Drift of steady states in Hamiltonian PDEs: two examples

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NLS evolution on the line

The Cauchy problem for the NLS flow:

$$\begin{cases} i\Psi_t = -\Delta\Psi - 2|\Psi|^2\Psi, \\ \Psi|_{t=0} = \Psi_0, \end{cases}$$

where $\Delta\Psi = \partial_x^2\Psi$ on the line $\mathbb{R}$.

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The Cauchy problem is locally and globally well-posed in $\Psi_0 \in H^1(\mathbb{R})$.
Moreover, the mass

$$Q(\Psi) = \|\Psi\|_{L^2}^2$$

and the energy

$$E(\Psi) = \|\Psi'\|_{L^2}^2 - \|\Psi\|_{L^4}^4,$$

are constants in time for $\Psi \in C(\mathbb{R}, H^1)$.

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are constants in time for $\Psi \in C(\mathbb{R}, H^1)$.

$E(\Psi)$ is coercive in H^1 thanks to Gagliardo–Nirenberg inequality:

$$\|\Psi\|_{L^4}^4 \leq C\|\partial_x\Psi\|_{L^2}\|\Psi\|_{L^2}^3,$$

where $C > 0$ is independent of Ψ.

Standing waves

Ground state is a standing wave of smallest energy E at fixed mass Q,

$$\mathcal{E} = \inf\{E(u) : \quad u \in H^1(\mathbb{R}), \quad Q(u) = \mu\}.$$

Euler–Lagrange equation for the standing waves:

$$-\Delta\Phi - 2|\Phi|^2\Phi = -\omega\Phi,$$

where $\omega > 0$ defines $\Psi(t,x) = \Phi(x)e^{i\omega t}$.

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Infimum $\mathcal{E}$ exists thanks to the same Gagliardo–Nirenberg inequality.

Theorem. (M.Weinstein, 1986) Infimum $\mathcal{E}$ is attained at the NLS soliton $\Phi_\omega(x) = \sqrt{\omega}\text{sech}(\sqrt{\omega}x)$ with $\omega = \mu^2/4$:

$$\mathcal{E} = \min_{u \in H^1(\mathbb{R})} E(u) = E(\phi).$$

Orbital stability of standing waves

From the constants of motion, we can define the Lyapunov functional

$$\Lambda_\omega(\Psi) := E(\Psi) + \omega Q(\Psi),$$

the critical points of which are the standing waves:

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The NLS soliton $\Phi_\omega(x) = \sqrt{\omega}\text{sech}(\sqrt{\omega}x)$ is a saddle point of $\Lambda_\omega(\Psi)$ for fixed $\omega > 0$. Moreover, it is a degenerate saddle point as $\Phi_\omega(x+a)e^{i\theta}$ is also a solution for every $\theta \in \mathbb{R}$ and $a \in \mathbb{R}$.

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Definition. For every $\epsilon > 0$, there is $\delta > 0$ such that for every $\Psi_0 \in H^1$ satisfying $\|\Psi_0 - \Phi_\omega\|_{H^1} < \delta$, the unique solution $\Psi \in C(\mathbb{R}, H^1)$ of the NLS equation satisfies

$$\inf_{\theta\in\mathbb{R},a\in\mathbb{R}} \|\Psi(t,\cdot) - \Phi_\omega(\cdot + a)e^{i\theta}\|_{H^1} < \epsilon,$$

where $\omega > 0$ is fixed.

Orbital stability of the NLS solitons on the line $\mathbb{R}$

Theorem. (Grillakis–Shatah–Strauss, 1987) The NLS soliton on the line $\mathbb{R}$ is orbitally stable for every $\omega > 0$.

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- Hessian $\Lambda''_\omega(\Phi_\omega)$ has exactly one simple negative eigenvalue and a double zero eigenvalue.
- Fixed $Q(\Psi) = \|\Psi\|^2_{L^2}$ produces the linear constraint $\langle U, \Phi_\omega \rangle_{L^2} = 0$ on $U = \mathrm{Re}(\Psi)$. Hessian $\Lambda''_\omega(\Phi_\omega)$ is non-negative under the constraint.
- The decomposition $\Psi(x) = e^{i\theta}\left[\Phi_\omega(x+a) + U(x+a) + iW(x+a)\right]$ is uniquely defined for $\theta \in \mathbb{R}$, $a \in \mathbb{R}$, and $\omega > 0$ subject to three constraints on U and W including $\langle U, \Phi_\omega \rangle_{L^2} = 0$. Hessian $\Lambda''_\omega(\Phi_\omega)$ is strictly positive under the three constraints.
- $U, W \in H^1$ and ω are controlled in the time evolution from energy estimates due to coercivity of the Lyapunov function in these variables.

Main Question

If the standing wave has a free parameter which is not supported by the corresponding symmetry of the PDE, does a drift along the parameter imply instability of the standing waves?

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Two answers:

NO Resonant normal forms (conformal cubic wave equation on three-sphere): with P. Bizon and D. Hunik (Jagiellonian University, Krakow, Poland) [Comm. Pure Appl. Math. **72** (2019), 1123–1151].

