Ground state on the bounded and unbounded graphs

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Summary

Background: Gross–Pitaevskii equation with potential wells

Nonlinear PDEs on metric graphs

Ground states on the unbounded graphs

Ground states on the dumbbell graph

Conclusion

Background: Gross–Pitaevskii equation with potential wells

In many problems (BECs, photonics, optics), wave dynamics is modeled with the nonlinear Schrödinger (Gross–Pitaevskii) equation

$$iu_t = -u_{xx} + V(x)u \pm |u|^{2p}u,$$

where $p > 0$ is the nonlinearity power and $V(x) : \mathbb{R} \mapsto \mathbb{R}$ is a trapping potential. The upper sign is defocusing (repelling) and the lower sign is focusing (attractive).

- Single-well potentials such as $V_0(x) = -\text{sech}^2(x)$.
- Double-well potentials such as

$$V(x;s) = \frac{1}{2}\left(V_0(x-s) + V_0(x+s)\right), \quad s \geq 0.$$

- Periodic potentials (optical lattices)

$$V(x+L) = V(x), \quad L > 0,$$

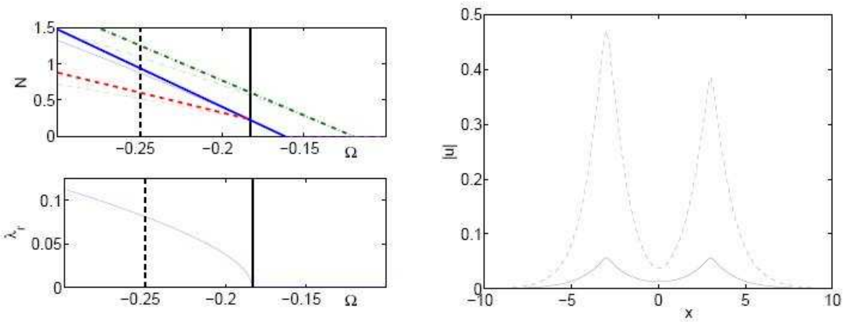
such as $V(x) = \sin^2(x)$.

Double-well potentials

Stationary solutions $u(x,t) = \phi(x)e^{-i\omega t}$ with $\omega \in \mathbb{R}$ satisfy a stationary Schrödinger equation with a double-well potential

$$\omega\phi = -\phi_{xx} + V(x;s)\phi - |\phi|^2\phi.$$

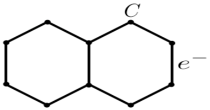
Let V_0 support exactly one negative eigenvalue of $L_0 = -\partial_x^2 + V_0(x)$ and s be large. The operator $L = -\partial_x^2 + V(x;s)$ has two negative eigenvalues with symmetric and anti-symmetric eigenfunctions. In the focusing case, the bifurcation diagram looks as



E. Kirr; P.G. Kevrekidis; J. Marzuola; M. Weinstein; R. Goodman; R. Fukuizumi; ...

Nonlinear PDEs on metric graphs

Graph models for the dynamics of constrained quantum particles were first suggested by Pauling and then used by Ruedenberg and Scherr in 1953 to study the spectrum of aromatic hydrocarbons.

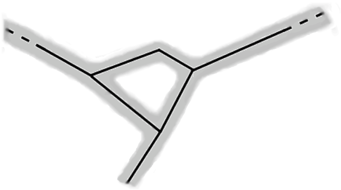


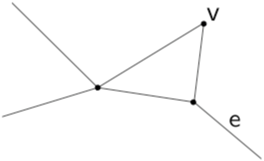
Nowadays graph models are widely used in the modeling of quantum dynamics of thin graph-like structures (quantum wires, nanotechnology, large molecules, periodic arrays in solids, photonic crystals...).

- G. Berkolaiko and P. Kuchment, *Introduction to Quantum Graphs* (AMS, Providence, 2013).
- P. Exner and H. Kovarík, *Quantum Waveguides*, (Springer, Heidelberg, 2015).

Metric Graphs

Graphs are one-dimensional approximations for constrained dynamics in which transverse dimensions are small with respect to longitudinal ones.



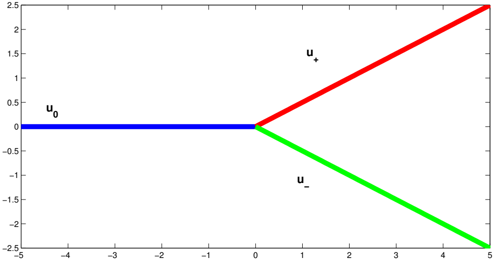


A metric graph Γ is given by a set of edges and vertices, with a metric structure on each edge. Proper boundary conditions are needed on the vertices to ensure that certain differential operators defined on graphs are self-adjoint.

