Drift and instability of steady states on star graphs

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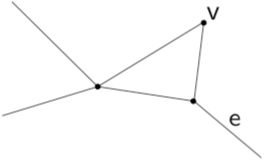
Nonlinear Schrödinger equation on a metric graph



A metric graph $\Gamma = \{E, V\}$ is given by a set of edges E and vertices V, with a metric structure on each edge.

Nonlinear Schrödinger equation on a graph Γ:

$$i\Psi_t = -\Delta\Psi - 2|\Psi|^2\Psi, \quad x \in \Gamma,$$

where Δ is the graph Laplacian and $\Psi(t,x)$ is defined componentwise on edges subject to Neumann–Kirchhoff boundary conditions at vertices:

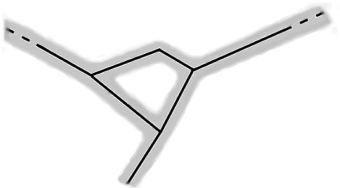
$$\begin{cases} \Psi(v) \text{ is continuous} & \text{for every } v \in V, \\ \sum_{e\sim v} \partial\Psi_e(v) = 0, & \text{for every } v \in V, \end{cases}$$

where $e \sim v$ denotes all edges $e \in E$ adjacent to $v \in V$.

Metric Graphs

Graph models are widely used in the modeling of quantum dynamics of thin graph-like structures (quantum wires, nanotechnology, large molecules, periodic arrays in solids, photonic crystals...).

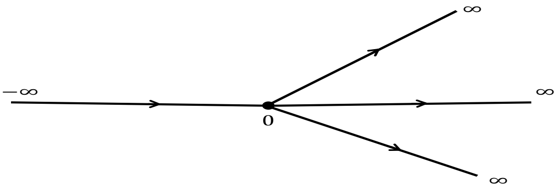
Graphs are one-dimensional approximations for constrained dynamics in which transverse dimensions are small with respect to longitudinal ones.



- G. Berkolaiko and P. Kuchment, *Introduction to Quantum Graphs* (AMS, Providence, 2013).
- P. Exner and H. Kovarík, *Quantum Waveguides* (Springer, 2015).

Example: a star graph

A star graph is the union of N half-lines connected at a single vertex. For $N = 2$, the graph is the line $\mathbb{R}$. For $N = 3$, the graph is a Y-junction.



Function spaces are defined componentwise:

$$L^2(\Gamma) = L^2(\mathbb{R}^-) \oplus \underbrace{L^2(\mathbb{R}^+) \oplus \cdots \oplus L^2(\mathbb{R}^+)}_{\text{(N-1) elements}},$$

subject to the Neumann–Kirchhoff conditions at a single vertex:

$$H^1_\Gamma := \{\Psi \in H^1(\Gamma) : \quad \psi_1(0) = \psi_2(0) = \cdots = \psi_N(0)\}$$

$$H^2_\Gamma := \{\Psi \in H^2(\Gamma) \cap H^1_\Gamma : \quad \psi_1'(0) = \sum_{j=2}^{N} \psi_j'(0)\},$$

Generalization of a star graph

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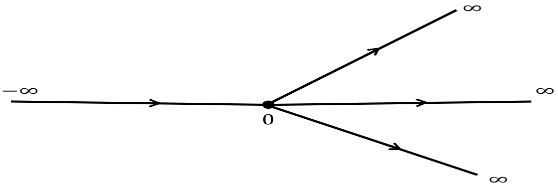

For given positive $(\alpha_1, \cdots, \alpha_N)$,

$$H^1_\Gamma := \{\Psi \in H^1(\Gamma) : \quad \alpha_1\psi_1(0) = \alpha_2\psi_2(0) = \cdots = \alpha_N\psi_N(0)\}$$

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Laplacian on the star graph

The Laplacian operator on the star graph Γ is defined by

$$\Delta\Psi = (\psi_1'', \psi_2'', \cdots, \psi_N'')$$

acting on functions in $L^2(\Gamma) = \oplus_{j=1}^N L^2(\mathbb{R}^+)$
(the first edge is reflected to $\mathbb{R}^+$ for convenience).

Lemma. $\Delta : H^2_\Gamma \subset L^2(\Gamma) \to L^2(\Gamma)$ is self-adjoint.

The Neumann–Kirchhoff boundary conditions are symmetric:

$$\langle \Phi, \Delta\Psi \rangle - \langle \Delta\Phi, \Psi \rangle = \sum_{j=1}^N \phi_j'(0)\psi_j(0) - \phi_j(0)\psi_j'(0) = 0.$$

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The generalized boundary conditions are also symmetric:

$$\left\{ \begin{array}{l} \alpha_1 \psi_1(0) = \alpha_2 \psi_2(0) = \cdots = \alpha_N \psi_N(0) \\ \alpha_1^{-1} \psi_1'(0) + \alpha_2^{-1} \psi_2'(0) + \cdots + \alpha_N^{-1} \psi_N'(0) = 0. \end{array} \right.$$

NLS evolution on the star graph

The Cauchy problem for the NLS flow:

$$\begin{cases} i\Psi_t = -\Delta\Psi - 2|\Psi|^2\Psi, \\ \Psi|_{t=0} = \Psi_0. \end{cases}$$

Lemma. The Cauchy problem is locally and globally well-posed for either $\Psi_0 \in H^1_\Gamma$ or for $\Psi_0 \in H^2_\Gamma$. Moreover, the mass

$$Q(\Psi) = \|\Psi\|^2_{L^2(\Gamma)}$$

and the energy

$$E(\Psi) = \|\Psi'\|^2_{L^2(\Gamma)} - \|\Psi\|^4_{L^4(\Gamma)},$$

are constants in time for $\Psi \in C(\mathbb{R}, H^1_\Gamma)$.

