w_2^2)dx\right| \leq Ce^{-\kappa R}\|V + iW\|_{\mathcal{H}}\|\eta_1\|_{L^2(|x|\geq R)}.$$

Orbital Stability

Theorem (Contreras–P–Plum, 2018)

Let $\Psi_0 \in \mathcal{D} \cap L^\infty(\mathbb{R})$. *There exists* $R_0 > 0$ *such that for any* $R > R_0$ *and for every* $\varepsilon > 0$, *there is* $\delta > 0$ *and real functions* $\alpha(t), \theta_1(t), \theta_2(t)$ *such that if* $\rho_R(\Psi_0, U) \leq \delta$, *then* $\sup_{t\in\mathbb{R}} \rho_R(\Psi(t), U_{\alpha(t),\theta_1(t),\theta_2(t)}) \leq \varepsilon$, *where*

$$U_{\alpha(t),\theta_1(t),\theta_2(t)} = (e^{-i\theta_1(t)} u_1(\cdot - \alpha(t)), e^{-i\theta_2(t)} u_2(\cdot - \alpha(t))).$$

Here

$$\rho_R(\Psi, \Phi) := \left\| \Psi - \Phi \right\|_{\mathcal{H}} + \sum_{j=1,2} \left\| |\psi_j| - |\varphi_j| \right\|_{L^2(|x|\geq R)}$$

and

$$\langle \Psi, \Phi \rangle_{\mathcal{H}} := \sum_{j=1}^{2} \int_{\mathbb{R}} \left[\frac{d\psi_j}{dx} \frac{d\bar{\varphi}_j}{dx} + (\gamma - 1)(1 - u_j^2)\psi_j \bar{\varphi}_j \right] dx.$$

Remarks

- Modulation parameters α, θ_1, and θ_2 in the orbit of domain walls

$$U_{\alpha(t),\theta_1(t),\theta_2(t)} = (e^{-i\theta_1(t)} u_1(\cdot - \alpha(t)), e^{-i\theta_2(t)} u_2(\cdot - \alpha(t)))$$

are uniquely determined by the projection algorithm.

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are uniquely determined by the projection algorithm.

- The time evolution of the modulation parameters is controlled:

$$|\alpha(t)| + |\theta_1(t)| + |\theta_2(t)| \leq C\varepsilon(1 + |t|), \quad t \in \mathbb{R}$$

for some $C > 0$.

Domain walls in external potentials

Consider the domain walls in external potentials:

$$
\begin{aligned}
i\partial_t \psi_1 &= -\partial_x^2 \psi_1 + V(x)\psi_1 + (|\psi_1|^2 + \gamma |\psi_2|^2)\psi_1, \\
i\partial_t \psi_2 &= -\partial_x^2 \psi_2 + V(x)\psi_2 + (\gamma |\psi_1|^2 + |\psi_2|^2)\psi_2,
\end{aligned}
$$

where $V \in C^2(\mathbb{R}) \cap L^1(\mathbb{R})$ is small in some sense.

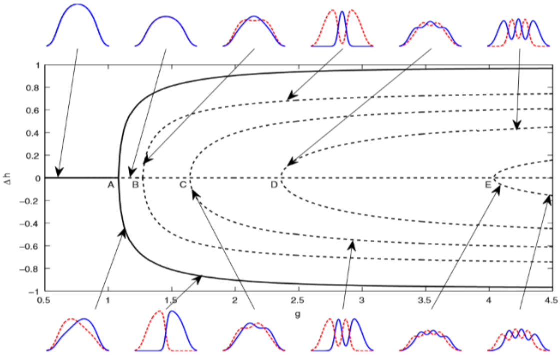
Domain walls (u_1, u_2) are pinned to the extremal points of the potential V and the pinning is stable at the maximum of the potential.
(Dror-Malomed-Zeng 2011, Alama–Bronsard–Contreras–P 2015).

For applications to Bose–Einstein condensates in magnetic traps, we need to consider $V(x) = x^2$ which violates assumptions on $V(x)$.

