Two-pulse solutions in the fifth-order KdV equation

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Background references:

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Yu. Kodama, D.P., J. Phys. A: Math. Gen. 38, 6129 (2005)

M. Chugunova, D.P., SIAM J. Math. Anal., submitted (2006)

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Background and motivations

Fifth-order KdV equation

$$u_t + u_{xxx} - u_{xxxxx} + 2uu_x = 0$$

has traveling wave solutions $u = \phi(z)$, $z = x - ct$, where $\phi(z)$ solves the fourth-order ODE

$$\phi^{(\text{iv})} - \phi'' + c\phi = \phi^2.$$

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Applications:

- capillary-gravity water waves (Craig–Groves, 1994)
- chains of coupled oscillators (Gorshkov–Ostrovsky, 1979)
- magneto–acoustic waves in plasma (Kawahara, 1972)

Solitary waves

Stability of the critical point $(0,0,0,0)$ in the fourth-order ODE:

$$\phi \sim e^{\kappa z}: \qquad \kappa^4 - \kappa^2 + c = 0.$$

Existence of localized solutions:

- $c < 0$ - no pulse solutions (Tovbis, 2000; Lombardi, 2000)
- $0 < c < \frac{1}{4}$ - unique one-pulse solution (Amick–Toland, 1992; Groves, 1998)
- $c > \frac{1}{4}$ - unique one-pulse and infinite countable set of two-pulse solutions (Buffoni–Sere, 1996)

$\Rightarrow$ The domain of our studies is $c > \frac{1}{4}$.

Mathematical problems

Numerical approximations of two-pulse solutions

- numerical shooting method and continuation techniques (Champneys, 1993)
- iterations in Fourier space (Petviashvili's method)

$\Rightarrow$ Iterations diverge for two-pulse solutions!

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Spectral stability of two-pulse solutions

- Lyapunov–Schmidt reductions (Sandstede, 1998)
- Count of eigenvalues in Pontryagin space (Krein's signatures)

$\Rightarrow$ The count of eigenvalues is inconclusive for two-pulse solutions!

Petviashvili's method

ODE for solitary waves

$$\phi^{(\mathrm{iv})} - \phi'' + c\phi = \phi^2, \qquad z \in \mathbb{R}$$

The ODE becomes the fixed-point problem in $H^2(\mathbb{R})$:

$$\hat{\phi}(k) = \frac{\widehat{\phi^2}(k)}{(c + k^2 + k^4)}, \qquad k \in \mathbb{R}$$

where $c > 0$ and $\hat{\phi}(k)$ is the Fourier transform of $\phi(z)$.

Iterations $\{\hat{u}_n(k)\}_{n=0}^{\infty}$ are defined recursively in $H^2_{\mathrm{ev}}(\mathbb{R})$:

$$\hat{u}_{n+1}(k) = M_n^2 \frac{\widehat{u_n^2}(k)}{(c + k^2 + k^4)}, \qquad M[\hat{u}_n] = \frac{\int_{\mathbb{R}} (c + k^2 + k^4) \left[\hat{u}_n(k)\right]^2 dk}{\int_{\mathbb{R}} \hat{u}_n(k) \widehat{u_n^2}(k) dk}$$

Convergence Theorem

- Let $\hat{\phi}(k)$ be a solution of the fixed-point problem in $H^2_{\rm ev}(\mathbb{R})$
- Let $\mathcal{H}$ be the Jacobian operator of the ODE at $\phi(z)$:
  $\mathcal{H} = c - \partial_z^2 + \partial_z^4 - 2\phi(z)$