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$E(\Psi)$ is coercive in $H^1(\Gamma)$ thanks to Gagliardo–Nirenberg inequality:

$$\|\Psi\|^4_{L^4(\Gamma)} \leq C_\Gamma \|\Psi'\|_{L^2(\Gamma)} \|\Psi\|^3_{L^2(\Gamma)},$$

where $C_\Gamma > 0$ depends on Γ only.

Ground state on the unbounded graphs

Ground state is a standing wave of smallest energy E at fixed mass Q,

$$\mathcal{E} = \inf\{E(u) : \quad u \in H^1_\Gamma, \quad Q(u) = \mu\}.$$

Euler–Lagrange equation for the standing waves:

$$-\Delta\Phi - 2|\Phi|^2\Phi = -\omega\Phi,$$

where $\omega > 0$ defines $\Psi(t,x) = \Phi(x)e^{i\omega t}$.

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Infimum $\mathcal{E}$ exists thanks to the same Gagliardo–Nirenberg inequality.

Theorem. (Adami–Serra–Tilli, 2015–2016) If G is unbounded and contains at least one half-line, then

$$\min_{\phi\in H^1(\mathbb{R}^+)} E(u;\mathbb{R}^+) \leq \mathcal{E} \leq \min_{\phi\in H^1(\mathbb{R})} E(u;\mathbb{R})$$

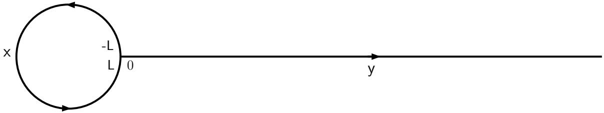
Infimum may not be attained by any of the standing waves Φ.

Ground state on the unbounded graphs

Theorem. (Adami–Serra–Tilli, 2015–2016) If Γ consists of only one half-line, then

$$\mathcal{E} < \min_{\phi \in H^1(\mathbb{R})} E(u; \mathbb{R})$$

and the infimum is attained.



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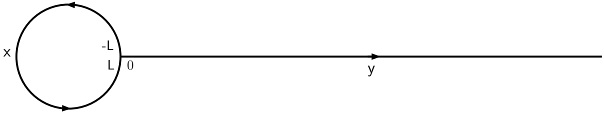


If Γ consists of more than two half-lines and is *connective to infinity*, then

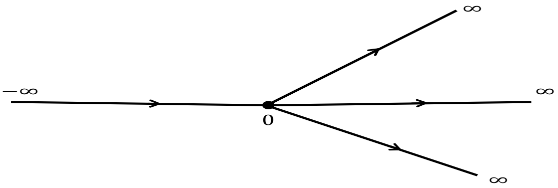
$$\mathcal{E} = \min_{\phi \in H^1(\mathbb{R})} E(u; \mathbb{R})$$

and the infimum is not attained. The reason is topological. By the symmetry rearrangements,

$$E(u; \Gamma) > E(\hat{u}; \mathbb{R}) \geq \min_{\phi \in H^1(\mathbb{R})} E(u; \mathbb{R}) = \mathcal{E}.$$

At the same time, a sequence of solitary waves escaping to infinity along one edge yields a sequence of functions that minimize $E(u; \Gamma)$ until it reaches $\mathcal{E}$.

Application to the star graphs



If $N \geq 3$, no ground state exists due to the same topological reason.
However, there exists a standing wave of the Euler–Lagrange equation:

$$-\Delta\Phi - 2|\Phi|^2\Phi = -\omega\Phi$$

in the form of the half-soliton:

$$\Phi(x) = \begin{bmatrix} \sqrt{\omega}\mathrm{sech}(\sqrt{\omega}x), & x \in (-\infty, 0), & j = 1, \\ \sqrt{\omega}\mathrm{sech}(\sqrt{\omega}x), & x \in (0, \infty), & 2 \leq j \leq N. \end{bmatrix}.$$

Theorem. (Adami *et al.*, 2012) (Kairzhan–P., JDE, 2018) Half-soliton is a saddle point of energy E at fixed mass Q. This saddle point is unstable in the time evolution of the NLS.

Uniqueness of the half-soliton

If N is odd, the half-soliton is unique.

