

Selection of the ground state in the Fisher-KPP equation on a compact metric graph

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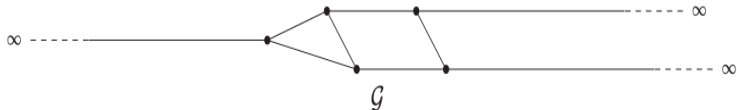
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1. Metric graphs

In many problems (BECs, photonics, optics, river networks, bio networks), wave dynamics can be considered trapped on a graph Γ that consists of a set of edges $E = \{e_j\}$ connected at the set of vertices $V = \{v_j\}$.

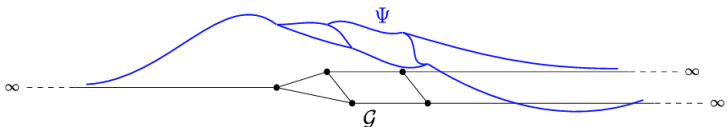


Compared to combinatorial graphs, edge lengths are important, whereas the angles between the edges at each vertex and the directions of orientation of each edge does not play any role. We can introduce a metric space on a metric graph, e.g. $L^p(\Gamma)$ defined pointwise on edges E with the norm

$$\|\Psi\|_{L^p(\Gamma)} = \left(\sum_{e_j \in E} \|\psi_j\|_{L^p(e_j)}^p \right)^{\frac{1}{p}},$$

for a piecewisely defined function $\Psi(x) : \Gamma \rightarrow \mathbb{C}$.

Laplace operator on a metric graph



One can define a Laplace operator $\Delta_\Gamma : \text{dom}(\Delta_\Gamma) \subset L^2(\Gamma) \rightarrow L^2(\Gamma)$ by the pointwise definition on edges $E = \{e_j\}$:

$$(\Delta_\Gamma \Psi)(x) = \frac{d^2 \psi_j}{dx^2}, \quad x \in e_j.$$

However, we need boundary conditions on vertices $V = \{v_j\}$ which connect components of Ψ on every edge incident to a particular vertex v . The *natural* conditions are called **Neumann–Kirchhoff boundary conditions**:

$$\begin{cases} \psi_e(v) = \psi_{e'}(v) \\ \sum_{e \prec v} \partial \psi_e(v) = 0, \end{cases} \quad \text{for every } e, e' \prec v,$$

where $\partial \psi_e(v)$ denotes outward derivative of ψ_e at v (which is $\pm \frac{d\psi_e}{dx}(v)$).

PDEs on a metric graph

The Laplace operator is self-adjoint on $L^2(\Gamma)$ subject to the Neumann–Kirchhoff (NK) boundary conditions:

$$\langle -\Delta_\Gamma \Psi, \Phi \rangle_{L^2(\Gamma)} = \langle \nabla_\Gamma \Psi, \nabla_\Gamma \Phi \rangle_{L^2(\Gamma)} = \langle \Psi, -\Delta_\Gamma \Phi \rangle_{L^2(\Gamma)}$$

for every $\Psi, \Phi \in \text{dom}(\Delta_\Gamma) = \{H^2(\Gamma) : \text{NK conditions}\}$. Similarly, one can define the form domain of Δ_Γ by $H_C^1(\Gamma) = \{H^1(\Gamma) : \text{continuity conditions}\}$.

Many nonlinear PDEs can be expressed by using the Laplace operator Δ_Γ :

$$\begin{aligned} \text{(NLS)} \quad & i\partial_t \Psi = -\Delta_\Gamma \Psi + |\Psi|^p \Psi, \\ \text{(Fisher–KPP)} \quad & \partial_t \Psi = \Delta_\Gamma \Psi + \Psi(1 - \Psi), \\ \text{(sine-Gordon)} \quad & \partial_t^2 \Psi = \Delta_\Gamma \Psi + \sin(\Psi), \\ \text{(Boussinesq)} \quad & i\partial_t \Psi = (1 - \Delta_\Gamma)^{-1} (\Delta_\Gamma \Psi + \Psi^2). \end{aligned}$$

However, some nonlinear PDEs are expressed by the first-order operators (Dirac) and third-order operators (KdV).