Kirchhoff boundary conditions:

- Functions in each edge have the same value at each vertex.
- Sum of fluxes (signed derivatives of functions) is zero at each vertex.

Example: Y junction graph



The Laplacian operator on the graph Γ is defined by

$$\Delta\Psi = \left[\begin{array}{ll} u_0''(x), & x \in (-\infty, 0), \\ u_\pm''(x), & x \in (0, \infty) \end{array} \right],$$

acting on functions in the form

$$\Psi = \left[\begin{array}{ll} u_0(x), & x \in (-\infty, 0) \\ u_\pm(x), & x \in (0, \infty) \end{array} \right],$$

in the domain

$$\mathcal{D}(\Gamma) = \left\{ \begin{array}{l} (u_0, u_+, u_-) \in H^2(\mathbb{R}^-) \times H^2(\mathbb{R}^+) \times H^2(\mathbb{R}^+) : \\ u_0(0) = u_+(0) = u_-(0), \quad u_0'(0) = u_+'(0) + u_-'(0) \end{array} \right\}.$$

Laplacian on the Y junction graph

Lemma
The operator $\Delta : \mathcal{D}(\Gamma) \to L^2(\Gamma)$ is self-adjoint.

The Kirchhoff boundary conditions are symmetric:

$$\begin{aligned} \langle \Phi, \Delta\Psi \rangle - \langle \Delta\Phi, \Psi \rangle &= \left[\left(\bar{v}_0' u_0 - \bar{v}_0 u_0' \right) - \left(\bar{v}_+' u_+ - \bar{v}_+ u_+' \right) - \left(\bar{v}_-' u_- - \bar{v}_- u_-' \right) \right]_{x=0} \\ &= \left[u_0 \left(\bar{v}_0' - \bar{u}_+' - \bar{u}_-' \right) - \bar{v}_0 \left(u_0' - u_+' - u_-' \right) \right]_{x=0} = 0, \end{aligned}$$

where $\Phi = (v_0, v_+, v_-)$ and $\Psi = (u_0, u_+, u_-)$ satisfy the Kirchhoff conditions:

$$\left\{ \begin{array}{l} u_0(0) = u_+(0) = u_-(0), \\ u_0'(0) = u_+'(0) + u_-'(0). \end{array} \right.$$

Moreover, Δ is self-adjoint under generalized Kirchhoff boundary conditions

$$\left\{ \begin{array}{l} \alpha_0 u_0(0) = \alpha_+ u_+(0) = \alpha_- u_-(0) \\ \alpha_0^{-1} u_0'(0) = \alpha_+^{-1} u_+'(0) + \alpha_-^{-1} u_-'(0), \end{array} \right.$$

where $\alpha_0, \alpha_+, \alpha_-$ are arbitrary nonzero parameters.

NLS on the Y junction graph

Consider the nonlinear Schrödinger (NLS) equation on the graph Γ:

$$i\partial_t u_0 + \partial_x^2 u_0 + 2|u_0|^2 u_0 = 0, \quad x < 0,$$
$$i\partial_t u_\pm + \partial_x^2 u_\pm + 2|u_\pm|^2 u_\pm = 0, \quad x > 0,$$

subject to the generalized Kirchhoff boundary conditions at $x = 0$.

The mass functional

$$Q = \int_{-\infty}^{0} |u_0|^2 dx + \int_0^{+\infty} |u_+|^2 dx + \int_0^{+\infty} |u_-|^2 dx$$

is constant in time t (related to the gauge symmetry).

The energy functional

$$E = \int_{-\infty}^{0} \left(|\partial_x u_0|^2 - |u_0|^4 \right) dx + \text{similar terms for } u_\pm,$$

is constant in time t (related to the time translation symmetry).

The momentum functional

$$P = i \int_{-\infty}^{0} (\bar{u}_0 \partial_x u_0 - u_0 \partial_x \bar{u}_0)\, dx + \text{similar terms for } u_\pm,$$

is no longer constant in time t because the spatial translation is broken.

NLS on the Y junction graph

Let us connect parameters $\alpha_0, \alpha_+, \alpha_-$ with the nonlinear coefficients as follows:

$$\begin{aligned} i\partial_t u_0 + \partial_x^2 u_0 + \alpha_0^2 |u_0|^2 u_0 = 0, &\quad x < 0, \\ i\partial_t u_\pm + \partial_x^2 u_\pm + \alpha_\pm^2 |u_\pm|^2 u_\pm = 0, &\quad x > 0, \end{aligned}$$

Let us also add the constraint

$$\frac{1}{\alpha_0^2} = \frac{1}{\alpha_+^2} + \frac{1}{\alpha_-^2}$$

in the generalized Kirchhoff boundary conditions:

$$\begin{cases} \alpha_0 u_0(0) = \alpha_+ u_+(0) = \alpha_- u_-(0) \\ \alpha_0^{-1} \partial_x u_0(0) = \alpha_+^{-1} \partial_x u_+(0) + \alpha_-^{-1} \partial_x u_-(0). \end{cases}$$

