Gaussian Solitary Waves in Granular Chains

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Introduction

- Granular crystal chains are chains of densely packed, elastically interacting particles.
- Recent works focus on solitary and periodic travelling waves in granular chains; said to be more relevant to physical experiments.
- Periodic travelling waves in granular chains were approximated numerically and analytically
  - K.R. Jayaprakash, Yu. Starosvetsky and A.F. Vakakis, *Phys. Rev. E* **83** (2011), 036606
  - G. James, *J. Nonlinear Sci.* **22** (2012), 813
  - M. Betti and D. Pelinovsky, *J. Nonlinear Sci.* **23** (2013), 619

Experimental setups (CalTECH)

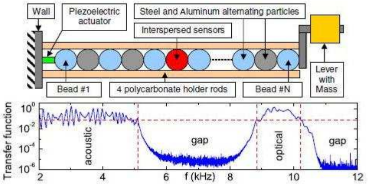

Figure : N. Boechler, G. Theocharis, S. Job, P.G. Kevrekidis, M.A. Porter, and C. Daraio, PRL **104**, 244302 (2010)

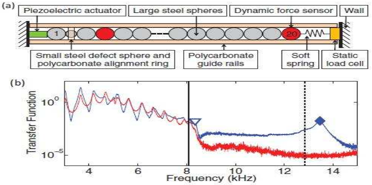

Figure : Y. Man, N. Boechler, G. Theocharis, P.G. Kevrekidis, and C. Daraio, Phys. Rev. E **85**, 037601 (2012)

On solitary travelling waves in granular chains

Proofs of existence of solitary waves were developed from the variational theory based on the differential–difference equation.

- G. Friesecke and J. Wattis, *Commun. Math. Phys.* **161** (1994), 391 - proof of existence for a general FPU lattice
- R. MacKay, *Phys. Lett. A* **251** (1999), 191 - adaptation of this method to granular chains
- J. English and R. Pego, *Proc. Amer. Math. Soc.* **133** (2005), 1763 - proof of the double-exponential tails of the solitary waves
- A. Stefanov and P. Kevrekidis, *J. Nonlinear Sci.* **22** (2012), 327 - proof of the bell-shaped profile of the solitary waves

The granular chain

Newton's equations define the FPU (Fermi-Pasta-Ulam) lattice:

$$\frac{d^2x_n}{dt^2} = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z},$$

where x_n is the displacement of the nth particle from a reference position versus time t.

The interaction potential for spherical beads is

$$V(x) = \frac{1}{1+\alpha}|x|^{1+\alpha}H(-x), \quad \alpha = \frac{3}{2},$$

where H is the step (Heaviside) function.

H. Hertz, *J. Reine Angewandte Mathematik* **92** (1882), 156

For the chains of hollow spherical particles of different width, we have other values of α in the range $1.2 \leq \alpha \leq 1.5$.

Travelling waves and the Boussinesq approximation

Using the relative displacements $u_n = x_n - x_{n-1}$ and applying the travelling wave reduction $u_n(t) = w_n(n-t)$, we obtain

$$\frac{d^2 w}{dz^2} = \Delta(w|w|^{\alpha-1}), \quad z \in \mathbb{R},$$

with $(\Delta w)(z) = w(z+1) - 2w(z) + w(z-1)$.

Expanding $\Delta = \partial_z^2 + \frac{1}{12}\partial_z^4$ and integrating twice, we obtain

$$w = w|w|^{\alpha-1} + \frac{1}{12}\frac{d^2}{dz^2} w|w|^{\alpha-1}, \quad z \in \mathbb{R},$$

which has compactons

$$w_{\mathrm{c}}(z) = \begin{cases} A\cos^{\frac{2}{\alpha-1}}(Bz), & |z| \leq \frac{\pi}{2B}, \\ 0, & |z| \geq \frac{\pi}{2B}, \end{cases}$$

where

$$A = \left(\frac{1+\alpha}{2\alpha}\right)^{\frac{1}{1-\alpha}}, \quad B = \frac{\sqrt{3}(\alpha-1)}{\alpha}.$$

Ill-posedness of the Boussinesq equation

The fully nonlinear Boussinesq equation takes the form

$$u_{tt} = (u|u|^{\alpha-1})_{xx} + \frac{1}{12}(u|u|^{\alpha-1})_{xxxx},$$

V.F. Nesterenko, *J. Appl. Mech. Tech. Phys.* **24** (1983), 733
K. Ahnert and A. Pikovsky, *Phys. Rev. E* **79** (2009), 026209.