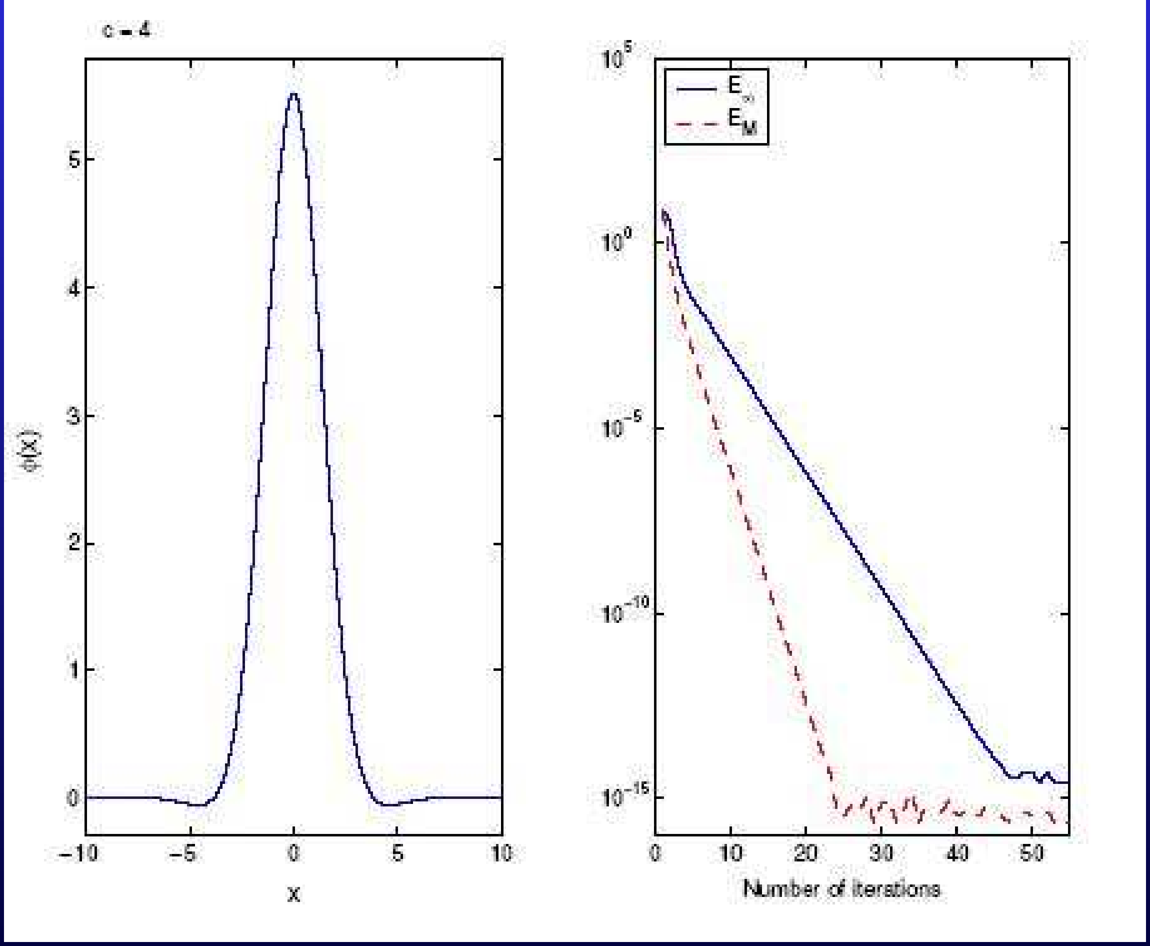
Theorem: *If $\mathcal{H}$ has exactly one negative eigenvalue and a simple zero eigenvalue and if*

$$\text{either} \quad \phi(z) \geq 0 \qquad \text{or} \quad \left| \inf_{z \in \mathbb{R}} \phi(z) \right| < \frac{c}{2},$$

then there exists an open neighborhood of $\hat{\phi}$ in $H^2_{\rm ev}(\mathbb{R})$, in which $\hat{\phi}$ is the unique fixed point and the sequence of iterations $\{\hat{u}_n(k)\}_{n=0}^{\infty}$ converges to $\hat{\phi}$.