Numerical results (motivations)

Consider the domain walls in the ε-perturbed system:

$$i\partial_t \psi_1 = -\partial_x^2 \psi_1 + x^2 \psi_1 + (|\psi_1|^2 + \gamma |\psi_2|^2)\psi_1,$$
$$i\partial_t \psi_2 = -\partial_x^2 \psi_2 + x^2 \psi_2 + (\gamma |\psi_1|^2 + |\psi_2|^2)\psi_2.$$



Navarro–Carretero-Gonzále–Kevrekidis 2008

Thomas–Fermi limit for BECs in harmonic potentials

Stationary system of Gross–Pitaevskii equations is

$$-\varepsilon^2 \partial_x^2 \psi_1 + x^2 \psi_1 + (\psi_1^2 + \gamma \psi_2^2 - 1)\psi_1 = 0,$$
$$-\varepsilon^2 \partial_x^2 \psi_2 + x^2 \psi_2 + (\gamma \psi_1^2 + \psi_2^2 - 1)\psi_2 = 0,$$

where the limit $\varepsilon \to 0$ is referred to as the Thomas–Fermi limit.

The energy is defined in $H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$:

$$G_\varepsilon(\Psi) = \frac{1}{2} \int_{\mathbb{R}} \Big[\varepsilon^2 (\psi_1')^2 + \varepsilon^2 (\psi_2')^2 + (x^2 - 1)(\psi_1^2 + \psi_2^2) + \frac{1}{2}(\psi_1^2 + \psi_2^2)^2 + (\gamma - 1)\psi_1^2 \psi_2^2 \Big] dx$$

All solutions decay like Hermite–Gauss functions at infinity.

Ground state of the scalar Gross–Pitaevskii theory

The scalar stationary Gross–Pitaevskii equation

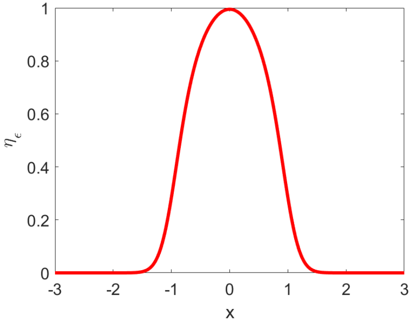
$$-\varepsilon^2 \eta_\varepsilon''(x) + (x^2 + \eta_\varepsilon^2(x) - 1)\eta_\varepsilon(x) = 0,$$

and the solution with $\eta_\varepsilon(x) > 0$ is referred to as the ground state.

The limiting TF cloud is

$$\lim_{\varepsilon \to 0} \eta_\varepsilon(x) = \sqrt{1 - x^2}\, 1_{\{|x|<1\}}$$

with the convergence:
$\|\eta_\varepsilon - \eta_0\|_{L^\infty} \leq C\varepsilon^{1/3}$,
$\|\eta_\varepsilon'\|_{L^\infty} \leq C\varepsilon^{-1/3}$.
Ignat–Millot (2006); Gallo–P (2011)



Domain walls in harmonic potentials

By using the transformation $\psi_{1,2}(x) = \eta_\varepsilon(x)\phi_{1,2}(x/\varepsilon)$ and changing the variables $x \to z := x/\varepsilon$, we obtain $G_\varepsilon(\Psi) = F_\varepsilon(\eta_\varepsilon) + \varepsilon J_\varepsilon(\Phi)$, where

$$J_\varepsilon(\Phi) = \frac{1}{2}\int_{\mathbb{R}} \eta_\varepsilon(\varepsilon z)^2 \left[(\phi_1')^2 + (\phi_2')^2\right] dz + \frac{1}{2}\int_{\mathbb{R}} \eta_\varepsilon(\varepsilon z)^4 \left[\frac{1}{2}(\phi_1^2 + \phi_2^2 - 1)^2 + (\gamma - 1)\phi_1^2\phi_2^2\right] dz.$$

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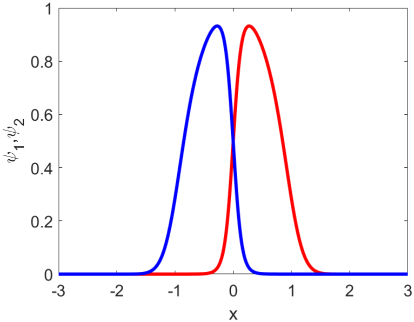
The limit $\varepsilon \to 0$ with $\eta_0(0) = 1$ recovers domain walls without harmonic potentials for $\gamma > 1$. By the Γ convergence theorem,

$$J_\varepsilon \to J_0 \;\text{ as } \varepsilon \to 0$$

Contreras–P–Slastikov (2022)