Cauchy problem for the Boussinesq equation is ill-posed.

Compare with the recent work on ill-posedness of degenerate dispersive equations:

D.M. Ambrose, G. Simpson, J.D. Wright, and D.G. Yang, *Nonlinearity* **25** (2012), 2655.

Linearized Boussinesq equation

Linearizing the Boussinesq equation at the compact solution

$$u(x,t) = w(x-t) + U(x-t)e^{\lambda t},$$

we arrive at the spectral problem

$$(\lambda - \partial_z)^2 U = \left(\partial_z^2 + \frac{1}{12}\partial_z^4\right)(k_\alpha U),$$

where

$$k_\alpha(z) := \alpha w^{\alpha-1}(z) = \alpha A^{\alpha-1}\cos^2(Bz)\mathbf{1}_{[-\frac{\pi}{2B},\frac{\pi}{2B}]}(z).$$

The spectral problem can be closed on the compact interval $\left[-\frac{\pi}{2B}, \frac{\pi}{2B}\right]$ subject to the boundary conditions

$$U\left(\pm\frac{\pi}{2B}\right) = 0, \quad U'\left(\pm\frac{\pi}{2B}\right) = 0.$$

Numerical results

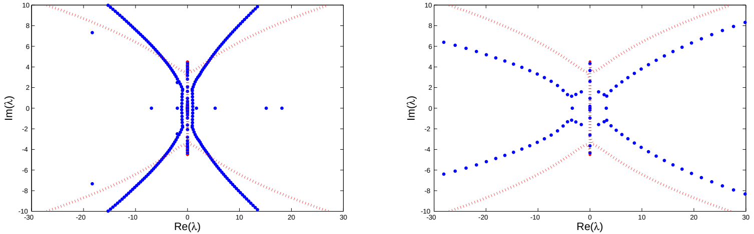

Figure : Eigenvalues of the spectral problem (blue dots) for $\alpha = 1.05$ (left) and $\alpha = 1.2$ (right). The red dotted curves show the continuous spectrum obtained in the limit case $\alpha \to 1^+$.

Korteweg–de Vries equation in the case of precompression

Consider again the FPU lattice

$$\frac{d^2 u_n}{dt^2} = V'(u_{n+1}) - 2V'(u_n) + V'(u_{n-1}), \quad n \in \mathbb{Z}.$$

If $V \in C^3$ with $V''(0) = \kappa > 0$ and $V'''(0) \neq 0$, then the asymptotic multi-scale expansion

$$u_n(t) = \kappa (4V'''(0))^{-1} \varepsilon^2 y(\xi, \tau) + \text{higher order terms},$$

where $\xi := \varepsilon(n - c_s t)$, $\tau := \varepsilon^3 c_s t/24$, and $c_s := \sqrt{\kappa}$ is the "sound velocity" of linear waves, shows that y satisfies the KdV equation

$$\partial_\tau y + 3y\,\partial_\xi y + \partial_\xi^3 y = 0.$$

The KdV equation admits the solitary waves $y = \mathrm{sech}^2((\xi - \tau)/2)$.

Relevant results

- The KdV equation can be justified at a time scale of order ε^{-3}.
  G. Schneider and C.E. Wayne, *International Conference on Differential Equations Appl.* **5** (1998) 69
  D. Bambusi, A. Ponno, *Comm. Math. Phys.* **264** (2006), 539

- Nonlinear stability of small amplitude FPU solitons can be proved.
  G. Friesecke and R.L. Pego, *Nonlinearity* **12** (1999), 1601; **15** (2002), 1343; **17** (2004), 207; **17** (2004), 229.

- Existence and stability of N-soliton solutions can be proved.
  A. Hoffman and C.E. Wayne, *Nonlinearity* **21** (2008), 2911; *J. Dyn. Diff. Equat.* **21** (2009), 343.
  T. Mizumachi, *Commun. Math. Phys.* **288** (2009), 125; *SIMA* **43** (2011), 2170; *Arch. Rat. Mech. Anal.* **207** (2013), 393.

