Evans function for Lax operators with algebraic potentials

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Reference:
J. Nonlinear Science, in print (2005)

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Algebraic solitons in integrable evolution equations

- Modified Korteweg–de Vries (mKdV) equation

$$u_t + 6u^2 u_x + u_{xxx} = 0, \quad u(x,t) = \frac{4(x-6t)^2 - 3}{4(x-6t)^2 + 1}$$

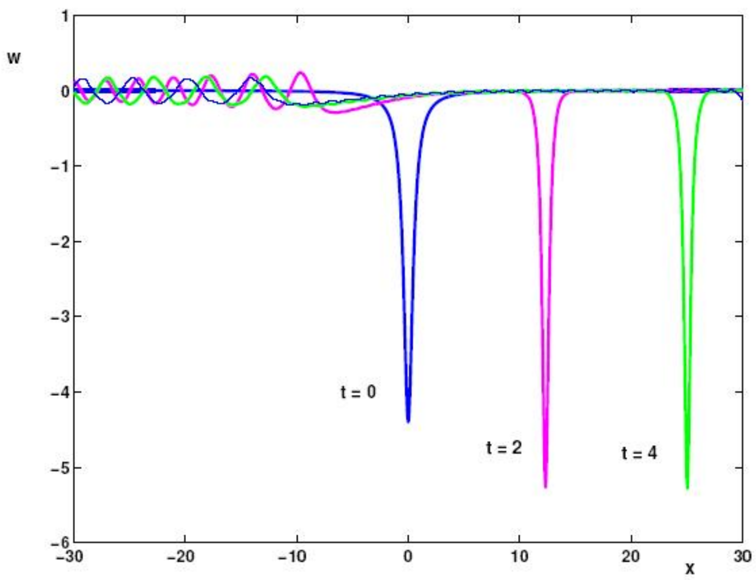
- Focusing nonlinear Schrödinger (NLS) equation

$$iu_t = u_{xx} + 2|u|^2 u, \quad u(x,t) = \frac{4x^2 + 16t^2 + 16it - 3}{4x^2 + 16t^2 + 1} e^{-2it}$$

- Massive Thirring model (MTM) equation

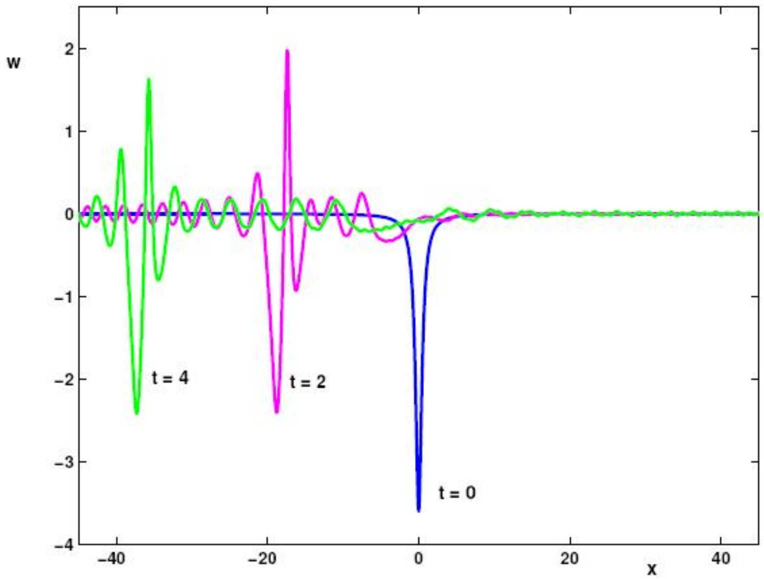
$$\begin{aligned} iv_t + w - 2|w|^2 v &= 0, \\ -iw_x + v - 2|v|^2 w &= 0 \end{aligned} \qquad v(x,t) = \frac{2\delta}{4\delta^2(x+\tau t) - i} e^{2i\delta^2(x-\tau t)}$$

Stability of algebraic solitons in nonlinear time-evolution



Modified KdV equation

- Travelling solitary wave



- Travelling breather

Methods of solution

- Linearized stability and a complete set of squared eigenfunctions

- Energy threshold and one-sided instability

$$P = \int_{\mathbb{R}} (u-1)^2 dx \geq P_0 = 2\pi$$

- Bifurcations in spectra of Lax operators

$$\boldsymbol{\psi}_x = \boldsymbol{\mathcal{L}}(\lambda; u)\boldsymbol{\psi}, \quad \boldsymbol{\psi}_t = \boldsymbol{\mathcal{A}}(\lambda; u)\boldsymbol{\psi},$$

where

$$\boldsymbol{\mathcal{L}}(\lambda; u) = \begin{bmatrix} 0 & -u \\ u & 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