Consider now generalized boundary conditions

$$\begin{cases} \alpha_1\psi_1(0) = \alpha_2\psi_2(0) = \cdots = \alpha_N\psi_N(0) \\ \alpha_1^{-1}\psi_1'(0) = \alpha_2^{-1}\psi_2'(0) + \cdots + \alpha_N^{-1}\psi_N'(0). \end{cases}$$

and generalized NLS equation $i\Psi_t = -\Delta\Psi - 2\alpha^2|\Psi|^2\Psi$, where $(\alpha_1, \ldots, \alpha_N)$ are positive.

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Lemma. If $\alpha_1^{-2} = \sum_{j=2}^N \alpha_j^{-2}$, then there exists a unique one-parameter family of solutions $\{\Phi(x;a)\}_{a\in\mathbb{R}}$ satisfying

$$\Phi(x;a) = \left[\begin{array}{ll} \alpha_1^{-1}\sqrt{\omega}\text{sech}(\sqrt{\omega}(x+a)), & x \in (-\infty, 0), \;\; j = 1, \\ \alpha_j^{-1}\sqrt{\omega}\text{sech}(\sqrt{\omega}(x+a)), & x \in (0, \infty), \;\; 2 \leq j \leq N. \end{array} \right].$$

Shifted standing waves

Example for $N = 3$:

$$\Phi(x;a) = \left[\begin{array}{ll} \alpha_1^{-1}\sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x+a)), & x \in (-\infty, 0), \\ \alpha_2^{-1}\sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x+a)), & x \in (0, \infty), \\ \alpha_3^{-1}\sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x+a)), & x \in (0, \infty). \end{array} \right].$$

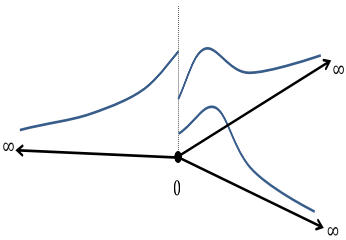

Figure: Schematic representation of the shifted standing waves on the star graph with $N = 3$, and either $a < 0$ (left) or $a > 0$ (right).

A hidden reason for existence of shifted states

Assume that $\Psi \in H^1_\Gamma$ satisfies the symmetry reduction:

$$\alpha_2 \psi_2(t,x) = \cdots = \alpha_N \psi_N(t,x), \quad x > 0.$$

If $\alpha_1^{-2} = \sum_{j=2}^N \alpha_j^{-2}$, the wave function

$$\varphi(t,x) = \begin{cases} \alpha_1 \psi_1(t,x), & x \leq 0, \\ \alpha_2 \psi_2(t,x), & x \geq 0, \end{cases}$$

satisfies the cubic NLS equation on the line $\mathbb{R}$:

$$i \frac{\partial \varphi}{\partial t} = -\frac{\partial^2 \varphi}{\partial x^2} - 2|\varphi|^2 \varphi, \quad x \in \mathbb{R},$$

which is translationally invariant in x.

D. Matrasulov–K. Sabirov–Z. Sobirov (2012,2016)

Momentum conservation

For a solution $\Psi \in C(\mathbb{R}, H^1_\Gamma)$, let us define the momentum of the NLS:

$$P(\Psi) = \mathrm{Im}\langle \Psi', \Psi \rangle_{L^2(\Gamma)}$$

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If $\alpha_1^{-2} = \sum_{j=2}^N \alpha_j^{-2}$, the map $t \mapsto P(\Psi)$ is monotonically increasing:

$$\frac{dP}{dt} = \frac{1}{2} \sum_{j=2}^{N} \sum_{i \neq j}^{N} \frac{\alpha_1^2}{\alpha_j^2 \alpha_i^2} \left| \alpha_j \psi_j'(0) - \alpha_i \psi_i'(0) \right|^2 \geq 0.$$

If in addition, the solution is symmetric and satisfies the NLS reduction:

$$\alpha_2 \psi_2(t,x) = \cdots = \alpha_N \psi_N(t,x), \quad x > 0,$$

then the momentum $P(\Psi)$ is constant in time.

Orbital stability of standing waves

From the constants of motion, we can define the Lyapunov functional

$$\Lambda_\omega(\Psi) := E(\Psi) + \omega Q(\Psi),$$

the critical points of which are the standing waves:

$$-\Delta\Phi - 2|\Phi|^2\Phi = -\omega\Phi.$$

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The NLS soliton $\Phi_\omega(x) = \sqrt{\omega}\text{sech}(\sqrt{\omega}x)$ on the line $\mathbb{R}$ is a saddle point of $\Lambda_\omega(\Psi)$ for fixed $\omega > 0$. Moreover, it is a degenerate saddle point as $\Phi_\omega(x+a)e^{i\theta}$ is also a solution for every $\theta \in \mathbb{R}$ and $a \in \mathbb{R}$.

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Definition. For every $\epsilon > 0$, there is $\delta > 0$ such that for every $\Psi_0 \in H^1$ satisfying $\|\Psi_0 - \Phi_\omega\|_{H^1} < \delta$, the unique solution $\Psi \in C(\mathbb{R}, H^1)$ of the NLS equation satisfies

$$\inf_{\theta\in\mathbb{R}, a\in\mathbb{R}} \|\Psi(t,\cdot) - \Phi_\omega(\cdot + a)e^{i\theta}\|_{H^1} < \epsilon,$$

where $\omega > 0$ is fixed.