YES Balanced star graphs (transmission problems): with A. Kairzhan (McMaster University) and R.H. Goodman (New Jersey Tech, USA) [SIAM J. Applied Dynamical Systems (2019), in press].

Resonant normal forms

In many infinite-dimensional Hamiltonian systems with spatial confinement,

- The system can be written in canonical coordinates;
- The resonant energy transfer can be isolated from the rest.

If the resonant energy transfer also involves infinitely many modes, this reductive technique leads to the infinite-dimensional resonant normal form.

Hamiltonian systems with ∞ degrees of freedom

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Resonant normal form with ∞ canonical coordinates

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Resonant normal form with ∞ canonical coordinates

Examples:

- Cubic Szegö equation
  Gerard, Grellier (2010, 2012, 2015)
- Rotating Bose–Einstein condensates in 2D
  Faou, Germain, & Hani (2016); Germain, Gerard, Thomann (2017); Biasi, Bizon, Craps, & Evnin (2017)

Resonant normal form for conformal flow on $\mathbb{S}^3$

- Background geometry: the Einstein cylinder $\mathcal{M} = \mathbb{R} \times \mathbb{S}^3$ with metric

$$g = -dt^2 + dx^2 + \sin^2 x \, d\omega^2, \qquad (t, x, \omega) \in \mathbb{R} \times [0, \pi] \times \mathbb{S}^2$$

  This spacetime has constant scalar curvature $R(g) = 6$.

- On $\mathcal{M}$ we consider a real scalar field ϕ satisfying

$$\Box_g \phi - \phi - \phi^3 = 0 \, .$$

- We assume that $\phi = \phi(t, x)$. Then, $\nu(t, x) = \sin(x)\phi(t, x)$ satisfies

$$\nu_{tt} - \nu_{xx} + \frac{\nu^3}{\sin^2 x} = 0$$

  with Dirichlet boundary conditions $\nu(t, 0) = \nu(t, \pi) = 0$.

- Linear eigenstates: $e_n(x) \sim \sin(\omega_n x)$ with $\omega_n = n + 1$ $(n = 0, 1, 2, ...)$

Time averaging

- Expanding $\nu(t,x) = \sum_{n=0}^{\infty} c_n(t) e_n(x)$ we get

$$\frac{d^2 c_n}{dt^2} + \omega_n^2 c_n = -\sum_{jkl} S_{njkl}\, c_j c_k c_l, \quad S_{jkln} = \int_0^{\pi} \frac{dx}{\sin^2 x} e_n(x) e_j(x) e_k(x) e_l(x)$$

- Using variation of constants

$$c_n = \beta_n e^{i\omega_n t} + \bar{\beta}_n e^{-i\omega_n t}, \qquad \frac{dc_n}{dt} = i\omega_n \left(\beta_n e^{i\omega_n t} - \bar{\beta}_n e^{-i\omega_n t}\right)$$

we factor out fast oscillations

$$2i\omega_n \frac{d\beta_n}{dt} = -\sum_{jkl} S_{njkl}\, c_j c_k c_l\, e^{-i\omega_n t}$$

- Each term in the sum has a factor $e^{-i\Omega t}$, where $\Omega = \omega_n \pm \omega_j \pm \omega_k \pm \omega_l$. The terms with $\Omega = 0$ correspond to resonant interactions.
- Let $\tau = \varepsilon^2 t$ and $\beta_n(t) = \varepsilon \alpha_n(\tau)$. For $\varepsilon \to 0$ the non-resonant terms $\propto e^{-i\Omega\tau/\varepsilon^2}$ are highly oscillatory and therefore negligible.

Resonant system

- Keeping only the resonant terms (and rescaling), we obtain

$$i(n+1)\frac{d\alpha_n}{d\tau} = \sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{njk,n+j-k}\,\bar{\alpha}_j\alpha_k\alpha_{n+j-k}\,,$$

  where $S_{njk,n+j-k} = \min\{n,j,k,n+j-k\}+1$.

- This system (labeled as conformal flow) provides an accurate approximation to the cubic wave equation on the timescale $\sim \varepsilon^{-2}$. Bizon–Craps–Evnin–Hunik–Luyten–Maliborski (2016)

- This is a Hamiltonian system

$$i(n+1)\frac{d\alpha_n}{d\tau} = \frac{1}{2}\frac{\partial H}{\partial \bar{\alpha}_n}$$

  with

$$H = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{njk,n+j-k}\bar{\alpha}_n\bar{\alpha}_j\alpha_k\alpha_{n+j-k}$$

Properties of conformal flow

- Symmetries

$$\begin{aligned} \text{Scaling:} \quad & \alpha_n(t) \to c\alpha_n(c^2 t), \quad c \in \mathbb{R} \\ \text{Global phase shift:} \quad & \alpha_n(t) \to e^{i\theta}\alpha_n(t), \quad \theta \in \mathbb{R} \\ \text{Local phase shift:} \quad & \alpha_n(t) \to e^{in\mu}\alpha_n(t), \quad \mu \in \mathbb{R} \end{aligned}$$

- Conserved quantities

$$Q = \sum_{n=0}^{\infty}(n+1)|\alpha_n|^2, \qquad E = \sum_{n=0}^{\infty}(n+1)^2|\alpha_n|^2$$

- The Cauchy problem is locally (and therefore also globally) well-posed for initial data in $\ell^{2,1}(\mathbb{Z})$ where H, Q, E are finite and conserved.