Example: a star graph

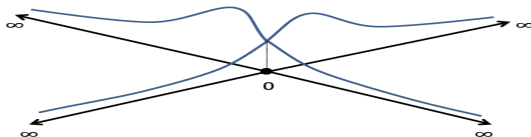
A **star graph** is the union of N half-lines connected at a single vertex with

$$L^2(\Gamma) = \underbrace{L^2(\mathbb{R}^+) \oplus \cdots \oplus L^2(\mathbb{R}^+)}_{N \text{ elements}},$$

with

$$H_C^1(\Gamma) = \{\Psi \in H^1(\Gamma) : \psi_1(0) = \psi_2(0) = \cdots = \psi_N(0)\},$$

$$\text{dom}(\Delta_\Gamma) = \{\Psi \in H^2(\Gamma) \cap H_C^1(\Gamma) : \sum_{j=1}^N \psi_j'(0) = 0\}.$$

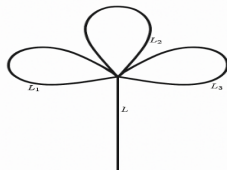
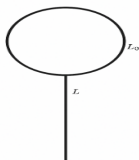


A. Kairzhan & D.P, JDE 264 (2018) 7357–7383;

A. Kairzhan & D.P, JPA: Math. Theor. 51 (2018) 095203;

A. Kairzhan, D.P. & R. Goodman, SIAM J. Dyn. Syst. 18 (2019) 1723–1755

Other graphs with loops (tadpole and flower)



- ▶ Variational methods were developed by R. Adami, S. Dovetta, E. Serre, P. Tilli, A. Pankov, . . .
- ▶ Dynamical stability (PDE) methods were developed by J. A. Pava, N. Goloshchapova, M. Cavalcante, R. Plaza, A. Ardila, . . .
- ▶ Dispersive estimates were developed by V. Banica, L. Ignat, A. Grecu, F. Mehmeti, K. Ammari, S. Nicaise, . . .
- ▶ Period function and ODE methods were developed in
A. Kairzhan, R. Marangell, D.P. & K. Xiao, JDE 271 (2021) 719-763
D. Noja & D.P., Calc. Var. 59 (2020) 173 (31 pages)
A. Kairzhan, D. Noja, & D.P., Standing waves on quantum graphs,
J. Phys. A: Math. Theor. 55 (2022) 243001

2. The Fisher–KPP model on metric graphs

The Fisher–KPP equation, the Keller–Segel systems, and other reaction–diffusion PDEs on a metric graph has been considered for modeling of populations, river networks, chemotaxis, reaction-diffusion systems:

$$(\text{Fisher-KPP}) : \quad \partial_t u = \Delta_\Gamma u + u(1 - u).$$

and

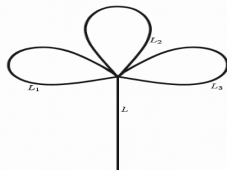
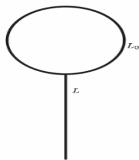
$$(\text{Keller–Segel}) : \quad \begin{cases} \partial_t u = \Delta_\Gamma u - \nabla_\Gamma \chi(u, v) \nabla_\Gamma v + f(u, v), \\ \tau \partial_t v = \Delta_\Gamma v + g(u, v). \end{cases}$$

The recent mathematical studies include the following:

- ▶ Existence and stability of steady states on compact metric graphs - E. Yanagida (2001), Y. Ishii & K. Kurata (2021), Y. Ishii (2022)
- ▶ Bifurcations and asymptotic stability of constant states - H. Shemtaga, W. Shen, & S. Sukhtaiev (2024, 2025)
- ▶ Spreading speeds of a propagation front (slowing down) - W-T. L. Fan, W. Hu, & G. Terlov (2021)
- ▶ Trapping of the spreading front at vertices - S. Jimbo & Y. Morita (2019, 2021, 2024)

Main problem related to the Fisher–KPP equation

Let Γ be a compact metric graph with finitely many edges of finite lengths. We assume that Γ is connected and there is at least one pendant (an edge with a boundary vertex not connected to any other edge).



We use the Neumann–Kirchhoff conditions on interior vertices and the Dirichlet condition on boundary vertices.

$$\partial_t u = \Delta_\Gamma u + u(1 - u) \quad \text{on } \Gamma.$$

$u = 0$ is a constant (trivial) state but $u = 1$ is no longer a valid state.