One-pulse solutions

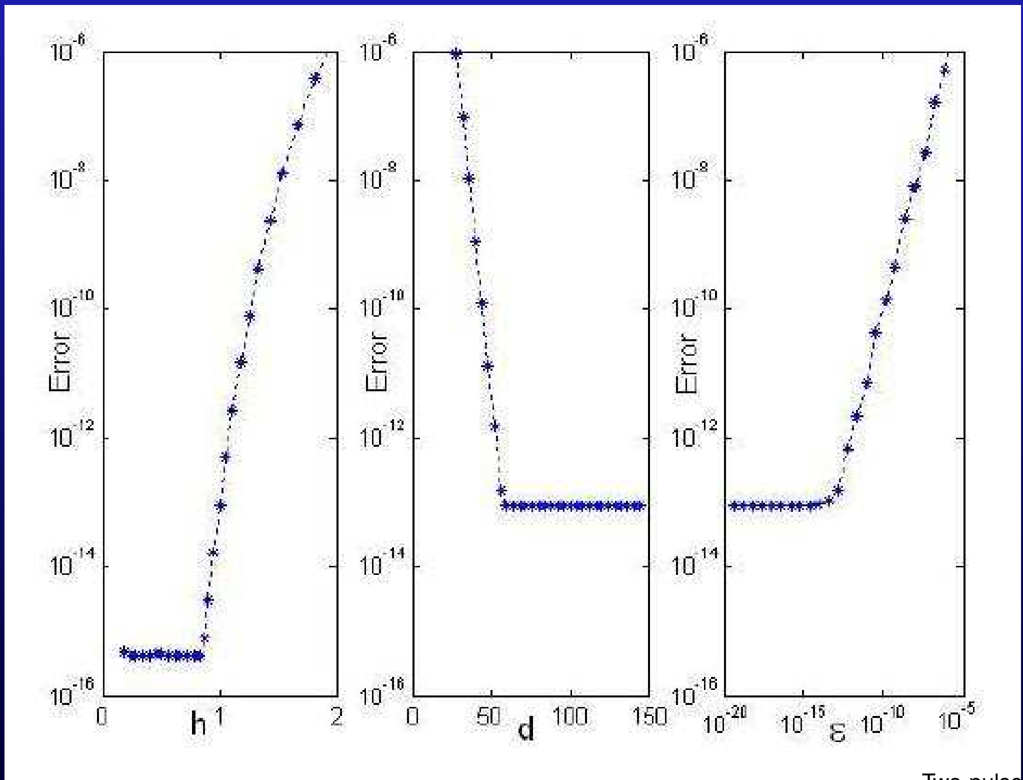
Let $\phi \equiv \Phi(z)$ be a one-pulse solution in $\mathcal{H} = c - \partial_z^2 + \partial_z^4 - 2\Phi(z)$. Then, $\mathcal{H}$ has exactly one negative eigenvalue and a simple kernel with $\Phi'(z)$ in $H^2(\mathbb{R})$.



Analysis of convergence

Numerical factors for numerical error:

- truncation of $z \in \mathbb{R}$ to the interval $z \in [-d, d]$
- truncation of Fourier series by the discrete sum with N terms
- small tolerance ε for $E_M = |M_n - 1|$ and $E_\infty = \|u_{n+1} - u_n\|_{L^\infty}$



Two-pulse solutions

Let $\phi \equiv \phi_n(z)$ be a two-pulse solution. Then,

$$\phi(z) = \Phi(z - s) + \Phi(z + s) + \varphi(z),$$

where $\|\varphi\|_{L^\infty} = \mathrm{O}(e^{-2\kappa s})$ and $|s - s_n| = \mathrm{O}(e^{-2\kappa s})$, where s_n is an extremum point of $W(2s)$ in