Energy inequality

Energy

$$H = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{njk,n+j-k} \bar{\alpha}_n \bar{\alpha}_j \alpha_k \alpha_{n+j-k}$$

Two mass quantities:

$$Q = \sum_{n=0}^{\infty} (n+1)|\alpha_n|^2, \qquad E = \sum_{n=0}^{\infty} (n+1)^2 |\alpha_n|^2$$

Theorem (Bizon–Hunik–P, 2019)

For every $\alpha \in \ell^{2,1/2}(\mathbb{N})$, it is true that $H(\alpha) \leq Q(\alpha)^2$. Moreover, $H(\alpha) = Q(\alpha)^2$ if and only if $\alpha_n = cp^n$ for some $c, p \in \mathbb{C}$ with $|p| < 1$.

Standing waves

A solution of the conformal flow is called a standing wave if $\alpha(t) = Ae^{-i\lambda t}$, where $(A_n)_{n\in\mathbb{N}}$ are time-independent and λ is real.

The amplitudes of the standing wave satisfy

$$(n+1)\lambda A_n = \sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{n,j,k,n+j-k}\,\bar{A}_j A_k A_{n+j-k}\,.$$

and they are critical points of energy H for fixed mass Q with the Lyapunov functional $K(\alpha) = \frac{1}{2}H(\alpha) - \lambda Q(\alpha)$.

Among the standing waves, there is a ground state

$$\alpha_n(t) = cp^n e^{-i\lambda t}, \quad \lambda = \frac{|c|^2}{(1-|p|^2)^2}, \quad c \in \mathbb{C}, \quad p \in \mathbb{C}$$

with $|p| < 1$ since $\alpha_n = cp^n$ attains $H(\alpha) \leq Q(\alpha)^2$.

Normalized ground state

Normalized ground state with $\lambda = 1$

$$A_n(p) = (1 - p^2)p^n, \quad p \in (0,1).$$

- The conserved quantities H, Q, E take values:

$$H(A) = 1, \quad Q(A) = 1, \quad E(A) = \frac{1+p^2}{1-p^2}.$$

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- If $\alpha = A(p) + a + ib$ is substituted into $K(\alpha) = \frac{1}{2}H(\alpha) - Q(\alpha)$, then

$$K(\alpha) - K(A(p)) = \langle L_+(p)a, a\rangle + \langle L_-(p)b, b\rangle + \mathcal{O}(\|a\|^3 + \|b\|^3).$$

  and the Hessian operator $(L_+(p), L_-(p))$ in $\ell^{2,1}(\mathbb{N}) \times \ell^{2,1}(\mathbb{N})$ admits a simple positive eigenvalue, a triple zero eigenvalue, and the rest of the spectrum is strictly negative.
  - $L_-(p)A(p) = 0$ and $L_-(p)MA(p) = 0$
  - $L_+(p)A'(p) = 0$ and $L_+(p)MA(p) = \lambda_*(p)MA(p)$.

  where $\lambda_*(p) = 2(1+p^2)/(1-p^2) > 0$ and $M = \mathrm{diag}(1, 2, \cdots)$.

Orbital stability of the ground state family

The ground state family

$$A_n(p) = (1-p^2)p^n, \quad p \in (0,1).$$

By the phase shift invariance, it defines the 2-dim orbit

$$\mathcal{A}(p) = \left\{ \left(e^{i\theta + i\mu n} A_n(p)\right)_{n\in\mathbb{N}} : (\theta,\mu) \in \mathbb{S}^1 \times \mathbb{S}^1 \right\}.$$

Theorem (Bizon–Hunik–P, 2019)

For every $p_0 \in (0,1)$ and every small $\epsilon > 0$, there is $\delta > 0$ such that for every initial data $\alpha(0) \in \ell^{2,1}(\mathbb{N})$ satisfying $\|\alpha(0) - A(p_0)\|_{\ell^{2,1}} \leq \delta$, the unique solution $\alpha(t) \in C(\mathbb{R}_+, \ell^{2,1})$ satisfies for all t

$$\inf_{\theta,\mu\in\mathbb{S}} \|\alpha(t) - e^{i\theta+i\mu n} A(p(t))\|_{\ell^{2,1/2}} \leq \epsilon,$$

and

$$\inf_{\theta,\mu\in\mathbb{S}} \|\alpha(t) - e^{i\theta+i\mu n} A(p(t))\|_{\ell^{2,1}} \lesssim \epsilon + (p_0 - p(t))^{1/2}$$

for some continuous function $p(t) \in [0, p_0]$.

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for some continuous function $p(t) \in [0, p_0]$.

Parameter $p(t)$ may drift towards smaller values compensated by the increasing $\ell^{2,1}$ distance between the solution and the orbit.