Then, the momentum is decreasing function of time:

$$\frac{dP}{dt} = -\frac{2\alpha_0^2}{\alpha_+^2 \alpha_-^2} \left|\alpha_+ \partial_x u_+(0) - \alpha_- \partial_x u_-(0)\right|^2 \leq 0.$$

Reflectionless scattering of solitary waves

The following reduction

$$\alpha_+ u_+(x) = \alpha_- u_-(x), \quad x \in \mathbb{R}^+,$$

is invariant with respect to the time evolution of the NLS equation. In this case, the momentum P is constant in time.

Then, the Kirchhoff boundary conditions imply

$$\left\{ \begin{array}{l} \alpha_0 u_0(0) = \alpha_+ u_+(0), \\ \alpha_0 \partial_x u_0(0) = \alpha_+ \partial_x u_+(0), \end{array} \right.$$

so that we can set the following function on the infinite line:

$$U(x,t) = \left\{ \begin{array}{ll} \alpha_0 u_0(x,t), & x < 0, \\ \alpha_\pm u_\pm(x,t), & x > 0. \end{array} \right.$$

The function U satisfies the integrable cubic NLS equation

$$i\partial_t U + \partial_x^2 U + |U|^2 U = 0, \quad x \in \mathbb{R},$$

where the vertex $x = 0$ does not appear as an obstacle in the time evolution.

D. Matrasulov; K. Sabirov; H. Uecker; D. Dytukh; J.G. Caputo;

Ground states on the unbounded graphs

The ground state is a standing wave of smallest energy E at a fixed value of mass Q,

$$\mathcal{E} = \inf\{E(u) : \quad u \in H^1(\Gamma), \quad Q(u) = \mu\}.$$

Euler–Lagrange equation is

$$-\Delta\phi - 2|\phi|^2\phi = \Lambda\phi \qquad \Lambda \in \mathbb{R}, \quad \phi \in \mathcal{D}(\Gamma),$$

where $\alpha_0^2 = \alpha_+^2 = \alpha_-^2 = 2$ are set for convenience.

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The infimum of $E(u)$ exists in $H^1_\mu(\Gamma)$ due to Gagliardo–Nirenberg inequality in 1D.

If G is unbounded and contains at least one half-line, then

$$\min_{\phi \in H^1_\mu(\mathbb{R}^+)} E(u;\mathbb{R}^+) \leq \mathcal{E} \leq \min_{\phi \in H^1_\mu(\mathbb{R})} E(u;\mathbb{R})$$

However, the infimum may not be achieved by any of the standing waves $\phi \in H^1_\mu(\Gamma)$.

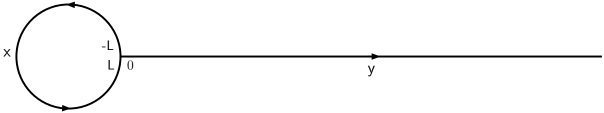
Adami–Cacciapuoti–Finco–Noja (2014,2016); Adami-Serra-Tilli (2015, 2016)

Ground states on the unbounded graphs

If G consists of either one half-line or two half-lines and a bounded edge, then

$$\mathcal{E} < \min_{\phi \in H^1_\mu(\mathbb{R})} E(u; \mathbb{R})$$

and the infimum is achieved.



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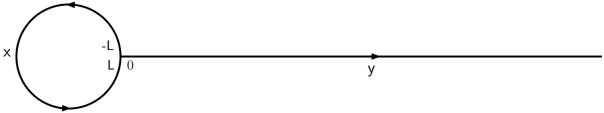


If G consists of more than two half-lines and is *connective to infinity*, then

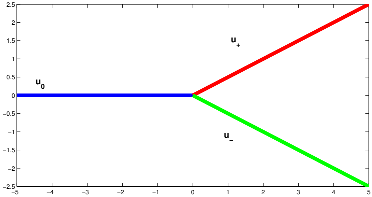
$$\mathcal{E} = \min_{\phi \in H^1_\mu(\mathbb{R})} E(u;\mathbb{R})$$

and the infimum is not achieved. The reason is topological. By the symmetry rearrangements,

$$E(u;\Gamma) > E(\hat{u};\mathbb{R}) \geq \min_{\phi \in H^1_\mu(\mathbb{R})} E(u;\mathbb{R}).$$

At the same time, a sequence of solitary waves escaping to infinity along one edge yields a sequence of functions that minimize $E(u;\Gamma)$ until it reaches $\mathcal{E}$.

Ground state on the Y junction graph



No ground state exists due to the same topological reason.