$$W = \int_{\mathbb{R}} \Phi^2(z)\Phi(z + 2s)dz.$$

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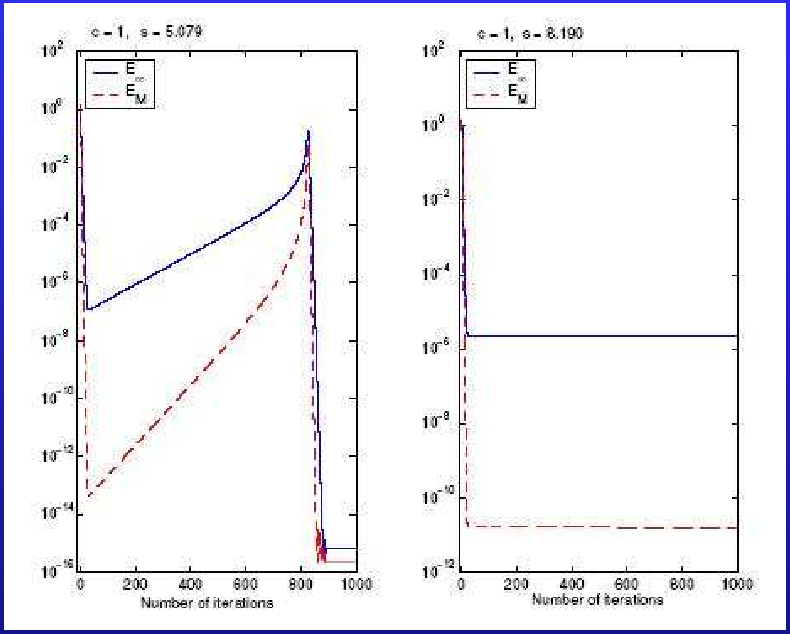
The operator $\mathcal{H}$ has two finite negative eigenvalues, a simple kernel with $\phi_n'(z)$, and a small eigenvalue μ in $H^2(\mathbb{R})$, such that

$$\left| \mu + \frac{2W''(2s_n)}{Q} \right| \leq C_n e^{-4\kappa s_n}, \qquad Q = \|\Phi'\|_{L^2}^2 > 0.$$