Coercivity of the energy in $\ell^{2,1/2}(\mathbb{N})$

Symplectically orthogonal subspace of $\ell^2(\mathbb{N})$:

$$[X_c(p)]^\perp := \left\{ a \in \ell^2(\mathbb{N}) : \quad \langle MA(p), a \rangle = \langle MA'(p), a \rangle = 0 \right\}.$$

There exists $C > 0$ such that

$$-\langle L_+(p)a, a \rangle \gtrsim \|a\|^2_{\ell^{2,1/2}}$$

$$-\langle L_-(p)b, b \rangle \gtrsim \|b\|^2_{\ell^{2,1/2}}$$

for every $a, b \in \ell^{2,1/2}(\mathbb{N}) \cap [X_c(p)]^\perp$.

Decomposition near the ground state

Assume that the initial data $\alpha(0) \in \ell^{2,1}(\mathbb{N})$ satisfy

$$\|\alpha(0) - A(p_0)\|_{\ell^{2,1}} \leq \delta,$$

for some $p_0 \in [0, 1)$ and a sufficiently small $\delta > 0$. Then, the corresponding unique global solution $\alpha(t) \in C(\mathbb{R}_+, \ell^{2,1})$ of the conformal flow can be represented by the decomposition

$$\alpha_n(t) = e^{i(\theta(t)+(n+1)\mu(t))} \left(c(t)A_n(p(t)) + a_n(t) + ib_n(t) \right),$$

$a, b \in [X_c(p)]^{\perp}$ satisfying for all t

$$|c(t) - 1| + \|a(t) + ib(t)\|_{\ell^{2,1/2}} \lesssim \delta.$$

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$a, b \in [X_c(p)]^\perp$ satisfying for all t

$$|c(t) - 1| + \|a(t) + ib(t)\|_{\ell^{2,1/2}} \lesssim \delta.$$

- The proof is based on the Lyapunov function

$$\Delta(c) := c^2 \left(Q(\alpha) - 1 \right) - \frac{1}{2} \left(H(\alpha) - 1 \right).$$

Control of the drift of $p(t)$ in time

Under the same assumptions,

$$p(t) \lesssim p_0 + \delta, \quad \|a(t) + ib(t)\|_{\ell^{2,1}} \lesssim \delta^{1/2} + |p_0 - p(t)|^{1/2}.$$

- The proof is based on the additional mass conservation:

$$E(\alpha(t)) = c(t)^2 \frac{1 + p(t)^2}{1 - p(t)^2} + \|a(t) + ib(t)\|^2_{\ell^{2,1}},$$

  which yields

$$\frac{2(p(t)^2 - p_0^2)}{(1 - p(t)^2)(1 - p_0^2)} + \|a(t) + ib(t)\|^2_{\ell^{2,1}} \lesssim \delta,$$

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Does the drift of $p(t)$ towards smaller values actually occur?

Refined control of the drift of $p(t)$ in time

In addition to the conservation of $H(\alpha)$, $Q(\alpha)$, and $E(\alpha)$, there exists another conserved quantity:

$$Z(\alpha) = \sum_{n=0}^{\infty} (n+1)(n+2)\bar{\alpha}_{n+1}\alpha_n.$$

Biasi–Bizon–Evrin (2018)

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For the normalized ground state,

$$H(A) = 1, \quad Q(A) = 1, \quad E(A) = \frac{1+p^2}{1-p^2}, \quad Z(A) = \frac{2p}{1-p^2},$$

and expansion near the ground state family gives

$$E(\alpha(t)) - |Z(\alpha(t))| \geq \frac{1-p(t)}{1+p(t)} c(t)^2,$$

which yields

$$\frac{p_0 - p(t)}{(1+p(t))(1+p_0)} \lesssim \delta,$$

or $p_0 - p(t) \lesssim \delta$.

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$$\inf_{\theta,\mu\in\mathbb{S}} \|\alpha(t) - e^{i\theta+i\mu n} A(p_0)\|_{\ell^{2,1}} \leq \epsilon,$$

Main Question

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Two answers:

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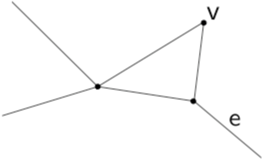
Nonlinear Schrödinger equation on a metric graph



A metric graph $\Gamma = \{E, V\}$ is given by a set of edges E and vertices V, with a metric structure on each edge.

Nonlinear Schrödinger equation on a graph Γ:

$$i\Psi_t = -\Delta\Psi - 2|\Psi|^2\Psi, \quad x \in \Gamma,$$

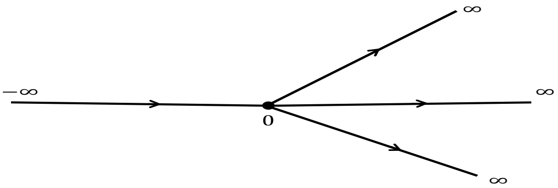
where Δ is the graph Laplacian and $\Psi(t,x)$ is defined componentwise on edges subject to Neumann–Kirchhoff boundary conditions at vertices:

$$\begin{cases} \Psi(v) \text{ is continuous} & \text{for every } v \in V, \\ \sum_{e\sim v} \partial\Psi_e(v) = 0, & \text{for every } v \in V, \end{cases}$$

where $e \sim v$ denotes all edges $e \in E$ adjacent to $v \in V$.