Korteweg–de Vries equation without precompression

Consider again the FPU lattice

$$\left(\frac{d^2}{dt^2} - \Delta\right) u_n = \Delta f_\alpha(u_n), \quad n \in \mathbb{Z},$$

where

$$f_\alpha(u) := u\,(|u|^{\alpha-1} - 1) = (\alpha - 1)\,u \ln|u| + \mathcal{O}((\alpha-1)^2).$$

Let $\alpha = 1 + \varepsilon^2$. Using the asymptotic multi-scale expansion

$$u_n(t) = v(\xi, \tau) + \text{higher order terms},$$

where $\xi := 2\sqrt{3}\varepsilon\,(n - t)$, $\tau := \sqrt{3}\varepsilon^3\, t$, we obtain the KdV equation with the logarithmic nonlinearity (log-KdV)

$$\partial_\tau v + \partial_\xi (v \log v) + \partial_\xi^3 v = 0.$$

Stationary solutions

Stationary log-KdV equation can be integrated once to get

$$\frac{d^2 v}{d\xi^2} + v \ln |v| = 0,$$

which admits the Gaussian solitons

$$v(\xi) = \sqrt{e}\, e^{-\xi^2/4}.$$

A. Chatterjee, PRE **59** (1999), 5912

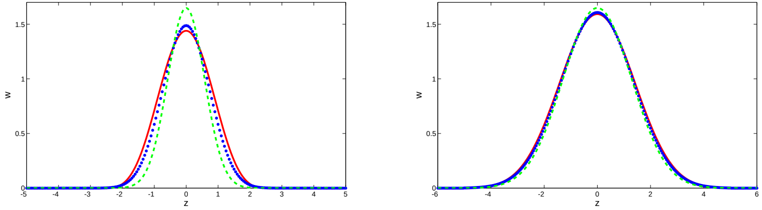

Figure : Solitary waves (blue) in comparison with the compactons (red) and the Gaussian solitons (green) for $\alpha = 1.5$ (left) and $\alpha = 1.1$ (right).

Convergence of the approximation

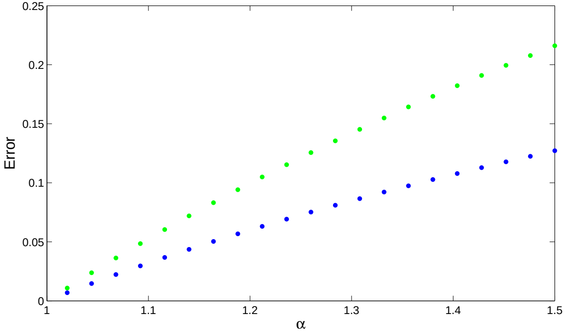

Figure : The L^∞ distance between solitary waves of the differential advance-delay equation and either the compactons (blue dots) or the Gaussian solitons (green dots) versus parameter α.

Numerical evidence of stability

Lattice of $N = 2000$ particles is excited with the initial condition of zero $x_n(0)$ and

$$\dot{x}_0(0) = 0.1, \quad \dot{x}_n(0) = 0 \;\text{ for all }\; n \geq 1.$$

A Gaussian solitary wave is formed asymptotically as t evolves.

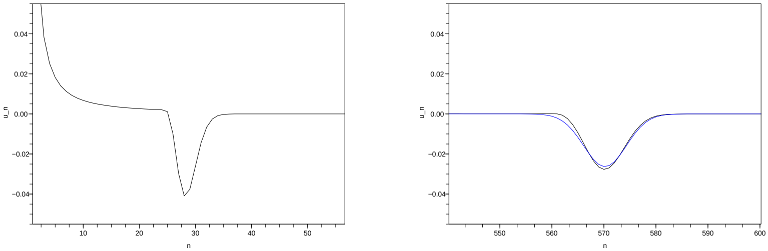

Figure : Formation of a localized wave in the Hertzian FPU lattice with $\alpha = 1.01$: left at $t \approx 30.5$, right at $t \approx 585.6$. The Gaussian approximation is shown by blue curve.

Summary of main results

The log-KdV equation

$$\partial_\tau v + \partial_\xi (v \log v) + \partial_\xi^3 v = 0.$$

1. Gaussian solitary wave is linearly orbitally stable in space $H^1(\mathbb{R})$.
2. For any initial data v_0 from the energy space X, there exists a global solution $v \in L^\infty(\mathbb{R}, X)$ such that the energy is not increasing in time.
3. The spectrum of the linearized operator in $L^2(\mathbb{R})$ is purely discrete and consists of a double zero eigenvalue and a symmetric sequence of simple purely imaginary eigenvalues $\{\pm i\omega_n\}_{n\in\mathbb{N}}$ such that $0 < \omega_1 < \omega_2 < \dots$ and $\omega_n \to \infty$ as $n \to \infty$. The eigenfunctions for nonze are smooth in ξ but decay algebraically as $|\xi| \to \infty$.