Question: existence, uniqueness, and asymptotic stability of the nonconstant (positive) states in dependence of the graph geometry (e.g. edge lengths).

The gradient system with the free energy

The Fisher–KPP equation is a gradient system with the energy

$$H(u) = \frac{1}{2} \int_{\Gamma} [(\nabla_{\Gamma} u)^2 - u^2] dx + \frac{1}{3} \int_{\Gamma} u^3 dx,$$

defined on

$$\mathcal{H}_0 = \{u \in H_0^1(\Gamma) : u \geq 0\}.$$

If $u_0 \in \mathcal{H}_0$, then there exists a unique global solution $u \in C^0([0, \infty), \mathcal{H}_0)$ with initial data $u|_{t=0} = u_0$, which converges to a global attractor $u_* \in \mathcal{H}_0$ as

$$u_* = \lim_{t \rightarrow +\infty} u(t, \cdot).$$

The attractor $u_* \in \mathcal{H}_0$ is a critical point of $H(u)$, that is, a solution of the steady state problem

$$-\Delta_{\Gamma} u = u(1 - u), \quad u \in D(\Delta_{\Gamma}).$$

Main question is now to characterize the existence, uniqueness, and asymptotic stability of $u_* \neq 0$ in dependence of the graph geometry.

3. Main results

Let $\{L_j\}$ be the set of lengths of edges of the graph Γ .

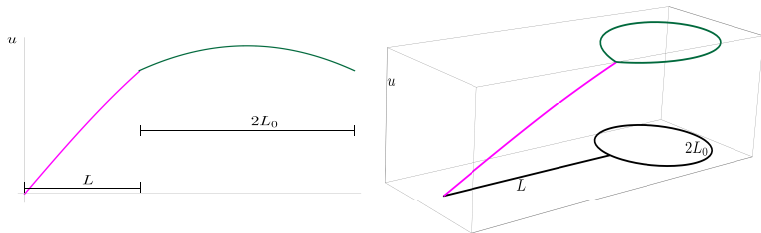
1. There is a unique global attractor for every $u_0 \in \mathcal{H}_0$. The attractor is either trivial or nontrivial depending whether the smallest eigenvalue $\lambda_0(\Gamma)$ of $-\Delta_\Gamma$ in $L^2(\Gamma)$ is greater or smaller than one.
2. There exists $L_0 > 0$ such that if $\max_j L_j < L_0$, then the trivial (zero) state is a global attractor and a minimizer of the energy $H(u)$ in $\mathcal{H}_0$ at the zero level.
3. For every j -th edge of length L_j , there is $L_j^{(0)} \in [0, \infty)$ such that for every $L_j > L_j^{(0)}$, a strictly positive state is a global attractor and a minimizer of the energy $H(u)$ in $\mathcal{H}_0$ at a negative level.
4. There exist $L_* > L_0$ such that if $L_{\min} := \min_j L_j > L_*$, then the unique, strictly positive state u_* satisfies

$$\|u_* - 1\|_{L^\infty(\Gamma_0)} \leq Ce^{-L_{\min}},$$

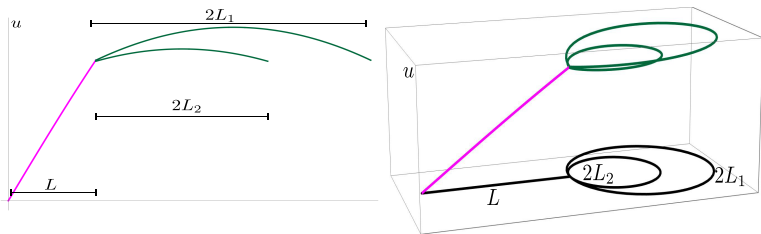
for some $C > 0$, where Γ_0 is the part of Γ without the pendants.

Positive states on the flower graphs

For the tadpole graph with a single loop, the unique positive state is



For the flower graph with two loops of different lengths, it is



4. Classical methods for the Fisher–KPP equation

The spectrum of $-\Delta_\Gamma$ in $L^2(\Gamma)$ consists of isolated positive eigenvalues, since Γ is compact. Furthermore, the Laplace equation associated with Δ_Γ enjoys the same maximum principle on Γ as in open regions of $\mathbb{R}^d$.