Example: a star graph

A star graph is the union of N half-lines connected at a single vertex. For $N = 2$, the graph is the line $\mathbb{R}$. For $N = 3$, the graph is a Y-junction.



Function spaces are defined componentwise:

$$L^2(\Gamma) = L^2(\mathbb{R}^-) \oplus \underbrace{L^2(\mathbb{R}^+) \oplus \cdots \oplus L^2(\mathbb{R}^+)}_{\text{(N-1) elements}},$$

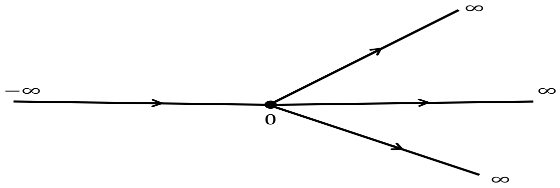
subject to the Neumann–Kirchhoff conditions at a single vertex:

$$H^1_\Gamma := \{\Psi \in H^1(\Gamma) : \quad \psi_1(0) = \psi_2(0) = \cdots = \psi_N(0)\}$$

$$H^2_\Gamma := \{\Psi \in H^2(\Gamma) \cap H^1_\Gamma : \quad \psi_1'(0) = \sum_{j=2}^{N} \psi_j'(0)\},$$

Generalization of a star graph

A star graph is the union of N half-lines connected at a single vertex. For $N = 2$, the graph is the line $\mathbb{R}$. For $N = 3$, the graph is a Y-junction.



For given positive $(\alpha_1, \cdots, \alpha_N)$,

$$H^1_\Gamma := \{\Psi \in H^1(\Gamma) : \quad \alpha_1\psi_1(0) = \alpha_2\psi_2(0) = \cdots = \alpha_N\psi_N(0)\}$$

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Steady states on star graphs

Theorem. (Adami–Serra-Tilli, 2015)
If $N \geq 3$ and $\alpha_1 = \cdots = \alpha_N = 1$, no ground state exists in

$$\mathcal{E} = \inf\{E(u) : \quad u \in H^1_\Gamma, \quad Q(u) = \mu\}.$$

A minimizing sequence escapes to infinity along the edges.

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There exists a standing wave of the Euler–Lagrange equation:

$$-\Delta\Phi - 2|\Phi|^2\Phi = -\omega\Phi$$

in the form of the half-soliton:

$$\Phi(x) = \left[\begin{array}{ll} \sqrt{\omega}\mathrm{sech}(\sqrt{\omega}x), & x \in (-\infty, 0), \;\; j = 1, \\ \sqrt{\omega}\mathrm{sech}(\sqrt{\omega}x), & x \in (0, \infty), \;\; 2 \leq j \leq N. \end{array} \right].$$

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Theorem. (Adami *et al.*, 2012) (Kairzhan–P., JDE, 2018) Half-soliton is a saddle point of energy E at fixed mass Q. This saddle point is unstable in the time evolution of the NLS.

Non-uniqueness of the half-soliton

Consider now generalized boundary conditions

$$\begin{cases} \alpha_1\psi_1(0) = \alpha_2\psi_2(0) = \cdots = \alpha_N\psi_N(0) \\ \alpha_1^{-1}\psi_1'(0) = \alpha_2^{-1}\psi_2'(0) + \cdots + \alpha_N^{-1}\psi_N'(0). \end{cases}$$

and generalized NLS equation $i\Psi_t = -\Delta\Psi - 2\alpha^2|\Psi|^2\Psi$, where $(\alpha_1, \dots, \alpha_N)$ are positive.

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If $\alpha_1^{-2} = \sum_{j=2}^N \alpha_j^{-2}$, then there exists a one-parameter family of solutions $\{\Phi(x;a)\}_{a\in\mathbb{R}}$ satisfying

$$\Phi(x;a) = \left[\begin{array}{ll} \alpha_1^{-1}\sqrt{\omega}\text{sech}(\sqrt{\omega}(x+a)), & x \in (-\infty, 0), \;\; j = 1, \\ \alpha_j^{-1}\sqrt{\omega}\text{sech}(\sqrt{\omega}(x+a)), & x \in (0, \infty), \;\; 2 \le j \le N. \end{array} \right].$$

D. Matrasulov–K. Sabirov–Z. Sobirov (2012)

Shifted standing waves

Example for $N = 3$:

$$\Phi(x;a) = \left[\begin{array}{ll} \alpha_1^{-1}\sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x+a)), & x \in (-\infty,0), \\ \alpha_2^{-1}\sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x+a)), & x \in (0,\infty), \\ \alpha_3^{-1}\sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x+a)), & x \in (0,\infty). \end{array} \right].$$

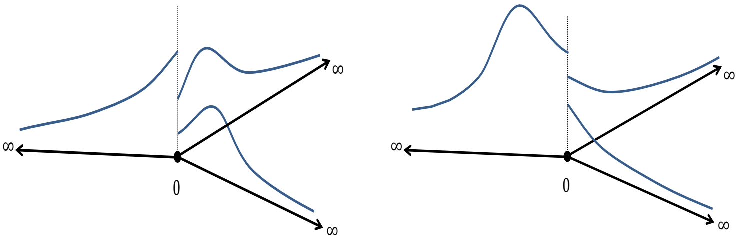

Figure: Schematic representation of the shifted standing waves on the star graph with $N = 3$, and either $a < 0$ (left) or $a > 0$ (right).