There exists a half-soliton solution to the Euler–Lagrange equation

$$-\Delta\phi - 2|\phi|^2\phi = \Lambda\phi \qquad \Lambda \in \mathbb{R} \quad \phi \in \mathcal{D}(\Gamma),$$

in the form

$$\phi(x) = \left[\begin{array}{ll} \phi_0(x) = \sqrt{|\Lambda|}\mathrm{sech}(\sqrt{|\Lambda|}x), & x \in (-\infty, 0) \\ \phi_\pm(x) = \sqrt{|\Lambda|}\mathrm{sech}(\sqrt{|\Lambda|}x), & x \in (0, \infty) \end{array} \right]$$

which satisfies the Kirchhoff boundary conditions.

Despite the half-soliton is not a ground state, it is a local constrained minimizer of $E(u)$ in $H^1_\mu(\Gamma)$, hence it is orbitally stable in the nonlinear dynamics.

Topic opened for discussions

Nonexistence of ground states on the unbounded graphs mean nothing in understanding the nonlinear dynamics of the NLS on the graphs.

Recall the NLS equation on the Y-junction graph:

$$
\begin{aligned}
-\partial_x^2 \phi_0 - \alpha_0^2 |\phi_0|^2 \phi_0 &= \Lambda \phi_0, \quad x < 0, \\
-\partial_x^2 \phi_\pm - \alpha_\pm^2 |\phi_\pm|^2 \phi_\pm &= \Lambda \phi_\pm, \quad x > 0,
\end{aligned}
$$

subject to the generalized Kirchhoff boundary conditions

$$
\left\{ \begin{array}{l} \alpha_0 \phi_0(0) = \alpha_+ \phi_+(0) = \alpha_- \phi_-(0) \\ \alpha_0^{-1} \phi_0'(0) = \alpha_+^{-1} \phi_+'(0) + \alpha_-^{-1} \phi_-'(0), \end{array} \right.
$$

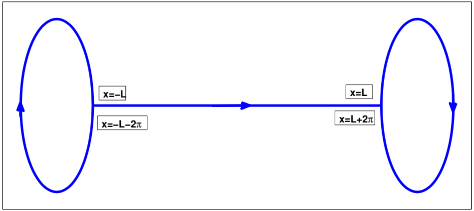
with the constraint $\alpha_0^{-2} = \alpha_+^{-2} + \alpha_-^{-2}$.

The existence problem is (almost) equivalent to the one on infinite line:

$$
-\partial_x^2 \phi - 2|\phi|^2 \phi = \Lambda \phi, \quad x \in \mathbb{R},
$$

for which the NLS soliton $\phi(x) = \sqrt{|\Lambda|} \mathrm{sech}(\sqrt{|\Lambda|} x)$ is the ground state of $E(u; \mathbb{R})$.

Dumbbell Graph



The PDE problem can be formulated in terms of components:

$$U = \begin{bmatrix} u_-(x), & x \in I_- := [-L-2\pi, -L], \\ u_0(x), & x \in I_0 := [-L, L], \\ u_+(x), & x \in I_+ := [L, L+2\pi], \end{bmatrix},$$

subject to the Kirchhoff boundary conditions at the two junctions, *i.e.*

$$\begin{cases} u_+(L+2\pi) = u_+(L) = u_0(L), \\ u_+'(L) - u_+'(L+2\pi) = u_0'(L) \end{cases}$$

J. Marzuola and D.P., Applied Math. Research Express **2016**, 98–145 (2016).

Standing waves on the dumbbell graph

The ground state is the standing wave of smallest energy E at a fixed value of Q,

$$\mathcal{E} = \inf\{E(u) : \quad u \in H^1(\Gamma), \quad Q(u) = \mu\}.$$

Euler–Lagrange equation is

$$-\Delta\phi - 2|\phi|^2\phi = \Lambda\phi \qquad \Lambda \in \mathbb{R}, \quad \phi \in \mathcal{D}(\Gamma).$$

The infimum of $E(u)$ exists in $H^1_\mu(\Gamma)$ and, because Γ is compact, it is achieved by a certain standing wave $\phi \in H^1_\mu(\Gamma)$.

Goals of our project are

- to understand construction of ground states on the dumbbell graph Γ;
- to compare it with the ground state in the double-well potential.

Spectrum of the Laplacian on Γ

The linear problem is

$$-\Delta u = \lambda u \qquad u \in \mathcal{D}(\Gamma).$$

Since $-\Delta : \mathcal{D}(\Gamma) \to L^2(\Gamma)$ is self-adjoint and positive, we have $\lambda \in \mathbb{R}$ and $\lambda \geq 0$. Since Γ is compact, the spectrum of $-\Delta$ consist of isolated eigenvalues λ.

