

Justification of the log-KdV equation in granular chains

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Introduction

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 - ▶ G. Friesecke and J. Wattis, *Commun. Math. Phys.* **161** (1994), 391 - proof of existence for a general FPU lattice
 - ▶ R. MacKay, *Phys. Lett. A* **251** (1999), 191 - adaptation of this method to granular chains
 - ▶ J. English and R. Pego, *Proc. Amer. Math. Soc.* **133** (2005), 1763 - proof of the double-exponential tails of the solitary waves
 - ▶ A. Stefanov and P. Kevrekidis, *J. Nonlinear Sci.* **22** (2012), 327; *Nonlinearity* **26** (2013), 539 - proof of the bell-shaped profile

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 - ▶ A. Stefanov and P. Kevrekidis, *J. Nonlinear Sci.* **22** (2012), 327; *Nonlinearity* **26** (2013), 539 - proof of the bell-shaped profile
- ▶ We consider solitary waves by simplifying the Fermi-Pasta-Ulam lattice to a Korteweg-de Vries equation.

The granular chain

Newton's equations define the FPU (Fermi-Pasta-Ulam) lattice:

$$\frac{d^2 x_n}{dt^2} = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z},$$

where x_n is the displacement of the n th particle from an equilibrium.

$$x_{n-2} x_{n-1} x_n \quad x_{n+1} x_{n+2}$$



The interaction potential for spherical particles is

$$V(x) = \frac{1}{1+\alpha} |x|^{1+\alpha} H(-x), \quad \alpha = \frac{3}{2},$$

where H is the step (Heaviside) function.

H. Hertz, *J. Reine Angewandte Mathematik* **92** (1882), 156

The logarithmic Korteweg–de Vries equation

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Consider the FPU lattice for relative displacements $u_n := x_{n+1} - x_n$,

$$\frac{d^2 u_n}{dt^2} - (\Delta u)_n = -(\Delta f_\alpha(u))_n, \quad n \in \mathbb{Z},$$

where $(\Delta u)_n = u_{n+1} - 2u_n + u_{n-1}$ and

$$f_\alpha(u) := u(|u|^{\alpha-1} - 1) = (\alpha - 1) u \log |u| + O((\alpha - 1)^2).$$

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Set $\alpha = 1 + \varepsilon^2$ and use the asymptotic multi-scale expansion

$$u_n(t) = -v(\xi, \tau) + \text{higher order terms},$$

where $\xi := 2\sqrt{3}\varepsilon(n - t)$ and $\tau := \sqrt{3}\varepsilon^3 t$. At $O(\varepsilon^4)$, we obtain formally the KdV equation with the logarithmic nonlinearity (log-KdV equation)

$$\partial_\tau v + \partial_\xi^3 v + \partial_\xi(v \log v) = 0.$$

Korteweg–de Vries equation for regular FPU lattices

If $V \in C^3$ with $V''(0) > 0$ and $V'''(0) \neq 0$, the same expansion reduces the FPU lattice to the standard KdV equation

$$\partial_\tau v + v \partial_\xi v + \partial_\xi^3 v = 0.$$

The KdV equation admits the solitary waves $v \sim \operatorname{sech}^2(\xi - c\tau)$.

- ▶ The KdV equation has been justified at a time scale of order ε^{-3} :
G. Schneider–C.E. Wayne (2000); D. Bambusi–A. Ponno (2006).
- ▶ Nonlinear stability of small amplitude FPU solitons was proved:
G. Friesecke–R.L. Pego (1999-2004).
- ▶ Existence and stability of N -soliton solutions has been proved:
A. Hoffman–C.E. Wayne (2009); T. Mizumachi (2009, 2013).

Stationary solutions

Stationary log-KdV equation can be integrated once to get

$$\frac{d^2 v}{d\xi^2} + v \log |v| = 0,$$

which admits the Gaussian solitons

$$v(\xi) = \sqrt{e} e^{-\xi^2/4}.$$

A. Chatterjee, PRE **59** (1999), 5912

G. James–D.P., Proc. Roy. Soc. A **470** (2014), 20130465

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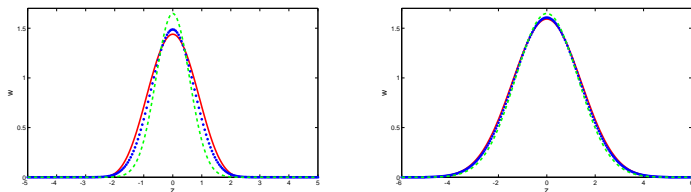


Figure : Solitary waves of the FPU chain (blue), Nesterenko compactons (red) and Gaussian solitons (green) for $\alpha = 1.5$ (left) and $\alpha = 1.1$ (right).