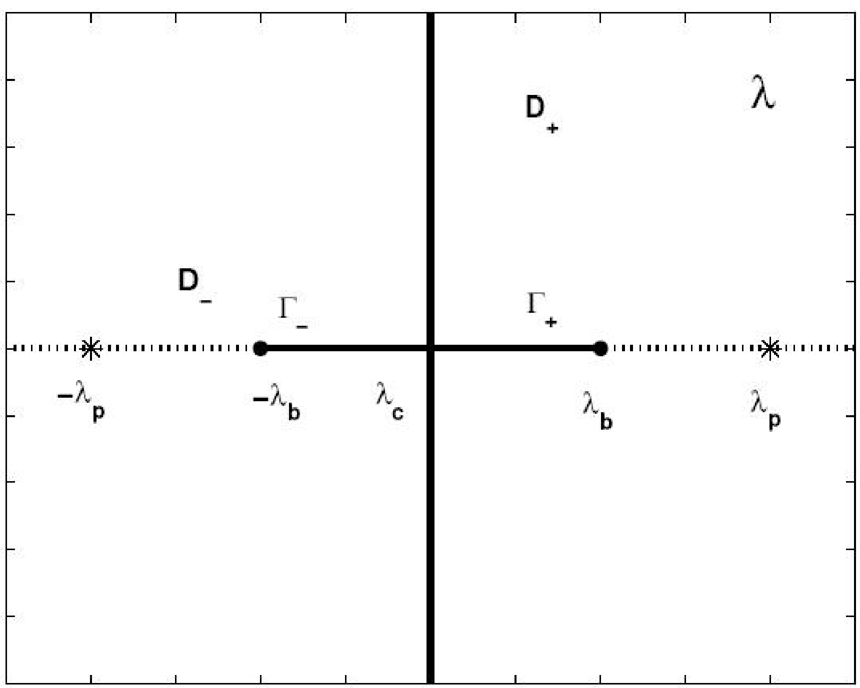
$$u(x,0) = 1 + w(x), \qquad \lim_{x \to \pm\infty} |x|^p w(x) = b_\infty, \quad p > 1$$

Spectral problem

$$\psi_1' = -(1 + w(x))\psi_2 + \lambda\psi_1, \quad \psi_2' = (1 + w(x))\psi_1 - \lambda\psi_2.$$

Fundamental solutions for $w(x) = 0$:

$$\boldsymbol{\psi}(x) = \mathbf{e}_{\pm}(\lambda)\, e^{\pm\kappa(\lambda)x}, \quad \kappa(\lambda) = \sqrt{\lambda^2 - 1}$$



Evans function

- When $w \in L^1(\mathbb{R})$ and $\lambda \in \mathbb{C} \setminus \{\pm 1\}$, there exist two sets of solutions:

$$\lim_{x \to -\infty} \boldsymbol{\phi}^{\pm}(x;\lambda) e^{\mp \kappa(\lambda) x} = \mathbf{e}_{\pm}(\lambda)$$

  and

$$\lim_{x \to +\infty} \boldsymbol{\psi}^{\pm}(x;\lambda) e^{\mp \kappa(\lambda) x} = \mathbf{e}_{\pm}(\lambda).$$

- Evans function for $\lambda \in \mathcal{D}_+$, where $\mathrm{Re}(\kappa(\lambda)) > 0$

$$E(\lambda) = \det(\boldsymbol{\phi}^{+}(x;\lambda), \boldsymbol{\psi}^{-}(x;\lambda))$$

- $E(\lambda)$ is analytic in $\lambda \in \mathcal{D}_+$
- $E(\lambda_p) = 0$ if λ_p is an isolated eigenvalue in $\mathcal{D}_+$
- $E(\lambda)$ is not analytic across $\lambda \in \Gamma_+$ if $w(x)$ decays algebraically

Explicit Evans function for algebraic potential

- Consider the AKNS problem with the algebraic potential

$$w_0(x) = -\frac{4}{1+4x^2}$$

- Fundamental solutions in $\lambda \in \mathcal{D}_+$:

$$\boldsymbol{\phi}^+(x;\lambda) = \frac{1}{\kappa(\lambda)} e^{\kappa(\lambda)x}\left[\left(\kappa(\lambda) + x w_0(x)\right)\mathbf{e}_+(\lambda) - \frac{1}{2} w_0(x)\boldsymbol{\xi}_+(\lambda)\right].$$

- Decaying eigenvector at $\lambda = 1$:

$$\boldsymbol{\phi}(x) = \begin{pmatrix} 2x-1 \\ 2x+1 \end{pmatrix} w_0(x).$$

- Evans function

$$E(\lambda) = 2\kappa(\lambda) = 2\sqrt{\lambda^2 - 1}$$

Technical problems and questions

- Consider a potential $w_\epsilon(x)$, such that $w_0(x)$ is the algebraic soliton.
- Although $E_0(\lambda)$ is bounded on $\lambda \in \Gamma_+$, $E_\epsilon(\lambda)$ may diverge for $\epsilon \neq 0$.
- Zero of $E(\lambda)$ at $\lambda = 1$ occurs on $\lambda \in \Gamma_+$, where $E(\lambda)$ is not analytic.
- How to define algebraic multiplicity of embedded eigenvalues?
- How to modify the Evans function for analysis of bifurcations?
- How to generalize analysis to other spectral systems (AKNS, ZS, KN)?