Energy functional

The log-KdV equation

$$\partial_\tau v + \partial_\xi (v \log v) + \partial_\xi^3 v = 0.$$

can be written in the Hamiltonian form

$$\partial_\tau v = \partial_\xi E'(v),$$

where the energy functional is

$$E(v) = \frac{1}{2} \int_{\mathbb{R}} \left[(\partial_\xi v)^2 - v^2 \left(\log v - \frac{1}{2} \right) \right] d\xi.$$

Gaussian wave $v_0 = e^{\frac{2-\xi^2}{4}}$ is a critical point of $E(v)$: $E'(v_0) = 0$.

Theorem 1 (G.James, D.P., 2014)

v_0 is linearly orbitally stable in space $H^1(\mathbb{R})$.

Linear operators and evolution

The Hessian operator at the critical point $v_0 = e^{\frac{2-\xi^2}{4}}$ is

$$L = E''(v_0) = -\partial_\xi^2 - 1 - \log(v_0) = -\frac{\partial^2}{\partial \xi^2} - \frac{3}{2} + \frac{\xi^2}{4}.$$

The operator L is self-adjoint in $L^2(\mathbb{R})$ with dense domain

$$D(L) = \{ u \in H^2(\mathbb{R}), \xi^2 u \in L^2(\mathbb{R}) \}.$$

The spectrum of L consists of simple eigenvalues at integers $n-1$, where $n \in \mathbb{N}_0$ (the set of natural numbers including zero).

Consider the time evolution of the perturbation u to v_0:

$$\partial_\tau u = \partial_\xi L u, \quad u(0) = u_0.$$

The solitary wave is linearly orbitally stable if for every $u_0 \in D(L)$ such that $\langle v_0, u_0 \rangle_{L^2} = 0$ there exists constant $C(u_0)$ such that

$$\|u(\tau)\|_{H^1} \leq C(u_0), \quad \tau \in \mathbb{R},$$

Symplectic decomposition

We know that $\partial_\xi L$ has a double zero eigenvalue because

$$L v_0' = 0, \quad \partial_\xi L\, v_0 = -v_0',$$

and no $u \in D(\partial_\xi L)$ exists in $\partial_\xi L u = v_0$ because $\|v_0\|_2^2 \neq 0$.

Using the decomposition

$$u(\xi,\tau) = a(\tau)\, v_0'(\xi) + b(\tau)\, v_0(\xi) + y(\xi,\tau)$$

with $\langle v_0, y \rangle_{L^2} = 0$ and $\langle \partial_\xi^{-1} v_0, y \rangle_{L^2} = 0$, we obtain

$$\frac{da}{d\tau} + b = 0, \quad \frac{db}{d\tau} = 0, \quad \frac{\partial y}{\partial \tau} = \partial_\xi L\, y.$$

If $\langle v_0, u_0 \rangle_{L^2} = 0$, then $b(\tau) = b(0) = 0$ and $a(\tau) = a(0)$.

Proof of linear orbital stability

Because v_0 and v_0' are eigenvectors of L for the negative and zero eigenvalues, L is strictly positive definite on ${v_0}^{\perp} \cap {v_0'}^{\perp} \subset L^2(\mathbb{R})$.

As a result, $\|y\|_L = \langle Ly, y \rangle_{L^2}^{1/2}$ defines a norm (equivalent to a weighted H^1-norm).

From the energy balance,

$$\frac{d}{d\tau} \frac{1}{2} \|y\|_L^2 = \langle Ly, \partial_\tau y \rangle_{L^2} = \langle Ly, \partial_\xi Ly \rangle_{L^2} = 0,$$

we obtain the Lyapunov stability of the zero equilibrium $y = 0$ in the constrained space $\langle v_0, y \rangle_{L^2} = 0$ and $\langle \partial_\xi^{-1} v_0, y \rangle_{L^2} = 0$. $\Box$

The constrained space corresponds to the modulation of the two parameters of the Gaussian solitary wave.