As a result, we can use the classical methods of analysis:

- ▶ Linearization and the lowest eigenvalue of $-\Delta_\Gamma$.
- ▶ Variational methods for the energy $H(u)$.
- ▶ Comparison principle with subsolutions and supersolutions.

This gives the main results (1), (2), and (3). For the exponential localization of the main result (4), we develop a novel method based on the period function. The period function method also gives the precise construction of the ground state, but for the simplest class of graphs (e.g., the flower graphs).

Classical methods: the lowest eigenvalue of $-\Delta_\Gamma$

Lemma

Let the set of lengths $\{L_j\}$ be fixed except for one edge parameterized as $[0, L]$. Let $\lambda(\Gamma) \geq 0$ be a simple eigenvalue of $-\Delta_\Gamma$ in $L^2(\Gamma)$ with the corresponding eigenfunction $\Psi : \Gamma \rightarrow \mathbb{R}$ in $D(\Delta_\Gamma)$. Then, $\lambda(\Gamma)$ is a C^1 function of L satisfying

$$\frac{d\lambda(\Gamma)}{dL} = -|\psi'(L)|^2 - \lambda(\Gamma)|\psi(L)|^2 \leq 0,$$

where $\psi(x) : [0, L] \rightarrow \mathbb{R}$ is the component of Ψ on the edge $[0, L]$.

Corollary

Let $\lambda_0(\Gamma)$ be the lowest eigenvalue of $-\Delta_\Gamma$. Then, $\Psi(x) > 0$ for all $x \in \Gamma$ except for the boundary vertices (under Dirichlet condition) and $\frac{d\lambda_0(\Gamma)}{dL} < 0$.

Corollary

The trivial state 0 in $\partial_t u = \Delta_\Gamma u + u(1 - u)$ is stable if $\lambda_0(\Gamma) \geq 1$ and unstable if $\lambda_0(\Gamma) < 1$.

Classical methods: the energy minimization

Lemma

If $\lambda_0(\Gamma) \geq 1$, then the infimum of $H(u)$ in $\mathcal{H}_0$ is attained at $u = 0$ for which $H(0) = 0$. If $\lambda_0(\Gamma) \in (0, 1)$, then the infimum of $H(u)$ in $\mathcal{H}_0$ is attained at u_ with $H(u_*) < 0$, where $u_* > 0$ for all points of Γ except for boundary vertices.*

This follows from the Rayleigh quotient which yields

$$\int_{\Gamma} (\nabla_{\Gamma} u)^2 dx \geq \lambda_0(\Gamma) \|u\|_{L^2(\Gamma)}^2, \quad \forall u \in H_0^1(\Gamma).$$

► If $\lambda_0(\Gamma) \geq 1$, then

$$H(u) \geq \frac{1}{2} \int_{\Gamma} [(\nabla_{\Gamma} u)^2 - u^2] dx \geq (\lambda_0(\Gamma) - 1) \|u\|_{L^2(\Gamma)}^2,$$

► If $\lambda_0(\Gamma) < 1$, then $H(u)$ is bounded from below:

$$H(u) \geq -\frac{1}{2} \int_{\Gamma} u^2 dx + \frac{1}{3} \int_{\Gamma} u^3 dx \geq -\frac{1}{6} \sum_j L_j.$$

Classical methods: the supersolution

Theorem

If $\lambda_0(\Gamma) \geq 1$, then $u = 0$ is a unique global attractor for every $u_0 \in \mathcal{H}_0$.

Let $\bar{u}(x, t) = Ce^{(1-\lambda_0(\Gamma))t}\varphi_0(x)$ with eigenfunction $\varphi_0 > 0$ and choose $C \gg 1$ such that $u_0(x) \leq C\varphi_0(x)$ for all $x \in \Gamma$. Then, we have

$$\bar{u}_t = (1 - \lambda_0(\Gamma)) \bar{u} \geq \Delta \bar{u} + \bar{u}(1 - \bar{u}),$$

so that

$$\bar{u}_t - \Delta \bar{u} - \bar{u}(1 - \bar{u}) \geq 0 = u_t - \Delta u - u(1 - u).$$