Two-pulse solutions

Iterations of the method



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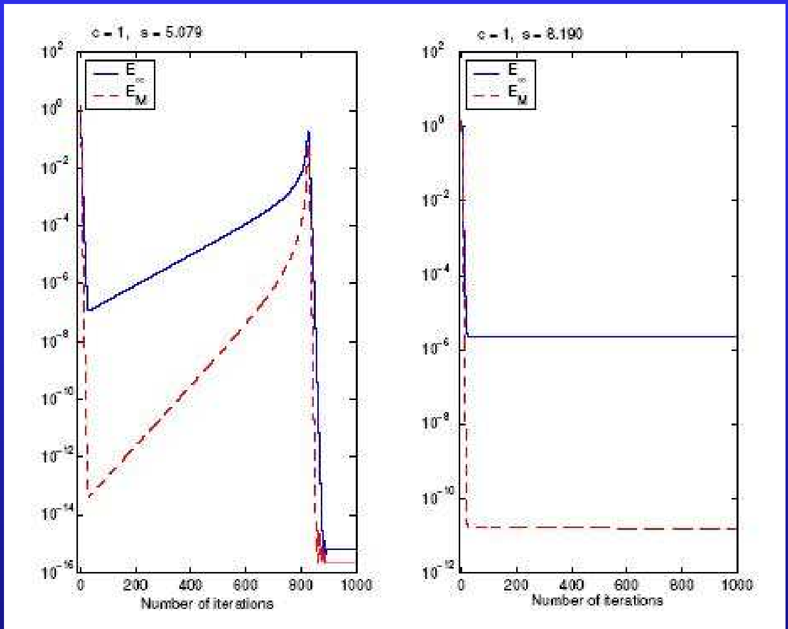
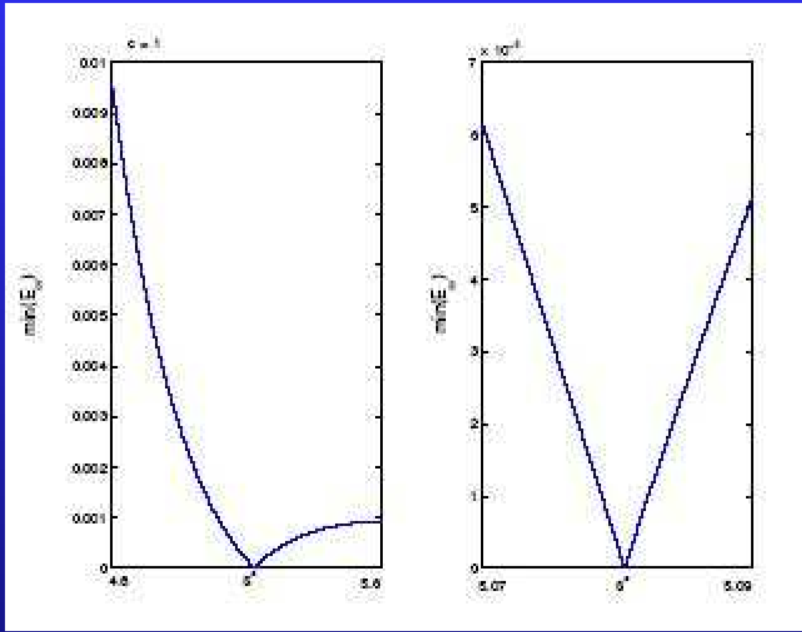
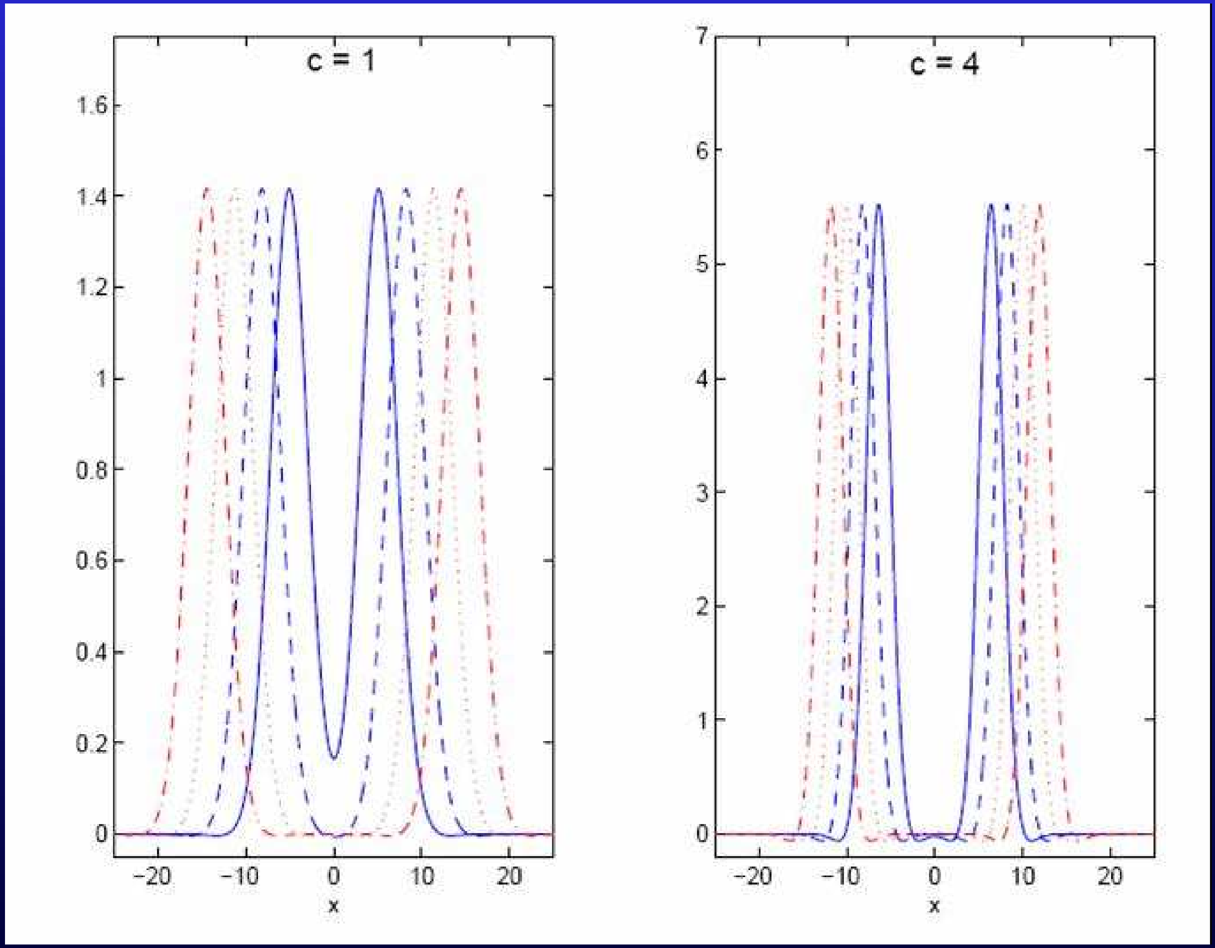