Global existence of solutions

The log-KdV equation

$$\partial_\tau v + \partial_\xi (v \log v) + \partial_\xi^3 v = 0$$

has the associated energy functional

$$E(v) = \frac{1}{2} \int_{\mathbb{R}} \left[(\partial_\xi v)^2 - v^2 \left(\log v - \frac{1}{2} \right) \right] d\xi,$$

defined in the function space

$$X := \left\{ v \in H^1(\mathbb{R}) : \quad v^2 \log |v| \in L^1(\mathbb{R}) \right\}.$$

Theorem 2 (R. Carles, D.P., 2014)

For any $v_0 \in X$, there exists a global solution $v \in L^\infty(\mathbb{R}, X)$ of the log–KdV equation such that

$$\|v(\tau)\|_{L^2} \leq \|v_0\|_{L^2}, \quad E(v(\tau)) \leq E(v_0), \quad \text{for all } \tau \in \mathbb{R}.$$

Proof of global existence

1. Construct an approximation of the logarithmic nonlinearity (Cazenave, 1980):

$$f_\varepsilon(v) = \begin{cases} v\log(v), & |v| \geq \varepsilon, \\ \left(\log(\varepsilon) - \frac{3}{4}\right) v + \frac{1}{\varepsilon^2} v^3 - \frac{1}{4\varepsilon^4} v^5, & |v| \leq \varepsilon, \end{cases}$$

hence $f_\varepsilon \in C^2(\mathbb{R})$ and $f_\varepsilon(v) \to v\log(v)$ as $\varepsilon \to 0$ for every $v \in \mathbb{R}$.

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2. Obtain existence of the global approximating solutions $v^\varepsilon \in C(\mathbb{R}, H^1(\mathbb{R}))$ of the generalized KdV equations

$$\begin{cases} v^\varepsilon_\tau + v^\varepsilon_{\xi\xi\xi} + f'_\varepsilon(v^\varepsilon) v^\varepsilon_\xi = 0, & \tau > 0, \\ v^\varepsilon|_{\tau=0} = v_0. \end{cases}$$

   (Kenig, Ponce, Vega, 1991).

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   (Kenig, Ponce, Vega, 1991).

3. Obtain uniform estimates for all $\varepsilon > 0$ and $\tau \in \mathbb{R}$:

$$\|v^\varepsilon(\tau)\|_{H^1} + \|(v^\varepsilon(\tau))^2 \log(v^\varepsilon(\tau))\|_{L^1} \leq C(v_0).$$

4. Pass to the limit $\varepsilon \to 0$ and obtain a global solution $v \in L^\infty(\mathbb{R}, X)$ of the log–KdV equation. $\square$

Uniqueness and global well-posedness

Lemma: *Assume that a solution $v \in L^\infty(\mathbb{R}, X)$ of the log–KdV equation satisfies the additional condition*

$$(\log|v|)_\xi \in L^\infty([-\tau_0, \tau_0] \times \mathbb{R}).$$

Then, the solution v is unique for every $\tau \in (-\tau_0, \tau_0)$, depends continuously on the initial data $v_0 \in X$, and satisfies $\|v(\tau)\|_{L^2} = \|v_0\|_{L^2}$ and $E(v(\tau)) = E(v_0)$ for all $\tau \in (-\tau_0, \tau_0)$.

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- $\partial_\xi \log|v|$ is unbounded as $|\xi| \to \infty$ for the Gaussian solitary wave.
- Nonlinear orbital stability of Gaussian solitary wave is conditional that the global solution $v \in L^\infty(\mathbb{R}, X)$ is unique and depends continuously on the initial data $v_0 \in X$.

Spectral stability

If $v = V(\xi)e^{\lambda\tau}$, we arrive to the linear eigenvalue problem

$$\partial_\xi L V = \lambda V.$$

Under the properties of L ($\sigma(L) = \{n-1, \quad n \in \mathbb{N}_0\}$), spectral stability of the Gaussian wave v_0 follows from an adaptation of recent works:

- T. Kapitula, A. Stefanov, *Stud. Appl. Math.* (2014).
- D.P., in *Spectral analysis, stability, and bifurcation in modern nonlinear physical systems* (Wiley–ISTE, 2014).