Orbital stability of the NLS solitons on the line $\mathbb{R}$

Theorem. (Grillakis–Shatah–Strauss, 1987; Weinstein, 1987) The NLS soliton on the line $\mathbb{R}$ is orbitally stable for every $\omega > 0$.

Orbital stability of the NLS solitons on the line $\mathbb{R}$

Theorem. (Grillakis–Shatah–Strauss, 1987; Weinstein, 1987) The NLS soliton on the line $\mathbb{R}$ is orbitally stable for every $\omega > 0$.

- Hessian $\Lambda''_\omega(\Phi_\omega)$ has exactly one simple negative eigenvalue and a double zero eigenvalue.
- Fixed $Q(\Psi) = \|\Psi\|^2_{L^2}$ produces the linear constraint $\langle U, \Phi_\omega \rangle_{L^2} = 0$ on $U = \mathrm{Re}(\Psi)$. Hessian $\Lambda''_\omega(\Phi_\omega)$ is non-negative under the constraint.
- The decomposition $\Psi(x) = e^{i\theta}\left[\Phi_\omega(x+a) + U(x+a) + iW(x+a)\right]$ is uniquely defined for $\theta \in \mathbb{R}$, $a \in \mathbb{R}$, and $\omega > 0$ subject to three constraints on U and W including $\langle U, \Phi_\omega \rangle_{L^2} = 0$. Hessian $\Lambda''_\omega(\Phi_\omega)$ is strictly positive under the three constraints.
- $U, W \in H^1$ and ω are controlled in the time evolution from energy estimates due to coercivity of the Lyapunov function.

Standing waves on the star graph

Shifted standing waves with parameters $\omega > 0$ and $a \in \mathbb{R}$:

$$\Phi_\omega(x;a) = \left[\begin{array}{ll} \alpha_1^{-1}\sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x+a)), & x \in (-\infty,0), \;\; j=1, \\ \alpha_j^{-1}\sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x+a)), & x \in (0,\infty), \;\; 2 \le j \le N. \end{array} \right]$$

Substituting $\Psi = \Phi_\omega + U + iW$ into $\Lambda_\omega(\Psi)$ yields

$$\Lambda_\omega(\Phi_\omega + U + iW) = \Lambda_\omega(\Phi_\omega) + \langle L_+(\omega,a)U, U\rangle_{L^2(\Gamma)} + \langle L_-(\omega,a)W, W\rangle_{L^2(\Gamma)} + \mathrm{O}(3),$$

where

$$\left\{ \begin{array}{l} L_-(\omega,a) = -\Delta + \omega - 2\alpha^2\Phi_\omega(\cdot;a)^2, \\ L_+(\omega,a) = -\Delta + \omega - 6\alpha^2\Phi_\omega(\cdot;a)^2. \end{array} \right.$$

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Spectral properties of $L_\pm(\omega,a)$:

- $\sigma_c(L_\pm) = [\omega,\infty)$ with $\omega > 0$.
- $L_- \ge 0$ and $\ker(L_-) = \mathrm{span}\{\Phi_\omega\}$.
- $\Phi'_\omega \in \ker(L_+)$

Negative eigenvalues of $L_+(\omega, a)$

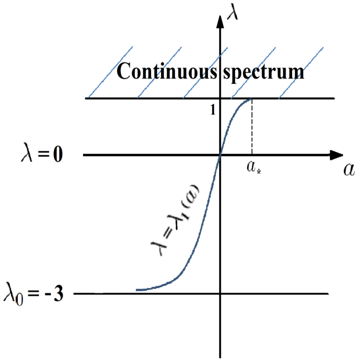

Figure: The spectrum of $L_+(\omega, a)$ for $\omega = 1$.

Theorem. (Kairzhan–P., JPA, 2018) Besides simple eigenvalues $\lambda_0 = -3\omega$ and $\lambda = 0$, there exists exactly one additional eigenvalue $\lambda_1(\omega, a)$ of multiplicity $N - 2$ such that $\lambda_1(\omega, a) > 0$ for $a > 0$ and $\lambda_1(\omega, a) < 0$.

Shifted standing waves

Recall the main example for $N = 3$:

$$\Phi(x;a) = \left[\begin{array}{ll} \alpha_1^{-1}\sqrt{\omega}\operatorname{sech}(\sqrt{\omega}(x+a)), & x \in (-\infty, 0), \\ \alpha_2^{-1}\sqrt{\omega}\operatorname{sech}(\sqrt{\omega}(x+a)), & x \in (0, \infty), \\ \alpha_3^{-1}\sqrt{\omega}\operatorname{sech}(\sqrt{\omega}(x+a)), & x \in (0, \infty). \end{array} \right].$$

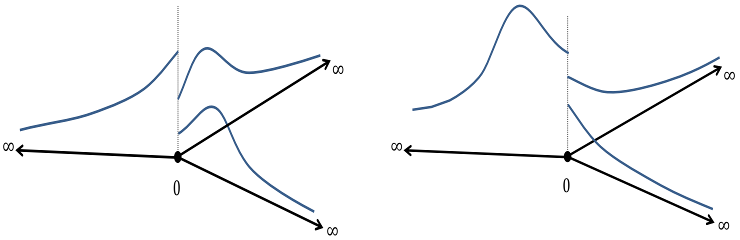

Figure: $L_+(\omega, a)$ has two negative eigenvalues for $a < 0$ (left) and one negative eigenvalue for $a > 0$ (right).