Minimum error for root search



Numerical algorithm

Theorem: *There exists $s = s_*$ near $s = s_n$ such that the iteration method with $u_0 = \Phi(z - s) + \Phi(z + s)$ converges to $\phi_n(z)$ in a local neighborhood of ϕ_n in $H^2_{\rm ev}(\mathbb{R})$.*



Spectral stability

Linearized problem for spectral stability

$$\partial_z \mathcal{H} v = \lambda v, \qquad v \in L^2(\mathbb{R})$$

Eigenvalues with $\mathrm{Re}(\lambda) > 0$ result in spectral instability.

Let $\phi \equiv \phi_n(z)$ be a two-pulse solution. There exists a pair of small eigenvalues λ of the linearized operator $\partial_z \mathcal{H}$, such that

$$\left| \lambda^2 + \frac{4W''(2s_n)}{P'(c)} \right| \leq C_n e^{-4\kappa s_n}, \qquad P'(c) = \frac{d}{dc} \|\Phi\|_{L^2}^2 > 0.$$

- $W''(2s_n) > 0$ - pair of purely imaginary eigenvalues
- $W''(2s_n) < 0$ - pair of real eigenvalues

Spectral stability theorem

Notations:

- N_{real} - the number of real positive eigenvalues
- N_{comp} - the number of complex eigenvalues in the first open quadrant
- N^{-}_{imag} - the number of simple positive imaginary eigenvalues with $(\mathcal{H}v, v) \leq 0$
- The kernel of $\mathcal{H}$ is simple and $P'(c) > 0$

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Theorem: Then,

$$N_{\text{real}} + 2N_{\text{comp}} + 2N^{-}_{\text{imag}} = n(\mathcal{H}) - 1,$$

where $n(\mathcal{H})$ is the number of negative eigenvalues of $\mathcal{H}$.

Applications of the Theorem

Counts of eigenvalues:

- One-pulse solutions

$$n(\mathcal{H}) = 1, \qquad N_{\rm real} = N_{\rm comp} = N^-_{\rm imag} = 0$$

The one-pulse solution is a ground state (Levandosky, 1999)

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- Two-pulse solutions with $W''(2s_n) < 0$

$$n(\mathcal{H}) = 2, \qquad N_{\text{real}} = 1, \qquad N_{\text{comp}} = N_{\text{imag}}^{-} = 0$$

The two-pulse solution with $W''(2s_n) < 0$ is spectrally unstable.

Applications of the Theorem

Counts of eigenvalues:

- Two-pulse solutions with $W''(2s_n) > 0$

$$n(\mathcal{H}) = 3, \qquad N_{\mathrm{real}} = 0, \qquad N_{\mathrm{comp}} + N_{\mathrm{imag}}^{-} = 1$$

The "standard" count is inconclusive for these solutions.

Applications of the Theorem

Counts of eigenvalues:

- Two-pulse solutions with $W''(2s_n) > 0$

$$n(\mathcal{H}) = 3, \qquad N_{\rm real} = 0, \qquad N_{\rm comp} + N^-_{\rm imag} = 1$$

The "standard" count is inconclusive for these solutions.

Theorem: Let λ be a simple purely imaginary eigenvalue of $\partial_z \mathcal{H}$ in $L^2(\mathbb{R})$. Then, it is structurally stable to parameter continuation, i.e. it remains purely imaginary eigenvalue upon an addition of a relatively compact perturbation to $\partial_z \mathcal{H}$.

$$N_{\rm comp} = 0, \qquad N^-_{\rm imag} = 1$$

The two-pulse solution with $W''(2s_n) > 0$ is spectrally stable.