Theorem 3 (R. Carles, D.P., 2014)

The spectrum of $\partial_x L$ in $L^2(\mathbb{R})$ is purely discrete and consists of a double zero eigenvalue and a symmetric sequence of simple purely imaginary eigenvalues $\{\pm i\omega_n\}_{n\in\mathbb{N}}$ such that $0 < \omega_1 < \omega_2 < \dots$ and $\omega_n \to \infty$ as $n \to \infty$. The eigenfunctions for nonzero eigenvalues are smooth in ξ but decay algebraically as $|\xi| \to \infty$.

Further remarks

- Because the spectrum of $\partial_x L$ is purely discrete, no asymptotic stability result can hold for Gaussian solitary waves.

- This agrees with the result of Cazenave for the log–NLS equation: the L^p norms at the solution v for any $p \geq 2$ including $p = \infty$ may not vanish as $t \to \infty$ (or in a finite time).

Further remarks

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- This agrees with the result of Cazenave for the log–NLS equation: the L^p norms at the solution v for any $p \geq 2$ including $p = \infty$ may not vanish as $t \to \infty$ (or in a finite time).
- Nonlinear analysis of perturbations to the Gaussian solitary wave becomes now problematic. If $v(\xi,\tau) := v_0(\xi) + w(\xi,\tau)$ is set, then w satisfies

  $$w_\tau = \partial_\xi L w - \partial_\xi N(w),$$

  where

  $$N(w) := w \log\left(1 + \frac{w}{v_0}\right) + v_0 \left[\log\left(1 + \frac{w}{v_0}\right) - \frac{w}{v_0}\right].$$

  However, w/v_0 may grow like an inverse Gaussian function of ξ.

Proof of spectral stability

The linear eigenvalue problem

$$AV = \lambda V, \quad A := \partial_\xi L = -\partial_\xi^3 + \frac{1}{4}(\xi^2 - 6)\partial_\xi + \frac{1}{2}\xi,$$

can be written in the equivalent form with the Fourier transform

$$\hat{A}\hat{V} = \lambda \hat{V}, \quad \hat{A} = \frac{i}{4}k\left(-\partial_k^2 + 4k^2 - 6\right).$$

with the natural choice $\lambda = \frac{i}{4}E$.

Eigenfunctions of A are defined in the domain $X_A := D(A) \cap \dot{H}^{-1}(\mathbb{R})$,

$$D(A) = \left\{u \in H^3(\mathbb{R}) : \quad \xi^2\partial_\xi u \in L^2(\mathbb{R}), \quad \xi u \in L^2(\mathbb{R})\right\}.$$

In the Fourier form, the domain X_A becomes

$$\hat{X}_A = \left\{\hat{u} \in H^1(\mathbb{R}) : \quad k\partial_k^2\hat{u} \in L^2(\mathbb{R}), \quad k^3\hat{u} \in L^2(\mathbb{R}), \quad k^{-1}\hat{u} \in L^2(\mathbb{R})\right\}.$$

Proof of spectral stability

The linear eigenvalue problem is

$$\frac{d^2\hat{u}}{dk^2} + \left(\frac{E}{k} + 6 - 4k^2\right)\hat{u}(k) = 0, \quad k \in \mathbb{R}.$$

- As $k \to 0$, two linearly independent solutions exist

$$\hat{u}_1(k) = k + \mathcal{O}(k^2), \quad \hat{u}_2(k) = 1 + \mathcal{O}(k\log(k)).$$

  The second solution does not belong to $\hat{X}_A$.

- As $|k| \to \infty$, the decaying solution satisfies

$$\hat{u}(k) = ke^{-k^2}\left(1 + \mathcal{O}(|k|^{-1})\right).$$

  The shooting problem is over-determined.

Proof of spectral stability

- The way around is the weak piecewise definition of the eigenfunction:

$$\hat{u}(k) = \begin{cases} \hat{u}_+(k), & k > 0, \\ 0, & k < 0, \end{cases} \quad \text{or} \quad \hat{u}(k) = \begin{cases} 0, & k > 0, \\ \hat{u}_-(k), & k < 0, \end{cases}$$

where $\hat{u}_\pm(0) = 0$, so that $\hat{u} \in \hat{X}_A$.

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  where $\hat{u}_\pm(0) = 0$, so that $\hat{u} \in \hat{X}_A$.

- For $\hat{u}_+$, we set $\hat{u}_+(k) = k^{1/2}\hat{v}_+(k)$ and obtain

$$k^{1/2}\left(-\frac{d^2}{dk^2} + 4k^2 - 6\right)k^{1/2}\hat{v}_+(k) = E\hat{v}_+(k), \quad k \in (0,\infty),$$

  which is now in the symmetric form. Hence $E \in \mathbb{R}$.