With $\lambda = \omega^2$ parametrization, one can construct eigenfunctions by

$$\begin{cases} u_0(x) = c_0 \cos(\omega x) + d_0 \sin(\omega x), & x \in I_0, \\ u_\pm(x) = c_\pm \cos(\omega x) + d_\pm \sin(\omega x), & x \in I_\pm. \end{cases}$$

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- Double eigenvalues $\{n^2\}_{n\in\mathbb{N}}$ with eigenfunctions supported in each ring.
- Simple eigenvalues $\{\omega_n^2\}_{n\in\mathbb{N}}$ with eigenfunctions symmetric on graph.
- Simple eigenvalues $\{\Omega_n^2\}_{n\in\mathbb{N}}$ with eigenfunctions anti-symmetric on graph.
- Zero eigenvalue with constant eigenfunction.

For both $L < \pi$ and $L \geq \pi$, we have the following ordering of eigenvalues:

$$0 < \Omega_1 < \omega_1 < 1 < \Omega_2 < \dots$$

Candidates for the ground state

We have the following ordering of eigenvalues:

$$0 < \Omega_1 < \omega_1 < 1 < \Omega_2 < \dots$$

- 0 is the lowest eigenvalue λ which corresponds to the constant eigenfunction.

  There exists the constant standing wave of $-\Delta\phi - 2|\phi|^2\phi = \Lambda\phi$:

  $$\phi(x) = p, \quad \Lambda = -2p^2, \quad \mu = 2(L + 2\pi)p^2, \quad p \in \mathbb{R}.$$

  The constant wave exists for every $\Lambda < 0$.

- Ω_1 gives the next eigenvalue $\lambda = \Omega_1^2$ with the odd eigenfunction.

  When the odd eigenfunction is superposed on the constant solution, it gives an asymmetric standing wave of $-\Delta\phi - 2|\phi|^2\phi = \Lambda\phi$.

- ω_1 gives the second next eigenvalue $\lambda = \omega_1^2$ with the even eigenfunction.

  When the even eigenfunction is superposed on the constant solution, it gives a symmetric standing wave of $-\Delta\phi - 2|\phi|^2\phi = \Lambda\phi$.

Bifurcation diagram: small μ

Theorem

There exist μ_1 and μ_2 ordered as $0 < \mu_1 < \mu_2 < \infty$ such that the ground state for $\mu \in (0, \mu_1)$ is given by the constant wave, which undertakes

- *the symmetry breaking bifurcation at μ_1 due to Ω_1,*
- *the symmetry preserving bifurcation at μ_2 due to ω_1.*

The asymmetric wave is the ground state for $\mu \gtrsim \mu_1$, while the symmetric wave is not the ground state for $\mu \approx \mu_2$.

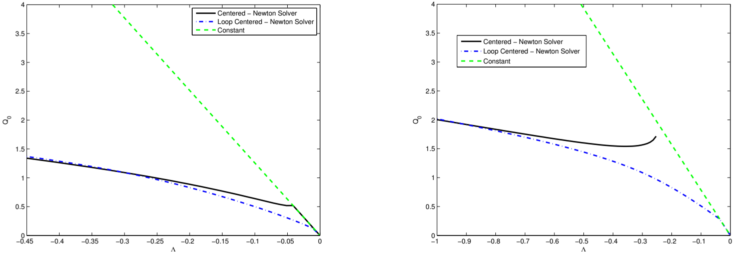


Figure : The bifurcation diagram for $L = 2\pi$ (left) and $L = \pi/2$ (right).

Election of candidates for the ground state

Question: Which standing wave is a ground state of $\mathcal{E}$ for large μ?

- Constant standing wave?
- Asymmetric standing wave?

Election of candidates for the ground state

Question: Which standing wave is a ground state of $\mathcal{E}$ for large μ?

- Constant standing wave?
- Asymmetric standing wave?

If your candidate is

- Symmetric standing wave (the correct choice!)

it was eliminated in primaries six months ago...

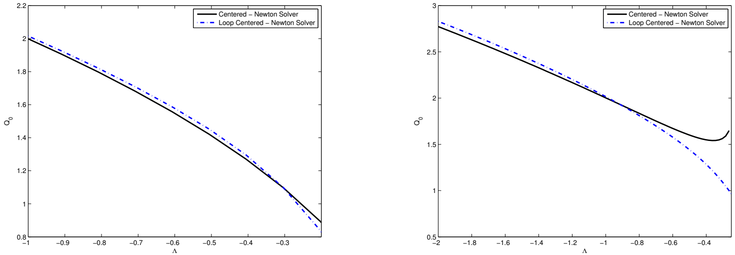
Bifurcation diagram: large μ

Theorem

There exist $\mu_* \in (\mu_2, \infty)$ s.t. for $\mu \in (\mu_*, \infty)$, two standing wave solutions exist:

- *an asymmetric wave localized in the ring,*
- *a symmetric wave localized at the center.*

The symmetric wave is the ground state but both are orbitally stable.