Numerical evidence of convergence of the approximation

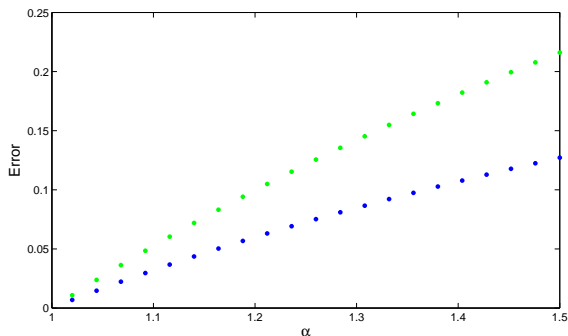


Figure : The L^∞ distance between solitary waves of the FPU chain and either Nesterenko compactons (blue dots) or Gaussian solitons (green dots) vs. α .

Numerical evidence of stability

Lattice of $N = 2000$ particles is excited with the initial impact

$$\dot{x}_n(0) = 0.1\delta_{n,0}, \quad \dot{x}_n(0) = 0 \text{ for all } n \geq 1.$$

A Gaussian solitary wave is formed asymptotically as t evolves.

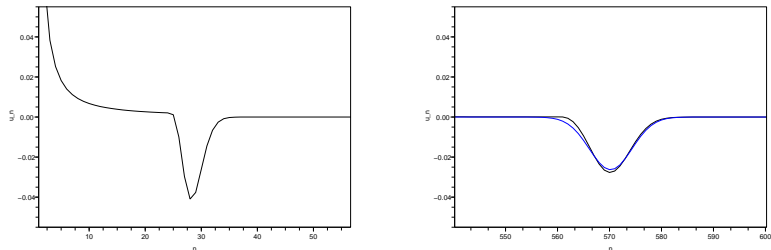


Figure : Formation of a Gaussian wave (blue curve) in the Hertzian FPU lattice with $\alpha = 1.01$: $t \approx 30.5$ (left) and $t \approx 585.6$ (right).

Precompression

Consider again the FPU lattice in the form

$$\frac{d^2 u_n}{dt^2} = -(\Delta |u|^{1+\varepsilon^2} H(-u))_n, \quad n \in \mathbb{Z}.$$

Let $u_n(t) = -v_0(1 + w_n(t'))$ and $t' = v_0^{\varepsilon^2/2} t$ for a fixed $v_0 > 0$. Then, the FPU lattice is written in the form

$$\frac{d^2 w_n}{dt^2} = (\Delta \tilde{V}_\varepsilon(w))_n, \quad n \in \mathbb{Z}$$

with the regularized potential

$$\tilde{V}_\varepsilon(w) := \frac{1}{2 + \varepsilon^2} \left[(1 + w)^{2 + \varepsilon^2} - 1 \right] - w, \quad w > -1$$

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The KdV scaling $w_n(t) \approx W(\xi, \tau)$ with $\xi := \varepsilon(n - t)$ and $\tau := \varepsilon^3 t$ yields the log-KdV equation

$$2\partial_\tau W + \frac{1}{12}\partial_\xi^3 W + \partial_\xi((1 + W)\log(1 + W)) = 0.$$

Stationary solutions

Traveling waves $W(\xi - \lambda\tau/2)$ satisfy the stationary log-KdV equation

$$\lambda W(x) = \frac{1}{12} W''(x) + (1 + W) \log(1 + W), \quad x \in \mathbb{R},$$

where $(1 + W) \log(1 + W) = W + W^2/2 + O(W^3)$.

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For any $\lambda > 1$, there exists a unique even solution $W \in H^1(\mathbb{R})$ of the stationary log-KdV equation. Moreover,

- ▶ $W(x) > 0$ for all $x \in \mathbb{R}$,
- ▶ $W(x) \rightarrow 0$ as $|x| \rightarrow \infty$ exponentially fast,
- ▶ $W \in H^\infty(\mathbb{R})$,
- ▶ W' vanishes only at one point on \mathbb{R} .

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The travelling solitary wave is orbitally stable in the log-KdV equation
J. Höwing, J. Diff. Eqs. **251** (2011), 2515.