Implication of the eigenvalue count for $N = 3$

Recall

$$\Lambda_\omega(\Phi_\omega + U + iW) = \Lambda_\omega(\Phi_\omega) + \langle L_+(\omega, a)U, U\rangle_{L^2(\Gamma)} + \langle L_-(\omega, a)W, W\rangle_{L^2(\Gamma)} + \mathrm{O}(3).$$

- $a < 0$: Φ_ω is a saddle point of Λ_ω with two negative eigenvalues and it remains a saddle point with one negative eigenvalue under the constraint of fixed $Q(\Psi) = \|\Psi\|^2_{L^2}$.

Shifted state with $a < 0$ is spectrally and nonlinearly unstable.

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- $a > 0$: Φ_ω is a saddle point of Λ_ω with one negative eigenvalue and it is a degenerate constrained minimizer under the constraint of fixed $Q(\Psi) = \|\Psi\|^2_{L^2}$ with double zero eigenvalue.

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  The shifted state with $a > 0$ is spectrally stable. Is it nonlinearly stable?

- $a = 0$: Φ_ω is a saddle point of Λ_ω with one negative eigenvalue and a triple zero eigenvalue.

  Is it a degenerate constrained minimizer? Is it nonlinearly stable?

Recap for shifted states with $a > 0$

Consider $\Phi_\omega(x; a)$ with $a > 0$ and recall

$$\Lambda_\omega(\Phi_\omega + U + iW) = \Lambda_\omega(\Phi_\omega) + \langle L_+(\omega, a)U, U\rangle_{L^2(\Gamma)} + \langle L_-(\omega, a)W, W\rangle_{L^2(\Gamma)} + \mathrm{O}(3).$$

- $L_- \geq 0$ and $\ker(L_-) = \mathrm{span}\{\Phi_\omega\}$.
- $\ker(L_+) = \mathrm{span}\{\Phi'_\omega\}$ and L_+ has one negative eigenvalue.

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- $\ker(L_+) = \mathrm{span}\{\Phi'_\omega\}$ and L_+ has one negative eigenvalue.
- Fixed $Q(\Psi) = \|\Psi\|^2_{L^2}$ produces the linear constraint $\langle U, \Phi_\omega\rangle_{L^2} = 0$ on $U = \mathrm{Re}(\Psi)$. Hessian $\Lambda''_\omega(\Phi_\omega)$ is non-negative under the constraint.

Recap for shifted states with $a > 0$

Consider $\Phi_\omega(x; a)$ with $a > 0$ and recall

$$\Lambda_\omega(\Phi_\omega + U + iW) = \Lambda_\omega(\Phi_\omega) + \langle L_+(\omega, a)U, U\rangle_{L^2(\Gamma)} + \langle L_-(\omega, a)W, W\rangle_{L^2(\Gamma)} + \mathrm{O}(3).$$

- $L_- \geq 0$ and $\ker(L_-) = \mathrm{span}\{\Phi_\omega\}$.
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- Fixed $Q(\Psi) = \|\Psi\|_{L^2}^2$ produces the linear constraint $\langle U, \Phi_\omega\rangle_{L^2} = 0$ on $U = \mathrm{Re}(\Psi)$. Hessian $\Lambda_\omega''(\Phi_\omega)$ is non-negative under the constraint.
- The decomposition $\Psi(x) = e^{i\theta}\left[\Phi_\omega(x; a) + U(x) + iW(x)\right]$ is uniquely defined for $\theta \in \mathbb{R}$, $a \in \mathbb{R}$, and $\omega > 0$ subject to three constraints on U and W including $\langle U, \Phi_\omega\rangle_{L^2} = 0$. Hessian $\Lambda_\omega''(\Phi_\omega)$ is strictly positive under the three constraints.

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- $U, W \in H^1_\Gamma$ and ω are controlled in the time evolution from energy estimates due to coercivity of the Lyapunov function.