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Numerical approximations: small μ

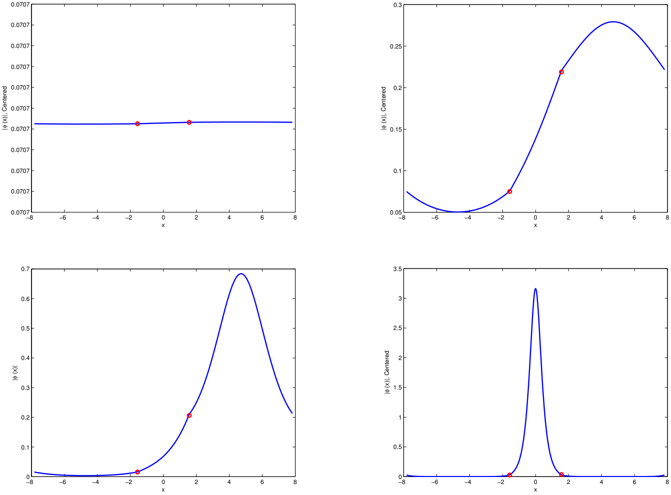


Figure : Ground states for $L = \pi/2$ and $\Lambda = -0.01$ (top left), $\Lambda = -0.1$ (top right), $\Lambda = -1.5$, and $\Lambda = -10.0$ (bottom right). Breaking points are marked by red dots.

Numerical approximations: large μ

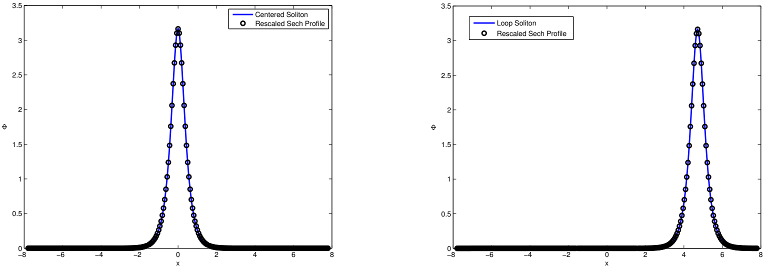


Figure : Comparison of the standing waves (solid line) localized in the central link (left) and in one of the rings (right) to the rescaled solitary wave profile (dots) for $L = \pi/2$ and $\Lambda = -10.0$.

The symmetric wave $\phi \in \mathcal{D}(\Gamma)$ is given by

$$\phi(x) = \sqrt{|\Lambda|}\text{sech}(\sqrt{|\Lambda|}x) + \tilde{\phi}(x), \quad x \in \Gamma,$$

where $\tilde{\phi} \to 0$ as $|\Lambda| \to \infty$ exponentially fast in the L^∞-norm.

Analytical approximations: large μ

After rescaling

$$\phi(x) = |\Lambda|^{1/2}\psi(z), \quad z = |\Lambda|^{1/2}x,$$

the existence problem becomes

$$-\Delta_z\psi + \psi - 2|\psi|^2\psi = 0, \qquad z \in J_- \cup J_0 \cup J_+,$$

where

$$\begin{aligned} J_- &:= \left[-(L+2\pi)|\Lambda|^{1/2}, -L|\Lambda|^{1/2}\right], \\ J_0 &:= \left[-L|\Lambda|^{1/2}, L|\Lambda|^{1/2}\right], \\ J_+ &:= \left[L|\Lambda|^{1/2}, (L+2\pi)|\Lambda|^{1/2}\right]. \end{aligned}$$

The analytical approximation is now $\psi_\infty(z) = \text{sech}(z)$.

Question: How to justify this approximation as $|\Lambda| \to \infty$?

Phase plane analysis

The first-order invariant for the second-order ODE with real ψ:

$$I := \left(\frac{d\psi}{dz}\right)^2 - \psi^2 + \psi^4 = \text{const.}$$

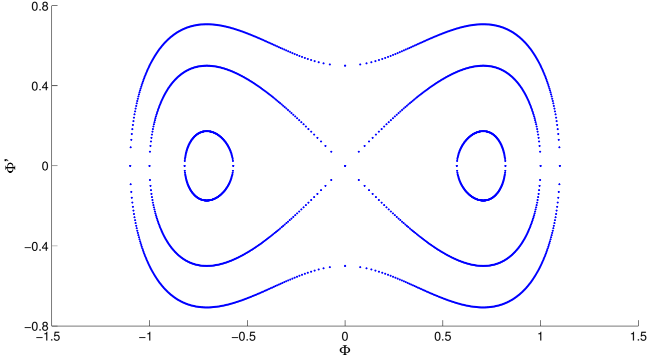


Figure : Level set on the phase plane (ψ, ψ').