Previous results

- Geometric construction based on re-scaling of differential equations
  B. Sandstede and A. Scheel, Disc. Cont. Dyn. Sys. (2004)

- Spectral analysis of Dirac and Schrodinger problems at low energy
  R. Newton, J. Math. Phys. (1986)
  M. Klaus, J. Math. Phys. (1988)
  M. Klaus, Inverse Problems (1988)

- Heuristic asymptotic multi-scale methods of the AKNS problem
  D.Pelinovsky, R. Grimshaw, Physics Letters A (1997)

Reformulation of the problem

- On $\lambda \in \Gamma_+$, let $\lambda = \sqrt{1-k^2}$, $0 \le k < 1$
- Consider a two-sheet Riemann surface

$$\mathrm{Re}(\kappa(\lambda)) > 0 : \quad -\pi < \arg(\lambda - 1) < \pi,$$
$$\mathrm{Re}(\kappa(\lambda)) < 0 : \quad \pi < \arg(\lambda - 1) < 3\pi$$

- Let $w \in L^1(\mathbb{R})$. Fundamental solutions satisfy the integral equations:

$$\boldsymbol{\phi}^{\pm}(x;k) = \mathbf{e}_{\pm}(k) e^{\pm ikx} - \int_{-\infty}^{x} K(x,s;k)\boldsymbol{\phi}^{\pm}(s;k)ds,$$

where $K(x,s;k)$ is a bounded kernel on $0 < k < 1$ and $(x,s) \in \mathbb{R}^2$.

- Evans function on $0 < k < 1$ can be extended to the first sheet:

$$G(k) = E(\lambda) = \det\left(\boldsymbol{\phi}^{+}(x;k), \boldsymbol{\psi}^{-}(x;k)\right)$$

Main results

$$\lim_{x \to \pm\infty} |x|^p w(x) = b_\infty, \quad p \geq 2$$

- The point $\lambda = 1$ is not an eigenvalue if $p > 2$ or if $p = 2$ and $b_\infty > -\frac{3}{8}$
- If $p = 2$ and $b_\infty < -\frac{3}{8}$, the point $\lambda = 1$ can be an eigenvalue of geometric multiplicity *one* and *finite* algebraic multiplicity
- The function $G(k)$ is C^0 at $k = 0$ if $p > 2$ and C^1 at $k = 0$ if $p > 3$
- Let $p = 2$, $b_\infty < -\frac{3}{8}$, and q_∞ be a positive root of $q(q+1) = 2|b_\infty|$. The renormalized Evans function $\hat{G}(k)$ is continuous at $k = 0$:

  $$\hat{G}(k) = k^{2q} G(k) = \alpha_0 + \mathrm{o}(1).$$

  If $\lambda = 1$ is an eigenvalue, then $\alpha = 0$ and

  $$\hat{G}(k) = \alpha_2 k^2 + \mathrm{o}(k^2).$$

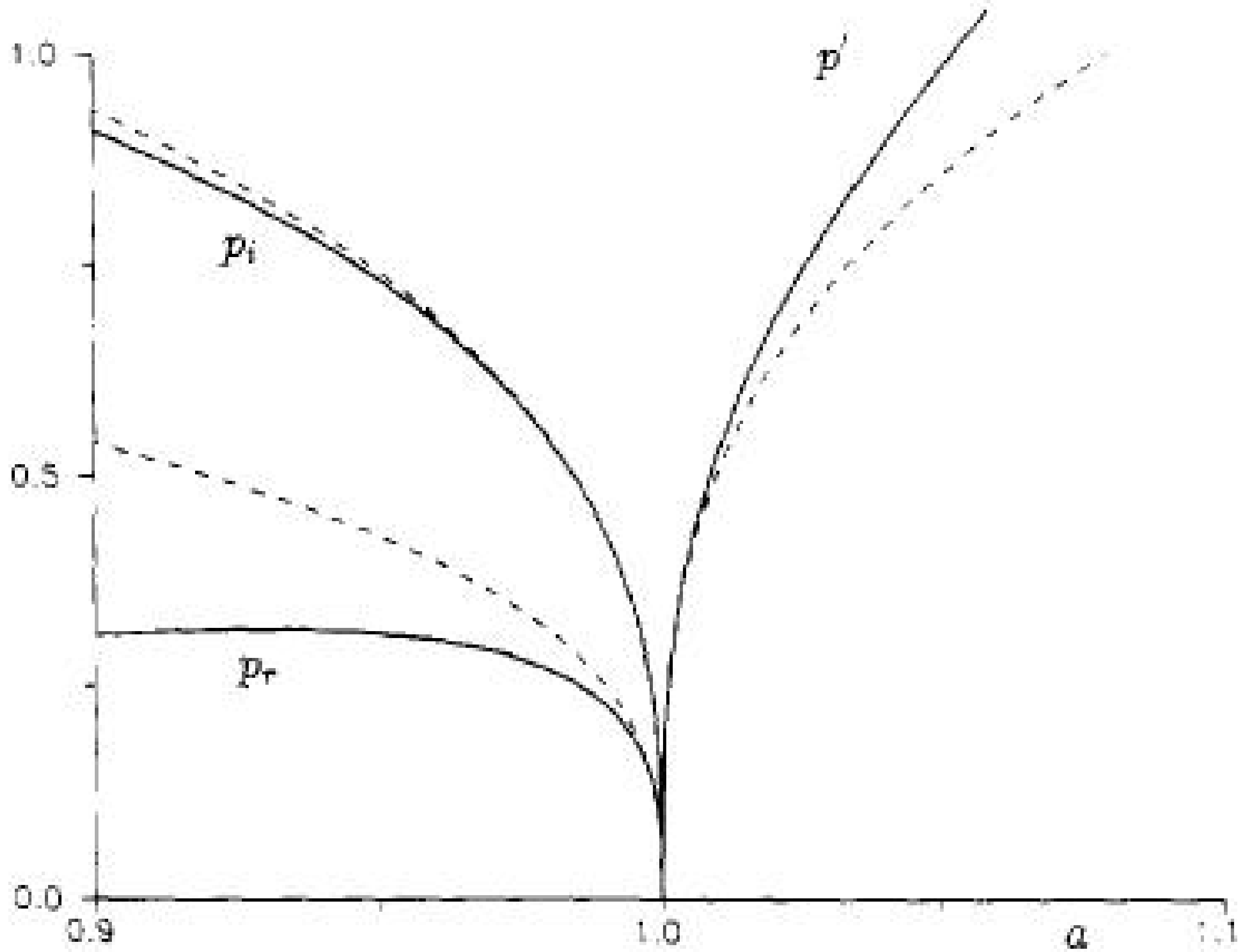
  If $\lambda = 1$ is both a resonance and eigenvalue, then $\alpha_2 = 0$.

Example of eigenvalue and resonance

- Consider the mKdV algebraic soliton:
$$w_0(x) = -\frac{4}{1+4x^2},$$
such that $p = 2$, $b_\infty = -1$, $q = 1$, $\alpha_0 = \alpha_2 = 0$, and
$$\hat{G}(k) = k^3 + \mathrm{o}(k^3)$$

- Let $w_\epsilon(x) = w_0(x) + \epsilon w_1(x)$ and $w_1(x)$ decays with $p > 2$. Then, $\hat{E}_\epsilon(\lambda)$ has a simple zero $\lambda \in \mathbb{R}$ near $\lambda = 1$ in $\mathcal{D}_+$ for small ϵ if
$$\epsilon \int_{-\infty}^{\infty} w_0(x) w_1(x) dx > 0$$
The function $\hat{E}_\epsilon(\lambda)$ has a pair of simple zeros $\lambda \in \mathbb{C}$ near $\lambda = 1$ in $\mathcal{D}_+$ for small ϵ if
$$\epsilon \int_{-\infty}^{\infty} w_0(x) w_1(x) dx < 0$$

Numerical illustration of the bifurcation



Ideas of analysis and proofs

- Reduce to a Schrödinger problem with a long-range potential:
$$U(x) = \frac{q(q+1)}{x^2} + W(x), \quad |x| \geq x_0 > 0$$
- Scattering of Jost functions associated with the long-range potential:
$$\psi(x) \to \sqrt{kx} H_{q+\frac{1}{2}}(kx), \quad 0 < k < 1$$
- The Jost functions are renormalized in the limit $k \to 0$:
$$\hat{\psi}(x) = \lim_{k \to 0^+} k^q \psi(x)$$
- Explicit calculations:
$$\alpha_0 = W[\hat{\psi}^+, \hat{\psi}^-], \quad \alpha_2 = -\int_{-\infty}^{\infty} \hat{\psi}^+(x)\hat{\psi}^-(x)dx.$$
- By the Implicit Function Theorem, we have near $\kappa = 0$ and $\epsilon = 0$:
$$\hat{G}_\epsilon(\kappa) = \kappa^3 + G_1\epsilon + \mathrm{o}(\epsilon).$$