Domain walls among other stationary states

Other positive solutions exist in the coupled system:

$$-\varepsilon^2 \partial_x^2 \psi_1 + x^2 \psi_1 + (\psi_1^2 + \gamma \psi_2^2 - 1)\psi_1 = 0,$$
$$-\varepsilon^2 \partial_x^2 \psi_2 + x^2 \psi_2 + (\gamma \psi_1^2 + \psi_2^2 - 1)\psi_2 = 0,$$

such as the uncoupled states $(\psi_1, \psi_2) = (\eta_\varepsilon, 0)$ and $(\psi_1, \psi_2) = (0, \eta_\varepsilon)$ or the symmetric state $(\psi_1, \psi_2) = (1+\gamma)^{-1/2}(\eta_\varepsilon, \eta_\varepsilon)$.

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By the same reason for $\gamma \in (0,1)$,

$$\begin{aligned} W(u_1, u_2) &= \frac{1}{2}\left(u_1^2 + u_2^2 - 1\right)^2 + (\gamma - 1) u_1^2 u_2^2 \\ &\geq -\frac{\gamma(1-\gamma)}{2(1+\gamma)^2} = W((1+\gamma)^{-1/2}, (1+\gamma)^{-1/2}) \end{aligned}$$

hence the symmetric state is the minimizer of G_ε for $\gamma \in (0,1)$.

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By the same reason for $\gamma > 1$,

$$W(u_1, u_2) = \frac{1}{2}\left(u_1^2 + u_2^2 - 1\right)^2 + (\gamma - 1)u_1^2 u_2^2$$
$$\geq 0 = W(1, 0) = W(0, 1)$$

hence the uncoupled states are the minimizers of G_ε for $\gamma > 1$.

Spaces for Minimization

Domain walls arise as minimizers in the energy space with the symmetry:

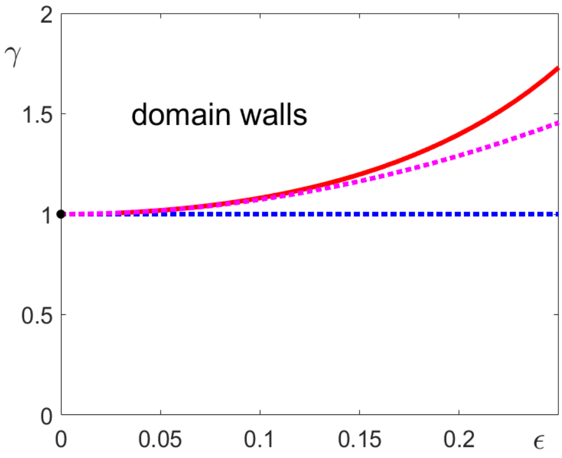
$$\mathcal{E}_s := \{\Psi \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R}) : \quad \psi_1(x) = \psi_2(-x), \;\; x \in \mathbb{R}\}.$$

Theorem (Contreras–P–Slastikov, 2022)

There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is $\gamma_0(\varepsilon) \in (1, \infty)$ such that the symmetric state is a global minimizer of the energy G_ε in $\mathcal{E}_s$ if $\gamma \in (0, \gamma_0(\varepsilon)]$ and a saddle point if $\gamma \in (\gamma_0(\varepsilon), \infty)$.

Domain wall states are global minimizers of the energy G_ε in $\mathcal{E}_s$ if $\gamma \in (\gamma_0(\varepsilon), \infty)$: one satisfies $\psi_1(x) > \psi_2(x) > 0$ for $x > 0$ and the other one obtained by the transformation $\psi_1 \leftrightarrow \psi_2$.

Bifurcation diagram



It is obvious that $\gamma_0(\varepsilon) \to 1$ as $\varepsilon \to 0$.