By the comparison principle, $\bar{u}$ is a supersolution to u satisfying $u(t, x) \leq \bar{u}(t, x)$ for all $(t, x) \in [0, \infty) \times \Gamma$ so that if $\lambda_0(\Gamma) > 1$, then

$$\lim_{t \rightarrow \infty} u(t, x) \leq \lim_{t \rightarrow \infty} \bar{u}(t, x) = 0, \quad \text{for every } x \in \Gamma.$$

Classical methods: the subsolution

Theorem

If $\lambda_0(\Gamma) < 1$, then the strictly positive ground state $u_ \in \mathcal{H}_0$ is a unique global attractor for every $u_0 \in \mathcal{H}_0 \setminus \{0\}$.*

Let $\underline{u}|_{t=0} = \varepsilon \varphi_0$ for some $\varepsilon > 0$ and choose $\varepsilon \ll 1$ such that

$$H(\underline{u}|_{t=0}) = \frac{1}{2} (\lambda_0(\Gamma) - 1) \varepsilon^2 \|\varphi_0\|_{L^2(\Gamma)}^2 + \frac{1}{3} \varepsilon^3 \|\varphi_0\|_{L^3(\Gamma)}^3 < 0.$$

The attractor for $\underline{u}$, denoted as $\underline{u}_*$, is nontrivial. Furthermore, $\underline{u}_* > 0$ in $\mathcal{H}_0$.

Assume first that $u_0(x) > 0$ for every $x \in \Gamma$ except for the boundary vertices. Then, there is $\varepsilon \ll 1$ so that $\varepsilon \varphi_0 \leq u_0$. By the comparison principle, $\underline{u}$ is a subsolution to u satisfying $\underline{u}(t, x) \leq u(t, x)$ for all $(t, x) \in [0, \infty) \times \Gamma$. The attractor for u , denoted as u_* , satisfies $\underline{u}_* \leq u_*$.

The attractor $\psi = u_*$ is the eigenfunction of the Schrödinger equation:

$$-\Delta_\Gamma \psi + (u_* - 1)\psi = 0, \quad \psi \in D(\Delta_\Gamma)$$

The uniqueness of u_* follows from Sturm's comparison theorem.

Further improvements

If u_0 vanishes at some points of Γ other than the boundary vertices, then the solution $u(t, \cdot)$ is advanced forward in time for some $t_0 > 0$: $u(t_0, \cdot) > 0$ by the strong maximum principle and $\exists \varepsilon \ll 1$: $\varepsilon \varphi_0 \leq u(t_0, \cdot)$.

Corollary

There exists $L_0 > 0$ such that if $\max_j L_j < L_0$, then $u = 0$ is globally asymptotically stable for initial data $u_0 \in \mathcal{H}_0$.

This follows the uniform scaling of eigenvalues $\lambda(\Gamma)$. If $L_j = L\ell_j$ with L -independent $\{\ell_j\}$, then $\lambda(\Gamma) = L^{-2}\lambda(\tilde{\Gamma})$ with L -independent $\lambda(\tilde{\Gamma})$.

Corollary

For every j -th edge of length L_j , $\exists L_j^{(0)} \in [0, \infty)$: for every $L_j > L_j^{(0)}$ with this j , $u = u_ \geq 0$ is globally asymptotically stable for initial data $u \in \mathcal{H}_0 \setminus \{0\}$.*

This follows from the monotone decrease of $\lambda_0(\Gamma)$ with $\lambda_0(\Gamma) \rightarrow 0$ as $L_j \rightarrow \infty$ for at least one L_j .

5. Novel methods based on the period function

Since Γ consists of edges $\{e_j\}$ (which are 1D intervals), we can consider the integral curves of the second-order equation

$$u''(x) + u(x) - u(x)^2 = 0, \quad u(x) : \mathbb{R} \rightarrow \mathbb{R}$$

to construct piecewisely smooth solutions to

$$-\Delta_\Gamma u - (1 - u)u = 0, \quad \psi \in D(\Delta_\Gamma).$$

We will work with $\tilde{u} = 1 - u$ satisfying

$$\tilde{u}''(x) - \tilde{u}(x) + \tilde{u}(x)^2 = 0, \quad \tilde{u}(x) : \mathbb{R} \rightarrow \mathbb{R},$$

so that $\tilde{u} = 1$ at the boundary vertices and $\tilde{u} \in (0, 1)$ everywhere in Γ .