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which is now in the symmetric form. Hence $E \in \mathbb{R}$.

- For $E = 0$, we have $\hat{v}_+ = k^{1/2}e^{-k^2} > 0$ for $k > 0$. By Sturm's Theorem, the set of eigenvalues $\{E_n\}_{n\in\mathbb{N}_0}$ satisfies $0 = E_0 < E_1 < E_2 < \ldots$ and $E_n \to \infty$ as $n \to \infty$. $\square$

Numerical illustration

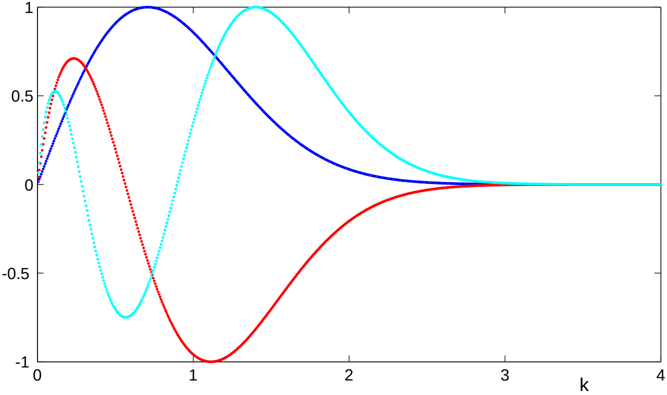

Figure : Eigenfunctions $\hat{u}$ of the spectral problem versus k for the first three eigenvalues $E_0 = 0$, $E_1 \approx 5.411$, and $E_2 \approx 12.308$.

Further development - justification of convergence

Writing the differential advance-delay equation

$$\frac{d^2 v}{dz^2} = \Delta v^{1+\varepsilon^2}, \quad z \in \mathbb{R},$$

the equivalent integral Fourier form, we obtain a fixed-point problem

$$\hat{v}(k) = \frac{4}{k^2} \sin^2\left(\frac{k}{2}\right) \widehat{v^{1+\varepsilon^2}}(k). \quad k \in \mathbb{R},$$

Expansion near $k = 0$ yields the stationary log–KdV equation

$$0 = -\frac{k^2}{12} \hat{v}(k) + \varepsilon^2 \widehat{v \log(v)}(k), \quad \text{as} \quad k \to 0.$$

Consider now solitary waves such that $v(z) \geq v_0 > 0$ for all $z \in \mathbb{R}$.

Theorem 4 (E. Dumas, D.P., 2014)

For sufficiently small ε, there exists a solution v in $H^1(\mathbb{R})$ near the solitary wave v_0 such that

$$\sup_{z \in \mathbb{R}} |v(z) - v_0(z)| \leq C_0 \varepsilon^{1/6}.$$

Further development - the KdV equation with compactons

Beyond order of $(\alpha-1)^2 = \varepsilon^4$, we can rewrite the nonlinearity of the differential advance-delay equation

$$\left(\frac{d^2}{dt^2} - \Delta\right) u_n = \Delta f_\alpha(u_n), \quad n \in \mathbb{Z},$$

in the equivalent form:

$$\begin{aligned} f_\alpha(u) &:= u(|u|^{\alpha-1} - 1) = (\alpha-1)\, u \ln|u| + \mathcal{O}((\alpha-1)^2) \\ &= \alpha\left(u - u|u|^{\frac{1}{\alpha}-1}\right) + \mathcal{O}((\alpha-1)^2). \end{aligned}$$

Consequently, we can derive the generalized KdV equation

$$\partial_\tau v + \partial_\xi^3 v + \frac{\alpha}{\alpha-1}\partial_\xi (v - v|v|^{\frac{1}{\alpha}-1}) = 0$$

at the same order as the log-KdV equation. The generalized KdV equation has exact compacton solutions.

Open questions

- Convergence of Gaussian waves and compactons in the generalized KdV equation to the solitary wave in the FPU chains.
- Orbital stability of Gaussian waves or compactons in the log-KdV and the generalized KdV equations.
- Transfer of orbital stability results to the solitary waves in the FPU chains with Hertzian potentials.
- Development of numerical methods for the log–KdV and generalized KdV equations.

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Merci beaucoup pour votre attention!