Second variation

Variational analysis is complemented by the study of the second variation:

$$G_\varepsilon''(\Psi) = \begin{pmatrix} -\varepsilon^2\partial_x^2 + x^2 + 3\psi_1^2 + \gamma\psi_2^2 - 1 & 2\gamma\psi_1\psi_2 \\ 2\gamma\psi_1\psi_2 & -\varepsilon^2\partial_x^2 + x^2 + \gamma\psi_1^2 + 3\psi_2^2 - 1 \end{pmatrix}$$
$$= \begin{pmatrix} -\varepsilon^2\partial_x^2 + x^2 + \frac{3+\gamma}{1+\gamma}\eta_\varepsilon^2 - 1 & \frac{2\gamma}{1+\gamma}\eta_\varepsilon^2 \\ \frac{2\gamma}{1+\gamma}\eta_\varepsilon^2 & -\varepsilon^2\partial_x^2 + x^2 + \frac{3+\gamma}{1+\gamma}\eta_\varepsilon^2 - 1 \end{pmatrix}.$$

Second variation

After rotation, it becomes

$$\begin{pmatrix} L_+ & 0 \\ 0 & L_- + 2\frac{1-\gamma}{1+\gamma}\eta_\varepsilon^2 \end{pmatrix}, \qquad \begin{array}{l} L_+ := -\varepsilon^2\partial_x^2 + x^2 - 1 + 3\eta_\varepsilon^2, \\ L_- := -\varepsilon^2\partial_x^2 + x^2 - 1 + \eta_\varepsilon^2, \end{array}$$

where $L_+ > 0$ and $L_- \geq 0$ because $L_-\eta_\varepsilon = 0$ Gallo–P, 2011

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Under the symmetry constraint in

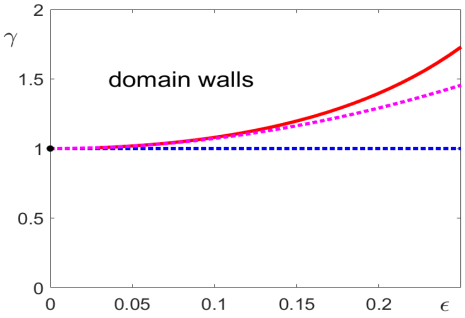
$$\mathcal{E}_s := \{\Psi \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R}) : \quad \psi_1(x) = \psi_2(-x), \ \ x \in \mathbb{R}\}.$$

and the rotation, the operator

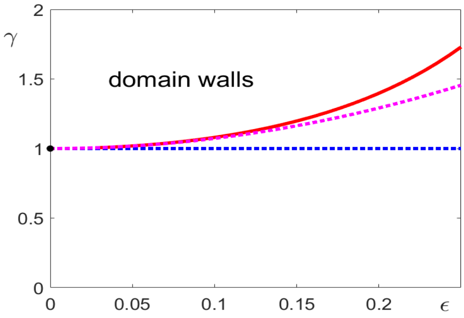
$$L_\gamma := L_- + 2\frac{1-\gamma}{1+\gamma}\eta_\varepsilon^2$$

is considered in $H^1_0(0,\infty)$ with Dirichlet condition at $x = 0$.
Bifurcation corresponds to the lowest eigenvalue of L_γ crossing 0.

More about bifurcation diagram



More about bifurcation diagram



For $\gamma = 1$, there is rotational symmetry with 1-parameter family

$$\psi_1(x) = \cos\theta\ \eta_\varepsilon(x), \quad \psi_2(x) = \sin\theta\ \eta_\varepsilon(x).$$

If $\gamma \neq 1$, however, only solutions with $\theta = \{0, \frac{\pi}{4}, \frac{\pi}{2}\}$ bifurcate from the family and they correspond to the uncoupled and symmetric states.

Proper analysis of bifurcation at $\gamma_0(\varepsilon)$ as $\varepsilon \to 0$

Let us rescale near $(\varepsilon, \gamma) = (0, 1)$:

$$z \mapsto y := z\sqrt{\gamma - 1}, \quad \Phi(z) = \Theta(y), \quad \varepsilon = \mu\sqrt{\gamma - 1},$$

so that $J_\varepsilon(\Phi) = \sqrt{\gamma - 1} I_{\mu,\gamma}(\Theta)$ is given by

$$I_{\mu,\gamma}(\Theta) = \frac{1}{2}\int_{\mathbb{R}} \eta_\varepsilon(\mu y)^2 \left[(\theta_1')^2 + (\theta_2')^2\right] dy$$
$$+ \frac{1}{4(\gamma - 1)}\int_{\mathbb{R}} \eta_\varepsilon(\mu y)^4 (\theta_1^2 + \theta_2^2 - 1)^2 dy + \frac{1}{2}\int_{\mathbb{R}} \eta_\varepsilon(\mu y)^4 \theta_1^2 \theta_2^2 dy.$$