Drift of the shifted states

Theorem. (Kairzhan–P–Goodman, 2019)
Fix $a_0 > 0$. For every $\mathbf{a} \in (0, a_0)$ there exists $\epsilon_0 > 0$ (sufficiently small) such that for every $\epsilon \in (0, \epsilon_0)$, there exists $\delta > 0$ and $T > 0$ such that for every initial datum $\Psi_0 \in H^1_\Gamma$ with $P(\Psi_0) > 0$ and

$$\inf_{\theta \in \mathbb{R}} \| \Psi_0 - e^{i\theta} \Phi_\omega(\cdot; a_0) \|_{H^1(\Gamma)} \leq \delta$$

the unique solution $\Psi \in C([0,T], H^1_\Gamma) \cap C^1([0,T], H^{-1}_\Gamma)$ to the NLS equation with the initial datum $\Psi(0,\cdot) = \Psi_0$ satisfies the bound

$$\inf_{\theta \in \mathbb{R}} \| \Psi(t,\cdot) - e^{i\theta} \Phi_\omega(\cdot; a(t)) \|_{H^1(\Gamma)} \leq \epsilon, \quad t \in [0,T],$$

where $a \in C^1([0,T])$ is a strictly decreasing function such that $\lim_{t \to T} a(t) = \mathbf{a}$.

A hidden reason for the drift

Recall that the momentum of the NLS:

$$P(\Psi) = \mathrm{Im}\langle \Psi', \Psi \rangle_{L^2(\Gamma)}$$

is no longer constant but is monotonically increasing:

$$\frac{dP}{dt} = \frac{1}{2} \sum_{j=2}^{N} \sum_{i \neq j}^{N} \frac{\alpha_1^2}{\alpha_j^2 \alpha_i^2} \left| \alpha_j \psi_j'(0) - \alpha_i \psi_i'(0) \right|^2 \geq 0.$$

For the solution uniquely decomposed as

$$\Psi(t,x) = e^{i\theta(t)} \left[\Phi_{\omega(t)}(x; a(t)) + U(t,x) + iW(t,x) \right],$$

the momentum is expanded as

$$P(\Psi) = -2\langle \Phi_\omega'(\cdot; a), W \rangle_{L^2(\Gamma)} + \mathcal{O}(\|U + iW\|_{H^1(\Gamma)}^2),$$

whereas the modulation equation for $a(t)$ reads as

$$\dot{a} = 2\langle \Phi_\omega'(\cdot; a), W \rangle_{L^2(\Gamma)} \left[1 + \mathcal{O}(\|U + iW\|_{H^1(\Gamma)}) \right] + \mathcal{O}(\|U + iW\|_{H^1(\Gamma)}^2),$$

so that $\dot{a} = -P(\Psi) + \mathcal{O}(\|U + iW\|_{H^1(\Gamma)}^2) < 0$ if $P(\Psi) \geq P(\Psi_0) > 0$.

Recap for half-solitons with $a = 0$

Consider $\Phi_\omega(x; a = 0)$ and recall

$$\Lambda_\omega(\Phi_\omega + U + iW) = \Lambda_\omega(\Phi_\omega) + \langle L_+(\omega, a)U, U\rangle_{L^2(\Gamma)} + \langle L_-(\omega, a)W, W\rangle_{L^2(\Gamma)} + \mathrm{O}(3).$$

- $L_- \geq 0$ and $\ker(L_-) = \mathrm{span}\{\Phi_\omega\}$.
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Saddle-point geometry

Consider the orthogonal decomposition in H^1_Γ,

$$\Psi = \Phi_\omega + c_1 U^{(1)} + c_2 U^{(2)} + \cdots + c_{N-1} U^{(N-1)} + U^\perp,$$

where $X_c = \text{span}\{\Phi_\omega, U^{(1)}, U^{(2)}, \dots, U^{(N-1)}\}$ and $U^\perp \in H^1_\Gamma \cap [X_c]^\perp$.

Theorem. (Kairzhan–P, 2018)
There exists $\delta > 0$ such that for every $c = (c_1, c_2, \dots, c_{N-1})^T \in \mathbb{R}^{N-1}$ satisfying $\|c\| \leq \delta$, there exists a unique minimizer $U^\perp \in H^1_\Gamma \cap [X_c]^\perp$ of the variational problem

$$M(c) := \inf_{U^\perp \in H^1_\Gamma \cap [X_c]^\perp} [\Lambda(\Psi) - \Lambda(\Phi_\omega)]$$

such that $\|U^\perp\|_{H^1(\Gamma)} \leq A\|c\|^2$ for a c-independent constant $A > 0$.
Moreover, $M(c)$ is sign-indefinite in c.

Minimization of the remainder term

Expanding for real $U \in H^1_\Gamma$:

$$\Lambda(\Phi_\omega + U) = \Lambda(\Phi_\omega) + \langle L_+ U, U\rangle_{L^2(\Gamma)} - 4\langle \alpha^2 \Phi_\omega U^2, U\rangle_{L^2(\Gamma)} + \mathrm{O}(\|U\|^4_{H^1}),$$

By minimizing $M(c) := \inf_{U^\perp \in H^1_\Gamma \cap [X_c]^\perp} [\Lambda(\Phi_\omega + U) - \Lambda(\Phi_\omega)]$. we obtain $F(U^\perp, c) = 0$ with

$$F(U^\perp, c) : X \times \mathbb{R}^{N-1} \mapsto Y, \quad X := H^1_\Gamma \cap [X_c]^\perp, \quad Y := H^{-1}_\Gamma \cap [X_c]^\perp,$$

$$F(U^\perp, c) := L_+ U^\perp - 6\Pi_c \alpha^2 \Phi_\omega \left(\sum_{j=1}^{N-1} c_j U^{(j)} + U^\perp \right)^2 + \mathrm{O}(\|U\|^3_{H^1}).$$

(i) F is a C^2 map from $X \times \mathbb{R}^{N-1}$ to Y;

(ii) $F(0,0) = 0$;

(iii) $D_{U^\perp} F(0,0) = \Pi_c L_+ \Pi_c : X \mapsto Y$ is invertible with a bounded inverse from Y to X;

(iv) $\Pi_c L_+ \Pi_c$ is strictly positive;

(v) $D_c F(0,0) = 0$.