Elliptic functions

Periodic solutions inside the positive homoclinic loop:

$$\psi(z) = \frac{1}{\sqrt{2-k^2}} \mathrm{dn}\left(\frac{z}{\sqrt{2-k^2}};k\right), \quad k \in (0,1).$$

Periodic solutions outside the homoclinic orbits

$$\psi(z) = \frac{k}{\sqrt{2k^2-1}} \mathrm{cn}\left(\frac{z}{\sqrt{2k^2-1}};k\right), \quad k \in (0,1),$$

where

$$\mathrm{dn}^2(\xi;k) + k^2 \mathrm{sn}^2(\xi;k) = 1, \quad \mathrm{cn}^2(\xi;k) + \mathrm{sn}^2(\xi;k) = 1,$$

As $k \to 1$, both converge to $\mathrm{sech}(z)$ in the formal expansion:

$$\mathrm{cn}(\xi;k) = \mathrm{sech}(\xi) - \tfrac{1}{4}(1-k^2)\left[\sinh(\xi)\cosh(\xi) - \xi\right]\tanh(\xi)\mathrm{sech}(\xi) + \mathcal{O}((1-k^2)^2),$$
$$\mathrm{dn}(\xi;k) = \mathrm{sech}(\xi) + \tfrac{1}{4}(1-k^2)\left[\sinh(\xi)\cosh(\xi) + \xi\right]\tanh(\xi)\mathrm{sech}(\xi) + \mathcal{O}((1-k^2)^2).$$

Approximation result

Theorem

There exist $\Lambda_* > 0$ *sufficiently large and* $C > 0$ *s.t. the stationary NLS equation with* $\Lambda \in (\Lambda_*, \infty)$ *admits a unique positive symmetric standing wave* ψ *with*

$$\psi_0(z) = \frac{k}{\sqrt{2k^2-1}} \mathrm{cn}\left(\frac{z}{\sqrt{2k^2-1}}; k\right), \quad z \in J_0,$$

such that

$$\|\psi_0 - \mathrm{sech}(\cdot)\|_{L^\infty(J_0)} \leq C e^{-L|\Lambda|^{1/2}},$$

and

$$\|\psi_+\|_{H^2(J_+)} + \|\psi_-\|_{H^2(J_-)} \leq C e^{-L|\Lambda|^{1/2}},$$

where the unique value for k satisfies the asymptotic expansion as $|\Lambda| \to \infty$*,*

$$\sqrt{1-k^2} = \frac{4}{\sqrt{3}} e^{-L|\Lambda|^{1/2}} \left[1 + \mathcal{O}(|\Lambda|^{1/2} e^{-2L|\Lambda|^{1/2}}, e^{-4\pi|\Lambda|^{1/2}})\right]$$

Proof of the approximation result

We have the system

$$-\partial_z^2\psi_0 + \psi_0 - \psi_0^3 = 0, \quad z \in J_0,$$
$$-\partial_z^2\psi_\pm + \psi_\pm - \psi_\pm^3 = 0, \quad z \in J_\pm.$$

The ODE for ψ_0 is solved with

$$\psi_0(z) = \frac{k}{\sqrt{2k^2-1}}\mathrm{cn}\left(\frac{z}{\sqrt{2k^2-1}}; k\right), \quad z \in J_0.$$

The symmetry suggests

$$\psi_-(-z) = \psi_+(z), \quad z \in J_+.$$

The ODE system is closed at the second-order equation

$$-\partial_z^2\psi_+ + \psi_+ - \psi_+^3 = 0, \quad z \in J_+,$$

subject to three boundary conditions

$$\psi_+(L\mu) = \psi_+(L\mu + 2\pi\mu) = p(k,\mu) := \frac{k}{\sqrt{2k^2-1}}\mathrm{cn}\left(\frac{L\mu}{\sqrt{2k^2-1}}; k\right)$$

and

$$\psi_+'(L\mu) - \psi_+'(L\mu + 2\pi\mu) = q(k,\mu) := \frac{k}{2k^2-1}\mathrm{sn}\left(\frac{L\mu}{\sqrt{2k^2-1}}; k\right)\mathrm{dn}\left(\frac{L\mu}{\sqrt{2k^2-1}}; k\right),$$

where $\mu = |\Lambda|^{1/2}$.

Proof of the approximation result

The range of k-values is defined by the condition $p(k, \mu) > 0$,
hence k belongs to the interval $(k_*(\mu), 1)$, where

$$\frac{L\mu}{\sqrt{2k_*^2 - 1}} = K(k_*) \quad \Rightarrow \quad \sqrt{1 - k_*^2} = 4e^{-L\mu} + \mathcal{O}(e^{-3L\mu}).$$

For every $k \in (k_*(\mu), 1)$, we solve the two-point boundary-value problem

$$-\partial_z^2 \psi_+ + \psi_+ - \psi_+^3 = 0, \quad z \in J_+,$$

subject to

$$\psi_+(L\mu) = \psi_+(L\mu + 2\pi\mu) = p(k, \mu).$$

Note that $p(k, \mu) = \mathcal{O}(e^{-L\mu})$ is small.