Reason for existence of shifted states

Assume that $\Psi \in H^1_\Gamma$ satisfies the symmetry reduction:

$$\alpha_2 \psi_2(t,x) = \cdots = \alpha_N \psi_N(t,x), \quad x > 0.$$

If $\alpha_1^{-2} = \sum_{j=2}^N \alpha_j^{-2}$, the wave function

$$\varphi(t,x) = \begin{cases} \alpha_1 \psi_1(t,x), & x \leq 0, \\ \alpha_2 \psi_2(t,x), & x \geq 0, \end{cases}$$

satisfies the cubic NLS equation on the line $\mathbb{R}$:

$$i \frac{\partial \varphi}{\partial t} = -\frac{\partial^2 \varphi}{\partial x^2} - 2|\varphi|^2 \varphi, \quad x \in \mathbb{R},$$

which is translationally invariant in x.

Momentum conservation

For a solution $\Psi \in C(\mathbb{R}, H^1_\Gamma)$, let us define the momentum of the NLS:

$$P(\Psi) = \mathrm{Im}\langle \Psi', \Psi \rangle_{L^2(\Gamma)}$$

Momentum conservation

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$$P(\Psi) = \mathrm{Im}\langle \Psi', \Psi \rangle_{L^2(\Gamma)}$$

If $\alpha_1^{-2} = \sum_{j=2}^N \alpha_j^{-2}$, the map $t \mapsto P(\Psi)$ is monotonically increasing:

$$\frac{dP}{dt} = \frac{1}{2} \sum_{j=2}^N \sum_{i \neq j}^N \frac{\alpha_1^2}{\alpha_j^2 \alpha_i^2} \left| \alpha_j \psi_j'(0) - \alpha_i \psi_i'(0) \right|^2 \geq 0.$$

If in addition, the solution is symmetric and satisfies the NLS reduction:

$$\alpha_2 \psi_2(t,x) = \cdots = \alpha_N \psi_N(t,x), \quad x > 0,$$

then the momentum $P(\Psi)$ is constant in time.

Stability of standing waves on the star graph

Shifted standing waves with parameters $\omega > 0$ and $a \in \mathbb{R}$:

$$\Phi_\omega(x;a) = \left[\begin{array}{ll} \alpha_1^{-1}\sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x+a)), & x \in (-\infty,0), \;\; j=1, \\ \alpha_j^{-1}\sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x+a)), & x \in (0,\infty), \;\; 2 \le j \le N. \end{array} \right]$$

Substituting $\Psi = \Phi_\omega + U + iW$ into $\Lambda_\omega(u) := E(u) + \omega Q(u)$ yields

$$\Lambda_\omega(\Phi_\omega + U + iW) - \Lambda_\omega(\Phi_\omega) = \langle L_+(\omega,a)U, U\rangle_{L^2(\Gamma)} + \langle L_-(\omega,a)W, W\rangle_{L^2(\Gamma)} + \cdots,$$

where

$$\left\{ \begin{array}{l} L_-(\omega,a) = -\Delta + \omega - 2\alpha^2\Phi_\omega(\cdot;a)^2, \\ L_+(\omega,a) = -\Delta + \omega - 6\alpha^2\Phi_\omega(\cdot;a)^2. \end{array} \right.$$

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Spectral properties of $L_\pm(\omega,a)$:

- $\sigma_c(L_\pm) = [\omega,\infty)$ with $\omega > 0$.
- $L_- \geq 0$ and $\ker(L_-) = \text{span}\{\Phi_\omega\}$.
- $\Phi'_\omega \in \ker(L_+)$

Negative eigenvalues of $L_+(\omega, a)$

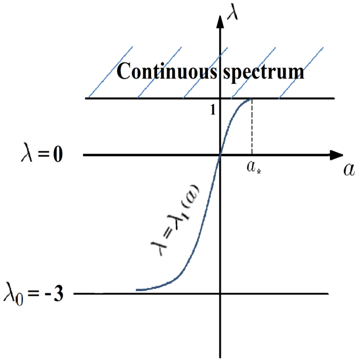

Figure: The spectrum of $L_+(\omega, a)$ for $\omega = 1$.

Theorem. (Kairzhan–P., JPA, 2018) Besides simple eigenvalues $\lambda_0 = -3\omega$ and $\lambda = 0$, there exists exactly one additional eigenvalue $\lambda_1(\omega, a)$ of multiplicity $N - 2$ such that $\lambda_1(\omega, a) > 0$ for $a > 0$ and $\lambda_1(\omega, a) < 0$.