Normal form

By the minimization problem, we obtain

$$\begin{aligned} M(c) &= \inf_{U^{\perp} \in H^1_{\Gamma} \cap [X_c]^{\perp}} \left[\Lambda(\Phi_\omega + U) - \Lambda(\Phi_\omega) \right] \\ &= M_0(c) + \mathrm{O}(\|c\|^4), \end{aligned}$$

where

$$M_0(c) := -4 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} c_i c_j c_k \langle \alpha^2 \Phi_\omega U^{(i)} U^{(j)}, U^{(k)} \rangle_{L^2(\Gamma)}.$$

Cubic $M_0(c)$, and hence $M(c)$, is sign-indefinite near $c = 0$.

Instability of half-solitons

Theorem. (Kairzhan–P–Goodman, 2019)
There exists $\epsilon > 0$ such that for every sufficiently small $\delta > 0$ there exists $V \in H^1_\Gamma$ with $\|V\|_{H^1_\Gamma} \leq \delta$ such that the unique solution

$$\Psi \in C(\mathbb{R}, H^1_\Gamma) \cap C^1(\mathbb{R}, H^{-1}_\Gamma)$$

to the NLS equation with the initial datum $\Psi(0, \cdot) = \Phi_\omega + V$ satisfies

$$\inf_{\theta \in \mathbb{R}} \|e^{-i\theta}\Psi(T, \cdot) - \Phi_\omega\|_{H^1(\Gamma)} > \epsilon \quad \text{for some} \;\; T > 0.$$

Time-dependent normal form

Time-dependent normal form is a Hamiltonian system with the conserved energy

$$
\begin{aligned}
H_0(c,b) &= \frac{1}{2}\sum_{j=1}^{N-1}\langle W^{(j)}, U^{(j)}\rangle_{L^2(\Gamma)} b_j^2 \\
&\quad -2\sum_{j=1}^{N-1}\sum_{k=1}^{N-1}\sum_{n=1}^{N-1}\langle \alpha^2 \Phi_\omega U^{(j)}, U^{(k)} U^{(n)}\rangle_{L^2(\Gamma)} c_j c_k c_n.
\end{aligned}
$$

For $N = 3$,

$$
\begin{cases} M_1 \ddot{c}_1 = -P_2 c_2^2, \\ M_2 \ddot{c}_2 = -2P_2 c_1 c_1 + R_2 c_2^2, \end{cases}
$$

where $M_1, M_2 > 0$ and $P_2 > 0$.

There exists an invariant reduction $c_1 = \gamma c_2$ for some $\gamma \neq 0$:

$$
2M_1 P_2 \gamma^2 - M_1 R_2 \gamma - M_2 P_2 = 0.
$$

Zero solution is unstable along the invariant reduction $c_1 = \gamma c_2$.

Numerical illustrations (Roy Goodman)

- Truncation of half-lines with Dirichlet boundary conditions
- No grid points on the vertex if the grid points are at $x_k = (k - \frac{1}{2})\Delta x$.
- Neumann–Kirchhoff boundary conditions are computed with a ghost point at $x_0 = -\frac{1}{2}\Delta x$.
- Second-order split-step method in time with Crank-Nicholson iterations for the linear part.
- Initial condition as $\Psi_0 = \Phi_\omega(\cdot; a) + \epsilon U_a$, where U_a is an eigenfunction for $L_+(\omega, a) U_a = \lambda_1(\omega, a) U_a$ with $\lambda_1(\omega, a) > 0$ for $a > 0$.

Shifted standing waves

Recall the shifted standing waves for $N = 3$:

$$\Phi(x;a) = \left[\begin{array}{ll} \alpha_1^{-1}\sqrt{\omega}\operatorname{sech}(\sqrt{\omega}(x+a)), & x \in (-\infty,0), \\ \alpha_2^{-1}\sqrt{\omega}\operatorname{sech}(\sqrt{\omega}(x+a)), & x \in (0,\infty), \\ \alpha_3^{-1}\sqrt{\omega}\operatorname{sech}(\sqrt{\omega}(x+a)), & x \in (0,\infty). \end{array} \right].$$

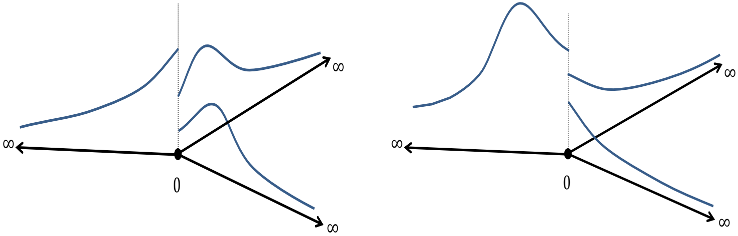


Figure: Schematic representation of the shifted standing waves on the star graph with $N = 3$, and either $a < 0$ (left) or $a > 0$ (right).