Integral curves on the phase plane $(\tilde{u}, \tilde{v})$ correspond to constant values of the first-order invariant

$$E(\tilde{u}, \tilde{v}) = \tilde{v}^2 - \tilde{u}^2 + \frac{2}{3}\tilde{u}^3, \quad \tilde{v} := \frac{d\tilde{u}}{dx}. \quad (1)$$

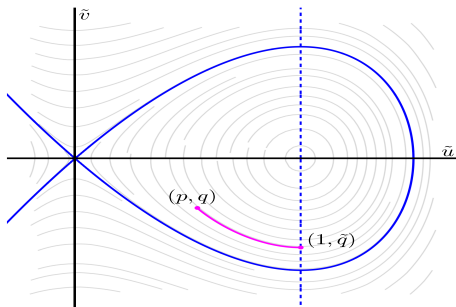
Period function I

Let $(p, q) \in (0, 1) \times (-\infty, 0)$ be a point on the phase plane $(\tilde{u}, \tilde{v})$ and consider an integral curve connecting $(1, \tilde{q})$ and (p, q) , where

$$E(p, q) = q^2 - p^2 + \frac{2}{3}p^3 = \tilde{q}^2 - \frac{1}{3}$$

The period function $T(p, q)$ is defined by

$$T(p, q) = \int_p^1 \frac{du}{v}, \quad v := \sqrt{E(p, q) + u^2 - \frac{2}{3}u^3}.$$



Main results about $T(p, q)$

Lemma

For every $(p, q) \in (0, 1) \times (-\infty, 0)$, we have

$$\frac{\partial T}{\partial p} < 0, \quad \frac{\partial T}{\partial q} > 0.$$

Two asymptotic limits near the saddle and center points are also important:

- ▶ As $(p, q) \rightarrow (0, 0)$, we have

$$T(p, q) = -\ln\left(\frac{p-q}{12}\right) - x_0 + \mathcal{O}(p),$$

where $x_0 = 2\operatorname{arccosh}\left(\frac{\sqrt{3}}{\sqrt{2}}\right)$.

- ▶ As $(p, q) \rightarrow (1, 0)$, we have

$$T(p, q) = \arcsin \frac{1-p}{\sqrt{(1-p)^2 + q^2}} + \mathcal{O}(|1-p|).$$

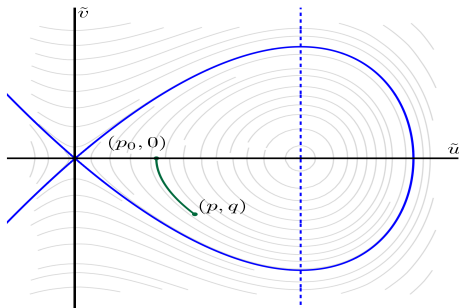
Period function II

Let $(p, q) \in (0, 1) \times (-\infty, 0)$ be a point on the phase plane $(\tilde{u}, \tilde{v})$ and consider an integral curve connecting (p, q) and $(p_0, 0)$, where

$$E(p, q) = q^2 - p^2 + \frac{2}{3}p^3 = -p_0^2 + \frac{2}{3}p_0^3.$$

The period function $T_0(p, q)$ is defined by

$$T_0(p, q) = \int_{p_0}^p \frac{du}{v}, \quad v := \sqrt{E(p, q) + u^2 - \frac{2}{3}u^3}.$$



Main results about $T_0(p, q)$

Lemma

For every $(p, q) \in (0, 1) \times (-\infty, 0)$, we have

$$\frac{\partial T_0}{\partial q} < 0.$$

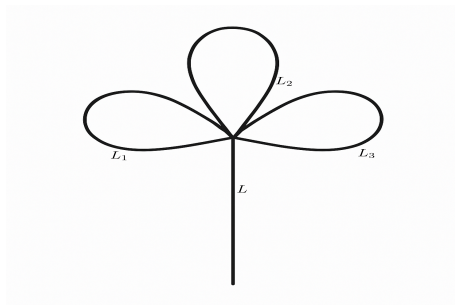
However, $T_0(p, q)$ is more difficult for analysis:

- ▶ $\frac{\partial T_0}{\partial p} < 0$ only for $p \in (0, \frac{1}{2})$ and is not monotone if $p \in (\frac{1}{2}, 1)$.
- ▶ The asymptotic expansion of $T_0(p, q)$ as $(p, q) \rightarrow (0, 0)$ does not make sense.
- ▶ As $(p, q) \rightarrow (1, 0)$, we only have the limit

$$\lim_{(p,q) \rightarrow (1,0)} T_0(p, q) = \frac{\pi}{2} - \arcsin \frac{1-p}{\sqrt{(1-p)^2 + q^2}},$$

but not the remainder term.

Ground state on the symmetric flower graph



Let $(p, q) \in (0, 1) \times (-\infty, 0)$ be parameters for the boundary conditions $p = \tilde{u}(L)$ and $q = \tilde{u}'(L)$ at the vertex. By uniqueness of integral curves, the steady state is necessarily described by the identical and even functions $\tilde{u}_j = \tilde{u}_0$ satisfying the boundary conditions $\tilde{u}_0(\pm L_0) = p$ and $\tilde{u}'_0(\pm L_0) = \mp \frac{q}{2N}$. The existence of the steady state is equivalent to finding a root $(p, q) \in (0, 1) \times (-\infty, 0)$ of the system of two equations:

$$T(p, q) = L, \quad T_0\left(p, \frac{q}{2N}\right) = L_0.$$

Applications of period functions: uniqueness

Here is the equivalent system of equations:

$$T(p, q) = L, \quad T_0\left(p, \frac{q}{2N}\right) = L_0,$$

for a given point $(L, L_0) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Theorem

There exists a simply connected region $\Omega \in \mathbb{R}^+ \times \mathbb{R}^+$ such that the positive steady state exists for every $(L, L_0) \in \Omega$ and is unique.

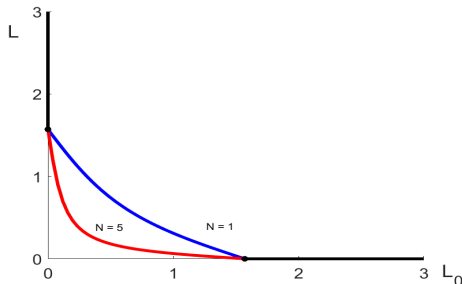
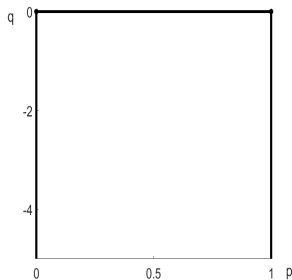
The result follows from the positivity of the Jacobian of the transformation $(p, q) \mapsto (L, L_0)$:

$$\frac{\partial T(p, q)}{\partial p} \frac{\partial T_0(p, \frac{q}{2N})}{\partial q} - \frac{\partial T(p, q)}{\partial q} \frac{\partial T_0(p, \frac{q}{2N})}{\partial p} > 0.$$

Applications of period functions: threshold on existence

Let $\Omega \subset \mathbb{R}^+ \times \mathbb{R}^+$ be the existence region in (L, L_0) . Then

- ▶ $L_0 = 0 : \Rightarrow [0, L]$ is an interval with Dirichlet and Neumann conditions. Then, $q = 0$ and $L > \frac{\pi}{2}$ for existence.
- ▶ $L = 0 : \Rightarrow [0, L_0]$ is an interval with Dirichlet and Neumann conditions. Then, $p = 1$ and $L_0 > \frac{\pi}{2}$ for existence.
- ▶ The lower boundary appears in the singular limit $(p, q) \rightarrow (1, 0)$, where $\cot(L) = 2N \tan(L_0)$. It is the same as $\lambda_0(\Gamma) = 1$ from classical theory.



Applications of period functions: limit of long graphs

Theorem

There exist $L_ > L_0$ such that if $L_{\min} := \min_j L_j > L_*$, then the positive ground state u_* is unique and satisfies*

$$\|u_* - 1\|_{L^\infty(\Gamma_0)} \leq Ce^{-L_{\min}},$$

where C is a positive constant and $\Gamma_0 = \Gamma \setminus P_0$ without the set of pendants P_0 .