Shifted standing waves

Recall the main example for $N = 3$:

$$\Phi(x;a) = \left[\begin{array}{ll} \alpha_1^{-1}\sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x+a)), & x \in (-\infty, 0), \\ \alpha_2^{-1}\sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x+a)), & x \in (0, \infty), \\ \alpha_3^{-1}\sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x+a)), & x \in (0, \infty). \end{array} \right].$$

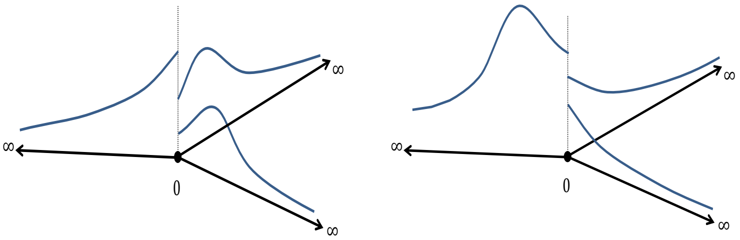

Figure: $L_+(\omega, a)$ has two negative eigenvalues for $a < 0$ (left) and one negative eigenvalue for $a > 0$ (right).

Implication of the eigenvalue count for $N = 3$

$$\Lambda_\omega(\Phi_\omega + U + iW) - \Lambda_\omega(\Phi_\omega) = \langle L_+(\omega, a)U, U\rangle_{L^2(\Gamma)} + \langle L_-(\omega, a)W, W\rangle_{L^2(\Gamma)} + \cdots$$

- $a < 0$: Φ_ω is a saddle point of Λ_ω with two negative eigenvalues and it remains a saddle point with one negative eigenvalue under the constraint of fixed $Q(\Psi) = \|\Psi\|^2_{L^2}$.

Shifted state with $a < 0$ is spectrally and nonlinearly unstable.

Implication of the eigenvalue count for $N = 3$

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- $a > 0$: Φ_ω is a saddle point of Λ_ω with one negative eigenvalue and it is a degenerate constrained minimizer under the constraint of fixed $Q(\Psi) = \|\Psi\|_{L^2}^2$ with double zero eigenvalue.

  The shifted state with $a > 0$ is spectrally stable. Is it nonlinearly stable?

Recap for spectrally stable shifted states with $a > 0$

$$\Lambda_\omega(\Phi_\omega + U + iW) - \Lambda_\omega(\Phi_\omega) = \langle L_+(\omega, a)U, U\rangle_{L^2(\Gamma)} + \langle L_-(\omega, a)W, W\rangle_{L^2(\Gamma)} + \cdots$$

- $L_- \geq 0$ and $\ker(L_-) = \text{span}\{\Phi_\omega\}$.
- $\ker(L_+) = \text{span}\{\Phi'_\omega\}$ and L_+ has one negative eigenvalue.

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- The decomposition $\Psi(x) = e^{i\theta}\left[\Phi_\omega(x;a) + U(x) + iW(x)\right]$ is uniquely defined for $\theta \in \mathbb{R}$, $a \in \mathbb{R}$, and $\omega > 0$ subject to three constraints on U and W including $\langle U, \Phi_\omega\rangle_{L^2} = 0$. Hessian $\Lambda_\omega''(\Phi_\omega)$ is strictly positive under the three constraints.

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- $U, W \in H^1_\Gamma$ and ω are controlled in the time evolution from energy estimates due to coercivity of the Lyapunov function.

Drift of the shifted states

Theorem. (Kairzhan–P–Goodman, 2019)
Fix $a_0 > 0$. For every $\mathbf{a} \in (0, a_0)$ there exists $\epsilon_0 > 0$ (sufficiently small) such that for every $\epsilon \in (0, \epsilon_0)$, there exists $\delta > 0$ and $T > 0$ such that for every initial datum $\Psi_0 \in H^1_\Gamma$ with $P(\Psi_0) > 0$ and

$$\|\Psi_0 - \Phi_\omega(\cdot; a_0)\|_{H^1(\Gamma)} \leq \delta$$

the unique solution $\Psi \in C([0,T], H^1_\Gamma) \cap C^1([0,T], H^{-1}_\Gamma)$ to the NLS equation with the initial datum $\Psi(0,\cdot) = \Psi_0$ satisfies the bound

$$\inf_{\theta \in \mathbb{R}} \|\Psi(t,\cdot) - e^{i\theta}\Phi_\omega(\cdot; a(t))\|_{H^1(\Gamma)} \leq \epsilon, \quad t \in [0,T],$$

where $a \in C^1([0,T])$ is a strictly decreasing function such that $\lim_{t\to T} a(t) = \mathbf{a}$.

Reason for the drift

Recall that the momentum of the NLS:

$$P(\Psi) = \mathrm{Im}\langle \Psi', \Psi \rangle_{L^2(\Gamma)}$$

is no longer constant but is monotonically increasing:

$$\frac{dP}{dt} = \frac{1}{2} \sum_{j=2}^{N} \sum_{i \neq j}^{N} \frac{\alpha_1^2}{\alpha_j^2 \alpha_i^2} \left| \alpha_j \psi_j'(0) - \alpha_i \psi_i'(0) \right|^2 \geq 0.$$

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For the solution uniquely decomposed as

$$\Psi(t,x) = e^{i\theta(t)} \left[\Phi_{\omega(t)}(x; a(t)) + U(t,x) + iW(t,x) \right],$$

the momentum is expanded as

$$P(\Psi) = -2\langle \Phi_\omega'(\cdot; a), W \rangle_{L^2(\Gamma)} + \mathcal{O}(\|U + iW\|_{H^1(\Gamma)}^2),$$

whereas the modulation equation for $a(t)$ reads as

$$\dot{a} = 2\langle \Phi_\omega'(\cdot; a), W \rangle_{L^2(\Gamma)} \left[1 + \mathcal{O}(\|U + iW\|_{H^1(\Gamma)}) \right] + \mathcal{O}(\|U + iW\|_{H^1(\Gamma)}^2),$$

so that $\dot{a} = -P(\Psi) + \mathcal{O}(\|U + iW\|_{H^1(\Gamma)}^2) < 0$ if $P(\Psi) \geq P(\Psi_0) > 0$.