Linear instability for $a < 0$

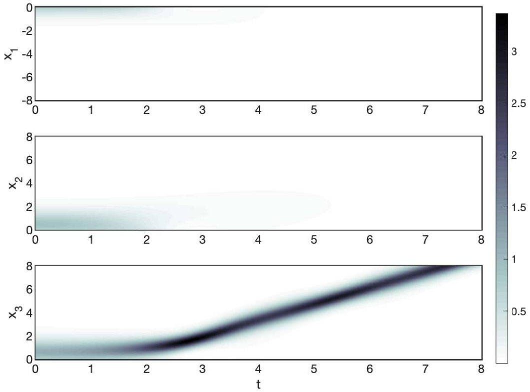
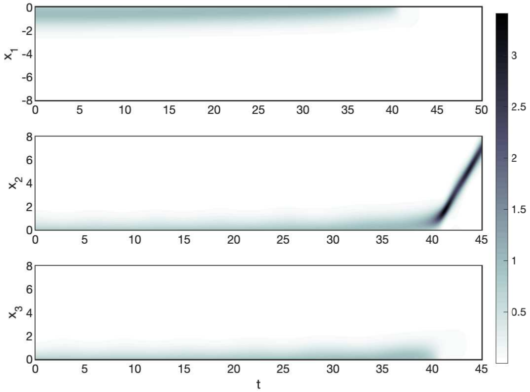


Figure: A numerical solution for $a = -0.55$ and $\epsilon = 0.1$. The colorbar corresponds to values of $|u|^2$. The three panels correspond to the solution on edges 1, 2, and 3 going down.

Drift instability for $a > 0$



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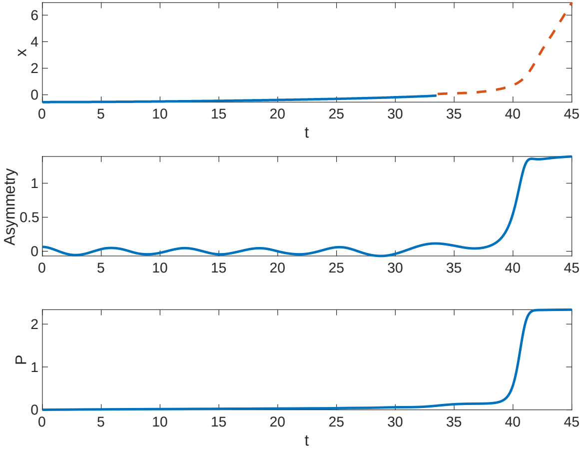

Figure: Postprocessed quantities form the same simulation. (Top) The position of the maximum of u. The solid line for $t < 33.5$ describes the position on the incoming edge one. The dashed line for $t > 33.5$ shows the position of the maximum on edge two. (Middle) The asymmetry, defined as $\|u_2\|_{L^2(\mathbb{R}+)} - \|u_3\|_{L^2(\mathbb{R}+)}$. (Bottom) The momentum $P(\Psi)$ versus time t.

Pushing experiments beyond the validity of the theorem

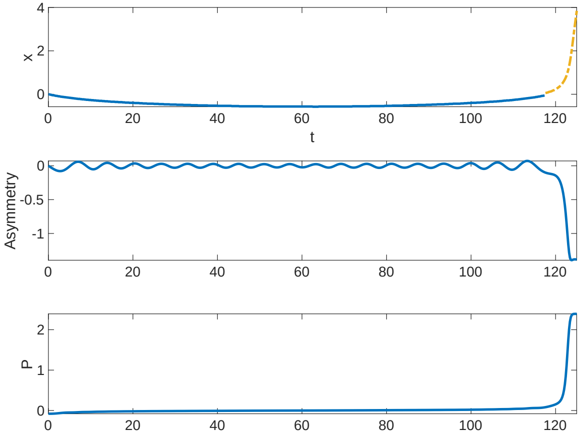

Figure: A numerical solution with $a = 0$. (Top) The position of the maximum of $|u|^2$, on edge one for $t < 117$ and on edge three (dashed) for $t > 117$. (Middle) Asymmetry of the solution between the two outgoing edges. (Bottom) The momentum $P(\Psi)$ versus time t.

Summary

- Infima of constrained energy may not be attained on unbounded graphs such as the star graphs.
- Standing wave solutions appear typically as saddle points of the constrained energy as hence they are unstable in the time evolution of the NLS flow.
- For the special case of reflectionless star graphs with translational symmetry, we showed that the spectrally and linearly stable standing waves are still nonlinearly unstable because of the drift instability.

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Thanks for listening. Questions???