Numerical spectrum

Exponentially weighted space

$$H^2_\alpha = \left\{ v \in H^2_{\mathrm{loc}}(\mathbb{R}) : \quad e^{\alpha z} v(z) \in H^2(\mathbb{R}) \right\}$$

Numerical spectrum

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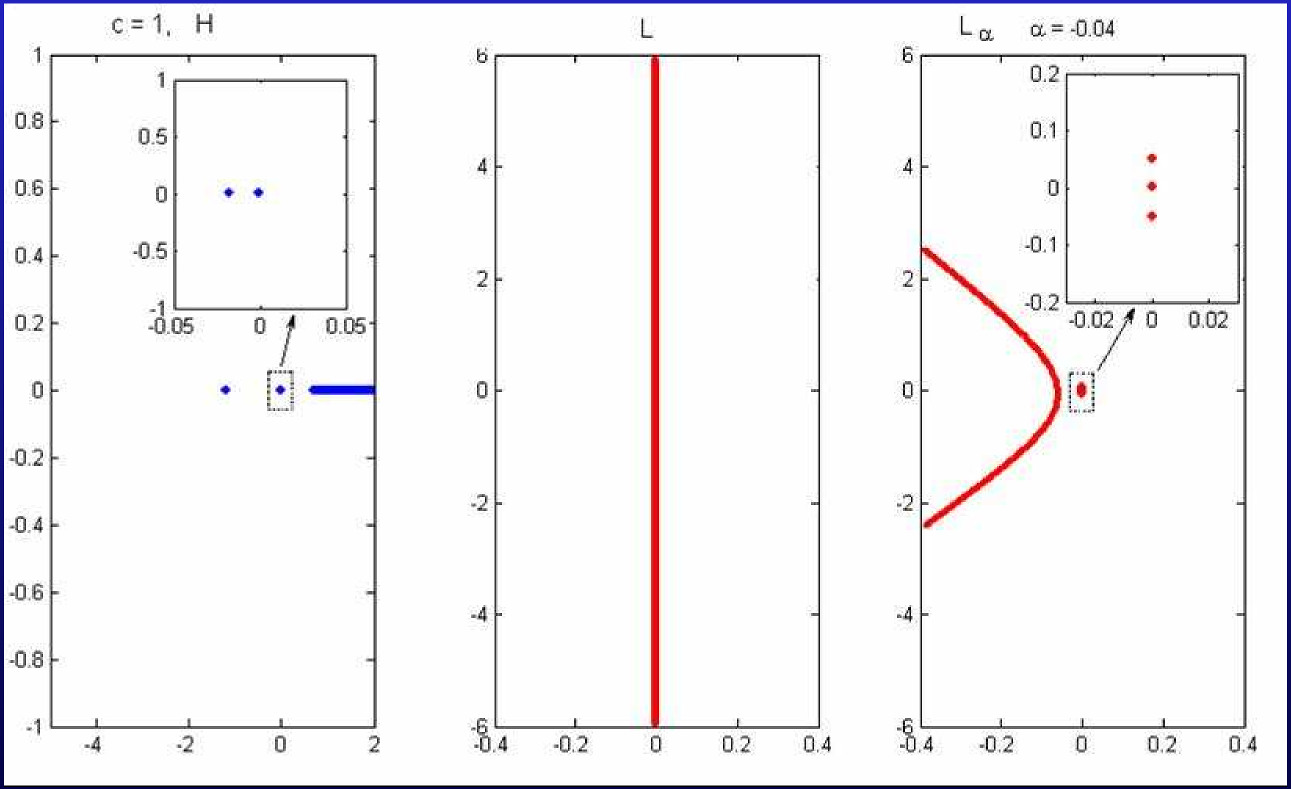