Main results

Theorem 1 (E.Dumas–D.P., 2014)

For every $\lambda > 1$, there exist positive constants ε_0 and C_0 s.t. for every $\varepsilon \in (0, \varepsilon_0)$, there exists a unique even travelling solution $w_{\text{stat}, \varepsilon}$ of the FPU lattice in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ s.t.

$$\sup_{z \in \mathbb{R}} |w_{\text{stat}, \varepsilon}(z) - W_{\text{stat}}(\varepsilon z)| \leq C_0 \varepsilon^{1/6},$$

where W_{stat} is the unique even solution to the log–KdV equation.

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Remarks:

- ▶ Moreover, $w_{\text{stat}, \varepsilon} \in H^\infty(\mathbb{R})$ and for every $k \in \mathbb{N}$,

$$\sup_{z \in \mathbb{R}} |\partial_z^k w_{\text{stat}, \varepsilon}(z) - \varepsilon^k \partial_x^k W_{\text{stat}}(\varepsilon z)| \leq C_k \varepsilon^{k+1/6}.$$

- ▶ Moreover, $w_{\text{stat}, \varepsilon}$ decays to zero exponentially fast at infinity.
- ▶ We have no proof that $w_{\text{stat}, \varepsilon}$ is positive.

Main results

Theorem 2 (E.Dumas–D.P., 2014)

For every $\tau_0 > 0$, there exist positive constants ε_0 , δ_0 and C_0 s.t. for all $\varepsilon \in (0, \varepsilon_0)$, when initial data $w_{\text{ini},\varepsilon} \in l^2(\mathbb{R})$ satisfy

$\delta := \|w_{\text{ini},\varepsilon} - w_{\text{trav},\varepsilon}(0)\|_{l^2} \leq \delta_0$, then the unique solution w_ε to the FPU lattice belongs to $C^1([-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}], l^2(\mathbb{Z}))$ and satisfies

$$\|w_\varepsilon(t) - w_{\text{trav},\varepsilon}(t)\|_{l^2} \leq C_0\delta, \quad t \in [-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}].$$

Main results

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For every $\tau_0 > 0$, there exist positive constants ε_0 , δ_0 and C_0 s.t. for all $\varepsilon \in (0, \varepsilon_0)$, when initial data $w_{\text{ini},\varepsilon} \in l^2(\mathbb{R})$ satisfy $\delta := \|w_{\text{ini},\varepsilon} - w_{\text{trav},\varepsilon}(0)\|_{l^2} \leq \delta_0$, then the unique solution w_ε to the FPU lattice belongs to $C^1([-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}], l^2(\mathbb{Z}))$ and satisfies

$$\|w_\varepsilon(t) - w_{\text{trav},\varepsilon}(t)\|_{l^2} \leq C_0\delta, \quad t \in [-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}].$$

Remarks:

- ▶ The travelling waves of the FPU lattice are stable w.r.t. modulations of any spatial scales, up to the time scale of $O(\varepsilon^{-3})$.
- ▶ The constant C_0 may grow exponentially fast in τ_0 .
- ▶ The travelling waves of the log-KdV equation are orbitally stable w.r.t. modulations at the spatial scale $O(\varepsilon^{-1})$ and time $O(\varepsilon^{-3})$.

Proof of (Approximation) Theorem 1

Travelling wave $w_n(t) = w_{\text{stat},\varepsilon}(z)$, $z = n - ct$ with $c^2 = 1 + \varepsilon^2\lambda$ satisfies the differential advance-delay equation

$$(1 + \varepsilon^2\lambda) \frac{d^2 w}{dz^2} = \Delta \tilde{V}'_{\varepsilon}(w)(z), \quad z \in \mathbb{R}.$$

Adopting the Fourier transform on $L^2(\mathbb{R})$ functions

$$\hat{w}(k) = \mathcal{F}(w)(k) := \int_{-\infty}^{\infty} w(z) e^{-ikz} dz,$$

we can rewrite the problem as the fixed-point equation

$$w(z) = \frac{1}{1 + \varepsilon^2\lambda} \int_{-1}^1 \Lambda(y) \tilde{V}'_{\varepsilon}(w(z-y)) dy, \quad z \in \mathbb{R},$$

where $\Lambda(z) = (1 - |z|)_+$ is the hat function.

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- ▶ The existence theory (Friesecke–Wattis, 1993) implies that $1 + \varepsilon^2\lambda = c^2 > \tilde{V}'_{\varepsilon}''(0) = 1 + \varepsilon^2$ (that is, $\lambda > 1$).
- ▶ If λ is much larger than 1, the travelling wave has large amplitude.