Numerical illustration: linear instability for $a < 0$

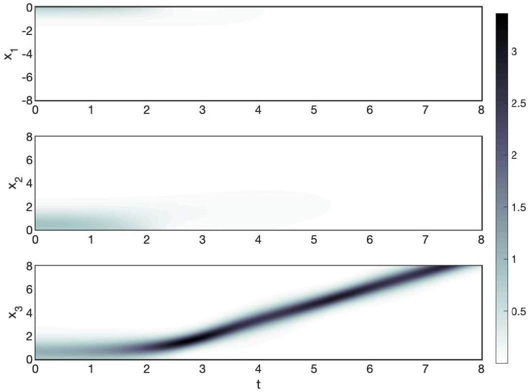

Figure: A numerical solution for $a = -0.55$ and $\epsilon = 0.1$. The colorbar corresponds to values of $|u|^2$. The three panels correspond to the solution on edges 1, 2, and 3 going down.

Numerical illustration: drift instability for $a > 0$

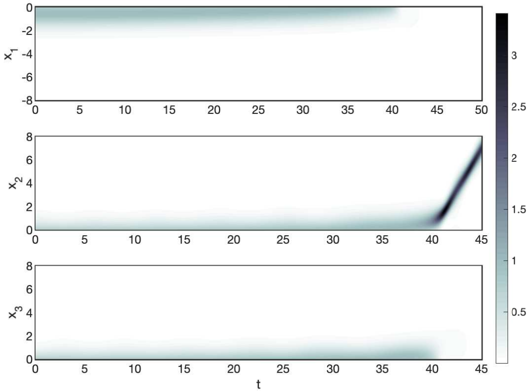

Figure: A numerical solution for $a = 0.55$ and $\epsilon = 0.1$. The colorbar corresponds to values of $|u|^2$.

Drift instability for $a > 0$

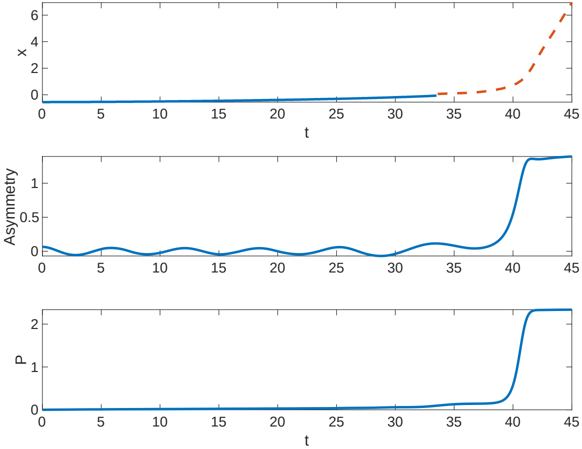

Figure: Postprocessed quantities form the same simulation. (Top) The position of the maximum of u. The solid line for $t < 33.5$ describes the position on the incoming edge one. The dashed line for $t > 33.5$ shows the position of the maximum on edge two. (Middle) The asymmetry, defined as $\|u_2\|_{L^2(\mathbb{R}+)} - \|u_3\|_{L^2(\mathbb{R}+)}$. (Bottom) The momentum $P(\Psi)$ versus time t.

Pushing experiments beyond the validity of the theorem

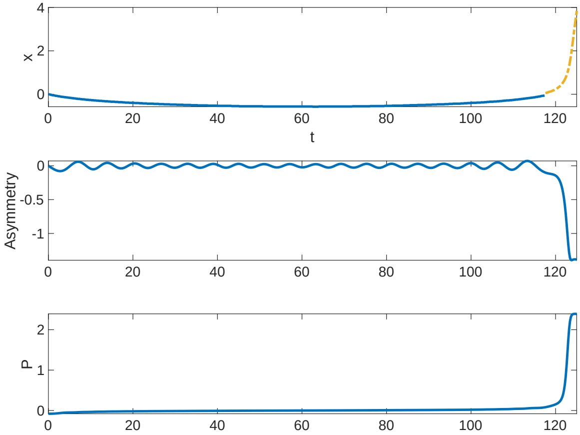

Figure: A numerical solution with $a = 0$. (Top) The position of the maximum of $|u|^2$, on edge one for $t < 117$ and on edge three (dashed) for $t > 117$. (Middle) Asymmetry of the solution between the two outgoing edges. (Bottom) The momentum $P(\Psi)$ versus time t.

Conclusion

- Resonant normal forms: drift along the degenerate ground state family is eliminated due to conserved quantities.

- PDEs on star graphs: drift along the shifted states leads to nonlinear instability of the standing wave.

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Thanks! Questions???