Let V_0 be the subset of interior vertices $V = \{v_j\}$ which are the boundary vertices between Γ_0 and P_0 . Denote $p_j = \tilde{u}(v_j)$ at $v_j \in V_0$ and write

$$\begin{cases} -\Delta_{\Gamma_0} \tilde{u} + \tilde{u} = \tilde{u}^2 & \text{in } \Gamma_0, \\ \tilde{u} & \text{satisfies NK conditions on } V \setminus V_0, \\ \tilde{u}(v_j) = p_j \geq 0, & v_j \in V_0, \end{cases}$$

and

$$\begin{cases} -\tilde{u}''(x) + \tilde{u}(x) = \tilde{u}(x)^2 & \text{in } P_0, \\ \tilde{u} & \text{satisfies Dirichlet condition on boundary vertices,} \\ \tilde{u}(v_j) = p_j \geq 0, & v_j \in V_0. \end{cases}$$

Dirichlet-to-Neumann map I

For

$$\begin{cases} -\Delta_{\Gamma_0} \tilde{u} + \tilde{u} = \tilde{u}^2 & \text{in } \Gamma_0, \\ \tilde{u} \text{ satisfies NK conditions on } V \setminus V_0, \\ \tilde{u}(v_j) = p_j \geq 0, & v_j \in V_0, \end{cases}$$

there exist $C_0 > 0$, $p_0 > 0$, $L_* > 0$ such that if $L_{\min} = \min_j L_j > L_*$, then for every $\vec{p}$ such that $\|\vec{p}\| \leq p_0$, there is a unique solution $\tilde{u} \in D(\Delta_{\Gamma_0})$ satisfying

$$\|\tilde{u}\|_{L^\infty(\Gamma_0)} \leq C_0 \|\vec{p}\|$$

and

$$|q_j - d_j p_j| \leq C_0 (\|\vec{p}\| e^{-L_{\min}} + \|\vec{p}\|^2), \quad v_j \in V_0,$$

where q_j is the Neumann data (the sum of outward derivatives from Γ_0) at the vertex v_j and d_j is the degree of the vertex $v_j \in V_0$.

Dirichlet-to-Neumann map II

For

$$\begin{cases} -\tilde{u}''(x) + \tilde{u}(x) = \tilde{u}(x)^2 & \text{in } P_0, \\ \tilde{u} & \text{satisfies Dirichlet condition on boundary vertices,} \\ \tilde{u}(v_j) = p_j \geq 0, & v_j \in V_0. \end{cases}$$

we denote $q_j = \tilde{u}'(L_j)$ and obtain from $T(p_j, q_j) = L_j$ in the limit of $L_{\min} = \min_j L_j \gg 1$ that

$$q_j = p_j - 12e^{-L_j - x_0} + \mathcal{O}(e^{-2L_j}), \quad x_0 = 2\operatorname{arccosh}\left(\frac{\sqrt{3}}{\sqrt{2}}\right).$$

Bringing both Dirichlet-to-Neumann maps together in the NK conditions:

$$d_j p_j + \mathcal{O}(\|\vec{p}\|e^{-L_{\min}} + \|\vec{p}\|^2) + m_j p_j - 12 \sum_{\ell_j \rightarrow v_j} e^{-L_j - x_0} + \mathcal{O}(e^{-2L_{\min}}) = 0.$$

By IFT, there exists a unique solution for small $\|\vec{p}\| \leq p_0$:

$$p_j = \frac{12}{d_j + m_j} \sum_{\ell_j \rightarrow v_j} e^{-L_j - x_0} + \mathcal{O}(e^{-2L_{\min}}).$$

6. Summary

We have considered the selection of the ground state on the compact metric graph with two groups of methods:

- ▶ Classical methods: linearization and the lowest eigenvalue of $-\Delta_\Gamma$, variational methods for the energy $H(u)$, and the comparison principle with subsolutions and supersolutions.
- ▶ Novel methods based on the period function for differential equations.

The period function methods give more precise information about construction of the ground state but their application is limited to either simplest graphs or to the asymptotic limit of long graphs. On the other hand, they work when the classical methods are not applicable.

THANK YOU FOR ATTENTION!