Partition in the Fourier space

In the equivalent Fourier form, we have

$$\hat{w}(k) = \frac{1}{1 + \varepsilon^2 \lambda} \hat{\Lambda}(k) \mathcal{F}(\tilde{V}'_\varepsilon(w))(k), \quad k \in \mathbb{R},$$

where

$$\hat{\Lambda}(k) := \frac{4}{k^2} \sin^2\left(\frac{k}{2}\right) = 1 - \frac{1}{12}k^2 + O(k^4)$$

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Let us divide \mathbb{R} into two sets: $I_p := [-\varepsilon^p, \varepsilon^p]$ and $J_p := \mathbb{R} \setminus I_p$, where $p > 0$ is to be defined. The solution is to be decomposed as

$\hat{w} = \hat{u} + \hat{v}$, where

$$\hat{u}(k) := \chi_{I_p}(k) \hat{w}(k), \quad \hat{v}(k) := \chi_{J_p}(k) \hat{w}(k).$$

For $\lambda > 1$, $R > 0$ and $r \in (-1, 0)$ (all ε -independent), we define the set

$$B_{R,r} := \left\{ u \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) : r \leq \inf_{\mathbb{R}} u, \quad \sup_{\mathbb{R}} u \leq R \right\},$$

Steps in the proof of Theorem 1

1. For any $u \in B_{R,r}$ and for any small ε , there exists a unique component v such that

$$\|v\|_{L^2 \cap L^\infty} \leq C_{R,r} \varepsilon^{2-2p} \|u\|_{L^2}, \quad p < 1,$$

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2. For any small ε , there exists a unique component u in $B_{R,r}$ near the solution $W_{\text{stat}}(\varepsilon \cdot)$ to the stationary log-KdV equation s.t.

$$\|u - W_{\text{stat}}(\varepsilon \cdot)\|_{L^2 \cap L^\infty} \leq C_{R,r,\lambda} \max\{\varepsilon^{4p-2}, \varepsilon^{2-2p}\} \|W_{\text{stat}}(\varepsilon \cdot)\|_{L^2},$$

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3. Since $\|W_{\text{stat}}(\varepsilon \cdot)\|_{L^2} = O(\varepsilon^{-1/2})$ as $\varepsilon \rightarrow 0$, we require

$$2 - 2p - \frac{1}{2} > 0 \quad \text{and} \quad 4p - 2 - \frac{1}{2} > 0 \quad \Rightarrow \quad p \in \left(\frac{5}{8}, \frac{6}{8}\right).$$

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4. The optimal value $p = 2/3$ yields $\|w_{\text{stat}} - W_{\text{stat}}(\varepsilon \cdot)\|_{L^\infty} \leq C_0 \varepsilon^{1/6}$.

Proof of (Stability) Theorem 2

The scalar FPU lattice equation can be written in the vector form

$$\begin{cases} \dot{w}_n = p_{n+1} - p_n, \\ \dot{p}_n = \tilde{V}'_\varepsilon(w_n) - \tilde{V}'_\varepsilon(w_{n-1}), \end{cases} \quad n \in \mathbb{Z}.$$

The energy functional is conserved at any $(w, p) \in C^1(\mathbb{R}, l^2(\mathbb{Z}))$:

$$H := \frac{1}{2} \sum_{n \in \mathbb{Z}} p_n^2 + \sum_{n \in \mathbb{Z}} \tilde{V}_\varepsilon(w_n).$$

Let $(w_{\text{trav}}, p_{\text{trav}}) \in C^1(\mathbb{R}, l^2(\mathbb{Z}))$ denote the travelling wave to the FPU lattice with the speed c . Then, $w_{\text{trav}}(t) = w_{\text{stat}}(n - ct)$ satisfy

$$\begin{cases} -cw'_{\text{stat}}(z) = p_{\text{stat}}(z+1) - p_{\text{stat}}(z), \\ -cp'_{\text{stat}}(z) = \tilde{V}'_\varepsilon(w_{\text{stat}}(n - ct)) - \tilde{V}'_\varepsilon(w_{\text{stat}}(n - 1 - ct)), \end{cases} \quad z \in \mathbb{R}.$$