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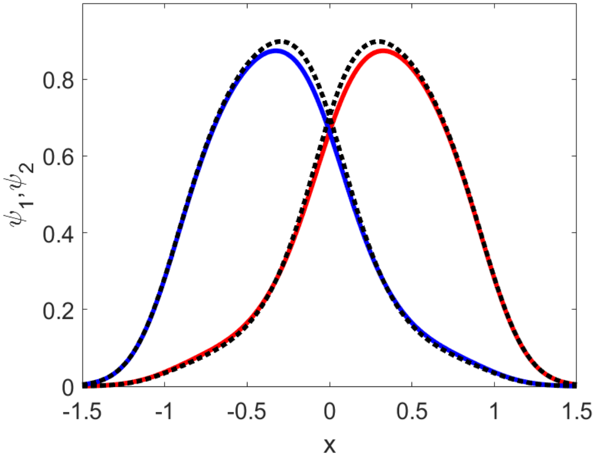
The Γ convergence as $\gamma \to 1$ gives $(\theta_1, \theta_2) = (\sin(u), \cos(u))$ with

$$I_{\mu,\gamma}(\Theta) \to I_{\mu,1}(\Theta) = \frac{1}{2}\int_{-\mu^{-1}}^{\mu^{-1}} \left[\eta_0(\mu y)^2 (u')^2 + \frac{1}{4}\eta_0(\mu y)^4 \sin^2(2u)\right] dy.$$

Comparison between numerical and asymptotic approximations for $\varepsilon = 0.1$ and $\gamma = 1.2$



Second variation again

Theorem (Contreras–P–Slastikov, 2022)

There exists $\mu_0 \in (0, \infty)$ such that the symmetric state is a global minimizer of the energy $I_{\mu,1}$ in $\mathcal{E}_s$ if $\mu \in [\mu_0, \infty)$ and a saddle point of the energy in $\mathcal{E}_s$ if $\mu \in (0, \mu_0)$. The domain wall states exist only if $\mu \in (0, \mu_0)$ and are global minimizers of the energy $I_{\mu,1}$ in $\mathcal{E}_s$.

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The symmetric state corresponds to the solution $u = \frac{\pi}{4}$ for $(\theta_1,\theta_2) = (\sin(u),\cos(u)) = (\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ in

$$I_{\mu,1}(\Theta) = \frac{1}{2}\int_{-\mu^{-1}}^{\mu^{-1}} \left[\eta_0(\mu y)^2 (u')^2 + \frac{1}{4}\eta_0(\mu y)^4 \sin^2(2u)\right] dy.$$

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The second variation of $I_{\mu,1}$ in $\mathcal{E}_s$ at $u = \frac{\pi}{4}$ gives

$$\delta^2 I_{\mu,1} = \int_0^{\mu^{-1}} \left[\eta_0(\mu y)^2 (\tilde{u}')^2 - \eta_0(\mu y)^4 \tilde{u}^2\right] dy,$$

where the perturbation $\tilde{u}$ satisfies $\tilde{u}(0) = 0$.

Second variation again

Theorem (Contreras–P–Slastikov, 2022)

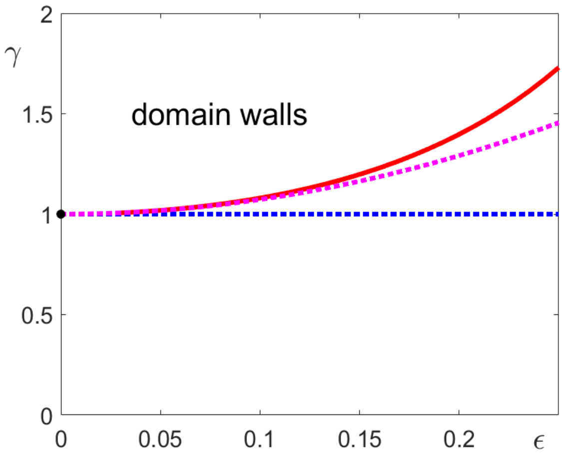
There exists $\mu_0 \in (0, \infty)$ such that the symmetric state is a global minimizer of the energy $I_{\mu,1}$ in $\mathcal{E}_s$ if $\mu \in [\mu_0, \infty)$ and a saddle point of the energy in $\mathcal{E}_s$ if $\mu \in (0, \mu_0)$. The domain wall states exist only if $\mu \in (0, \mu_0)$ and are global minimizers of the energy $I_{\mu,1}$ in $\mathcal{E}_s$.