There exists a unique solution of the boundary value problem such that $\|\psi_+\|_{H^2(J_+)} = \mathcal{O}(p(k, \mu)) = \mathcal{O}(e^{-L\mu})$. More explicitly,

$$\psi_+(z) = p(k, \mu) \frac{\cosh(z - L\mu - \pi\mu)}{\cosh(\pi\mu)} + \mathcal{O}(e^{-3L\mu}).$$

Proof of the approximation result

Finally, we find a root of $k \in (k_*(\mu), 1)$ from the last boundary condition

$$q(k,\mu) = \psi'_+(L\mu) - \psi'_+(L\mu + 2\pi\mu) = 2p(k,\mu) + \mathcal{O}(e^{-3L\mu}).$$

There exists a unique value of k in this interval.

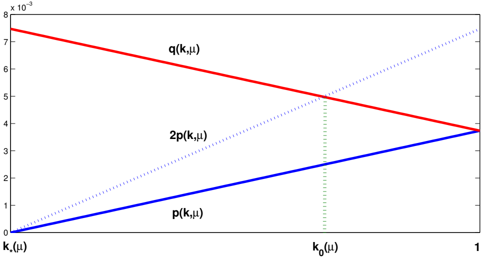


Figure : The graphs of p (solid blue), q (solid red), and $2p$ (dashed blue) versus k in $(k_*(\mu), 1)$ for $\mu = 2$ and $L = \pi$. The dotted line shows a graphical solution of the equation $q = 2p$.

Further applications of the method

- Similarly, one can construct the asymmetric wave with

$$\psi_+(z) = \frac{1}{\sqrt{2-k^2}} \mathrm{dn}\left(\frac{z - L|\Lambda|^{1/2} - \pi|\Lambda|^{1/2}}{\sqrt{2-k^2}}; k \right)$$

  and exponentially small ψ_0 and ψ_-.

- For fixed Λ, the symmetric wave has a smaller mass Q but the difference is exponentially small.

- For fixed mass Q, the symmetric wave has a smaller energy E.

- The symmetric wave is a critical point of energy with only one negative eigenvalue of the Hessian operator.

- The asymmetric wave is a critical point of energy with only one negative eigenvalue of the Hessian operator.

The second eigenvalue of the Hessian operator

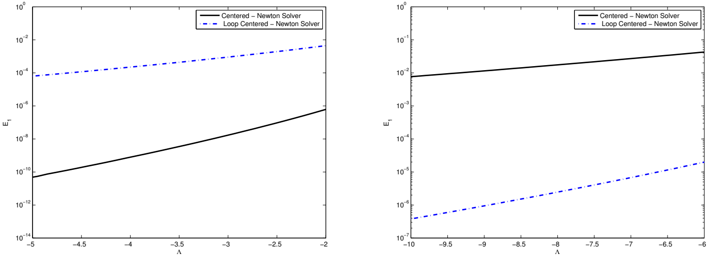
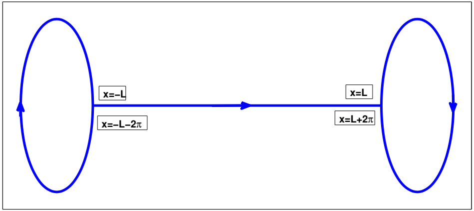


Figure : Left: the second eigenvalue of the Hessian operator for the symmetric and asymmetric waves plotted with respect to Λ for $L = 2\pi$. Right: the same for $L = \pi/2$.

Summary on the dumbbell graph



The ground state on the dumbbell graph:

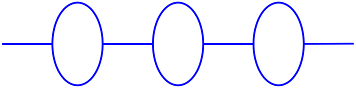
- small mass Q - constant wave;
- intermediate mass Q - asymmetric wave localized in a ring;
- large mass Q - symmetric wave localized in a center.

Both the asymmetric and symmetric waves are orbitally stable for large μ.

Other graphs

The NLS equation on the periodic quantum graph Γ:

$$i\partial_t U + \partial_x^2 U + |U|^2 U = 0, \quad x \in \Gamma.$$



- Two standing localized waves exist at the points of symmetry of Γ.
- Open question: which of the two standing waves is a ground state for $\Lambda < 0$?
- The two standing waves follow from justification of the effective NLS equation

$$i\partial_\tau A + \beta \partial_\xi^2 A + \gamma |A|^2 A = 0, \quad x \in \mathbb{R}.$$

S. Gills, D.P., and G. Schneider, Nonlin. Diff. Equations Applic. (2017)
D.P. and G. Schneider, Annales Henri Poincaré (2017)