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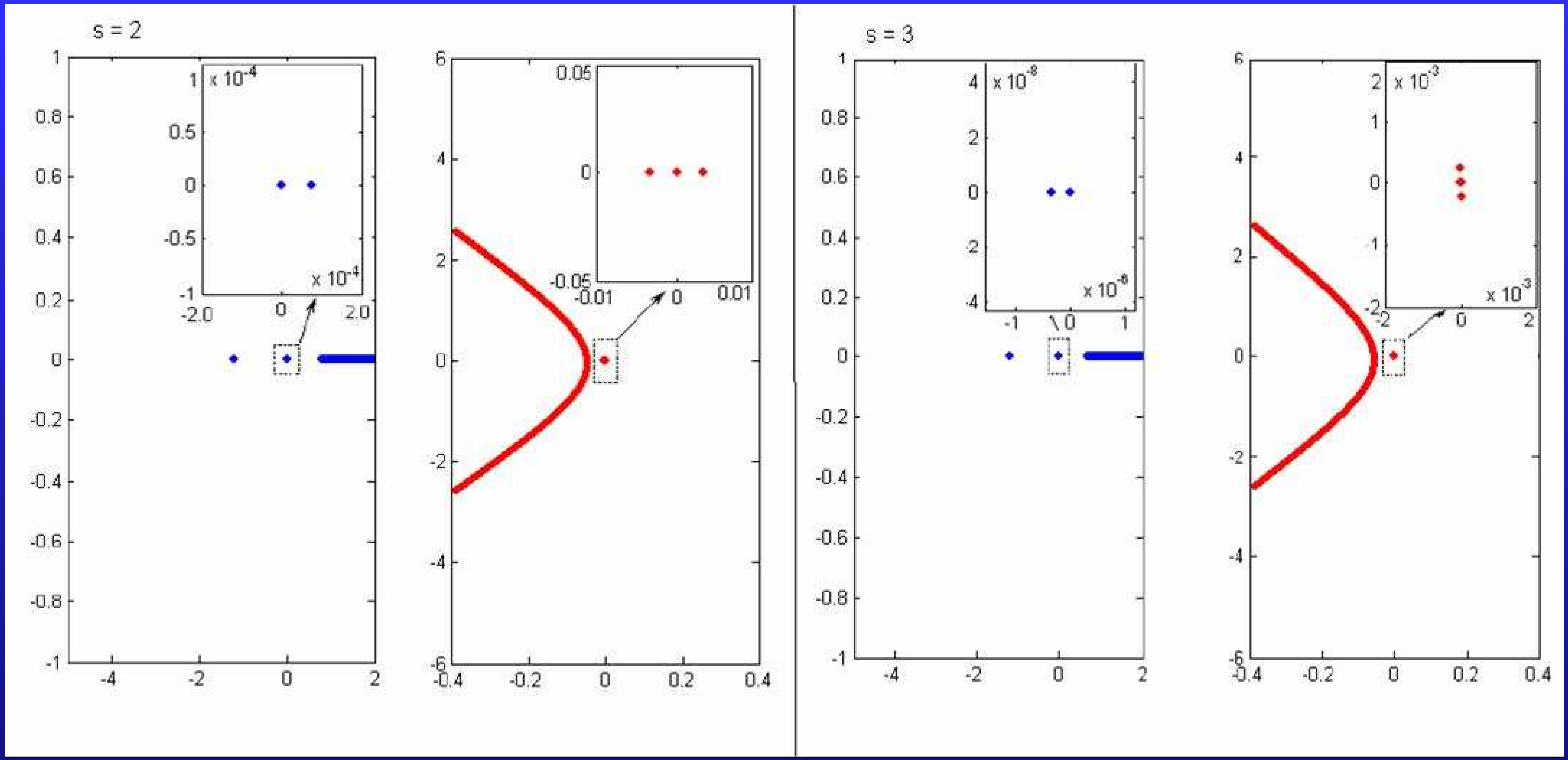

Conclusions

Outcomes of our work:

- Application of Pontryagin spaces to KdV equations
- Numerical approximations of two-pulse solutions
- Proof of structural stability of embedded eigenvalues

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Open problems:

- Error bounds on validity of the Newton's particle law:

$$P'(c)\ddot{L} = -W''(L),$$

where $L(t) = 2s$ is the distance between two pulses.

- Numerical approximations of three- and multi-pulse solutions
- Proof of asymptotic stability of multi-pulse solutions

Software for relevant computations