Bifurcation corresponds to the lowest eigenvalue $\nu = \mu^{-2}$

$$-\frac{d}{dx}\left[(1-x^2)\frac{dv}{dx}\right] = \nu(1-x^2)^2 v(x), \quad 0 < x < 1.$$

It is found at $\nu_0 \approx 7.29$ which determines $\gamma_0(\varepsilon) = 1 + \nu_0 \varepsilon^2 + \mathcal{O}(\varepsilon^4)$.

Bifurcation diagram



The magenta line corresponds to $\gamma_0(\varepsilon) = 1 + \nu_0 \varepsilon^2$.

Numerical approximations of domain walls

In the energy space with the symmetry,

$$\mathcal{E}_s := \{\Psi \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R}) : \quad \psi_1(x) = \psi_2(-x), \;\; x \in \mathbb{R}\},$$

we can introduce the parameter $\alpha := \psi_1(0) = \psi_2(0)$ and consider minimizers of energy G_ε in $\mathcal{E}_s(\alpha)$ for fixed $\alpha > 0$.

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Theorem (Contreras–P–Slastikov, 2022)

Fix $\varepsilon > 0$, $\gamma > 0$, and $\alpha > 0$. Let $\Psi \in \mathcal{E}_s(\alpha)$ be a critical point of the energy G_ε satisfying $\psi_1(x) > \psi_2(x) > 0$ for all $x \in (0,\infty)$. Then

$\Psi = (\psi_1, \psi_2)$ and $\Psi' = (\psi_2, \psi_1)$ are the only global minimizers of the energy G_ε in $\mathcal{E}_s(\alpha)$.

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Two equivalent criteria for the minimizers $\Psi_\alpha = (\psi_1, \psi_2)$ of G_ε in $\mathcal{E}_s(\alpha)$ to become the domain wall solutions:

- Split function $S_\varepsilon(\alpha) := \psi_1'(0) + \psi_2'(0)$ vanishes
- Energy $G_\varepsilon(\Psi_\alpha)$ is minimal.

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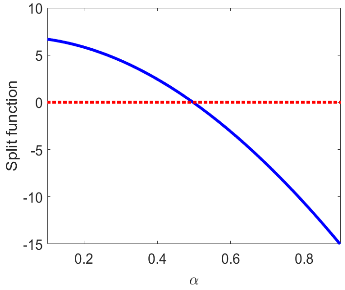
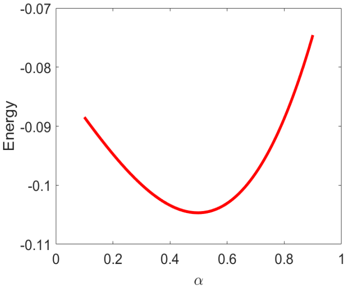
For fixed $\varepsilon = 0.1$ and $\gamma = 3$:





Summary

Domain walls $\Psi = (\psi_1, \psi_2)$ of the coupled Gross–Pitaevskii equations:

$$-\varepsilon^2 \partial_x^2 \psi_1 + x^2 \psi_1 + (\psi_1^2 + \gamma \psi_2^2 - 1)\psi_1 = 0,$$
$$-\varepsilon^2 \partial_x^2 \psi_2 + x^2 \psi_2 + (\gamma \psi_1^2 + \psi_2^2 - 1)\psi_2 = 0,$$

- minimizers of energy G_ε in the energy space with symmetry
- orbitally stable in a weighted $H^1(\mathbb{R})$ space
- persist under harmonic potentials
- asymptotically stable (conjecture)
- do not travel in space (conjecture)