Decomposition and the energy method

For any fixed c , we decompose

$$w(t) = w_{\text{trav}}(t) + \mathcal{W}(t), \quad \rho(t) = \rho_{\text{trav}}(t) + \mathcal{P}(t),$$

such that $H = H_0 + H_1 + H_2 + H_R$ with

$$H_0 = \frac{1}{2} \sum_{n \in \mathbb{Z}} \rho_{\text{stat}}^2(n - ct) + \sum_{n \in \mathbb{Z}} \tilde{V}_\varepsilon(w_{\text{stat}}(n - ct)),$$

$$H_1 = \sum_{n \in \mathbb{Z}} \rho_{\text{stat}}(n - ct) \mathcal{P}_n + \sum_{n \in \mathbb{Z}} \tilde{V}'_\varepsilon(w_{\text{stat}}(n - ct)) \mathcal{W}_n,$$

$$H_2 = \frac{1}{2} \sum_{n \in \mathbb{Z}} \mathcal{P}_n^2 + \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{V}''_\varepsilon(w_{\text{stat}}(n - ct)) \mathcal{W}_n^2,$$

and

$$|H_R| \leq C_\rho \sup_{z \in \mathbb{R}} |\tilde{V}'''_\varepsilon(w_{\text{stat}}(z))| \|\mathcal{W}\|_{\rho^2}^3 \leq C_\rho \varepsilon^2 \|\mathcal{W}\|_{\rho^2}^3,$$

as long as $\|\mathcal{W}\|_{\rho^2} \leq \rho$. Here we recall that

$$\tilde{V}''_\varepsilon(w) = (1 + \varepsilon^2)(1 + w)^{\varepsilon^2}, \quad \tilde{V}'''_\varepsilon(w) = \varepsilon^2(1 + \varepsilon^2)(1 + w)^{\varepsilon^2 - 1}.$$

Energy estimates

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- ▶ H_1 is controlled in terms of H_2 :

$$\frac{dH_1}{dt} = \frac{c}{2} \sum_{n \in \mathbb{Z}} w'_{\text{stat}}(n - ct) \tilde{V}_\varepsilon'''(w_{\text{stat}}(n - ct)) (\mathcal{W}_n^2 + O(\mathcal{W}_n^3)).$$

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$$\frac{dH_1}{dt} = \frac{c}{2} \sum_{n \in \mathbb{Z}} w'_{\text{stat}}(n-ct) \tilde{V}_\varepsilon'''(w_{\text{stat}}(n-ct)) (\mathcal{W}_n^2 + O(\mathcal{W}_n^3)).$$

Hence, we have

$$\left| \frac{dH_1}{dt} \right| \leq C_\rho \varepsilon^3 (1 + \rho) \|\mathcal{W}\|_{\rho^2}^2 \leq 2C_\rho \varepsilon^3 (1 + \rho) H_2,$$

and

$$H_1(t) - H_1(0) \geq -2C_\rho \varepsilon^3 (1 + \rho) \int_0^{|t|} H_2(t') dt'.$$

End of the proof of Theorem 2

By using the energy expansion, we have

$$H - H_0 - H_1(0) \geq -2C_\rho \varepsilon^3 (1 + \rho) \int_0^{|t|} H_2(t') dt' + H_2(t)(1 - C_\rho \varepsilon^2 \rho).$$

By Gronwall's inequality, we obtain

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Remark: The proof of nonlinear stability uses the KdV limit scaling of small ε , but does not rely on the stability of KdV travelling waves.

Justification result

Theorem 3 (Schneider-Wayne, 2000; E.Dumas–D.P., 2014)

Let $W \in C([- \tau_0, \tau_0], H^s(\mathbb{R}))$ be a solution to the log-KdV equation for some integer $s \geq 6$ and some $\tau_0 > 0$. Assume that there exists $r_W > -1$ such that $W \geq r_W$. Then there exist positive constants ε_0 and C_0 s.t. for all $\varepsilon \in (0, \varepsilon_0)$, when initial data $w_{\text{ini}, \varepsilon} \in l^2(\mathbb{R})$ are given s.t.

$$\|w_{\text{ini}, \varepsilon} - W(\varepsilon \cdot, 0)\|_{\rho} \leq \varepsilon^{3/2},$$

the unique solution w_ε to the FPU lattice belongs to $C^1([- \tau_0 \varepsilon^{-3}, \tau_0 \varepsilon^{-3}], l^2(\mathbb{Z}))$ and satisfies

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Remarks:

- ▶ The proof relies on the energy method as in Theorem 2.
- ▶ The result suggests correlation between stability of KdV and FPU travelling waves but C_0 may grow exponentially fast in τ_0 .

Discussion

Consider the FPU lattice

$$\frac{d^2 w_n}{dt^2} = (\Delta \tilde{V}_\varepsilon(w))_n, \quad n \in \mathbb{Z},$$

with the nonlinear potential

$$\tilde{V}_\varepsilon(w) = \frac{1}{2} w^2 + \frac{\varepsilon^2}{p+1} w^{p+1}, \quad \text{for an integer } p \geq 2.$$

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- ▶ The question of stability of FPU solitons is still opened for $p \geq 3$...