Standing waves on quantum graphs: variational methods and the period function

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Background: Nonlinear Schrödinger equation

In many problems (BECs, photonics, optics), wave dynamics is modeled with the (focusing) nonlinear Schrödinger equation

$$iu_t = -u_{xx} + V(x)u - |u|^{2p}u,$$

where $p > 0$ is nonlinearity power, $V(x) : \mathbb{R} \mapsto \mathbb{R}$ is the trapping potential.

- Single-well potentials such as $V_0(x) = -\text{sech}^2(x)$.
- Double-well potentials such as

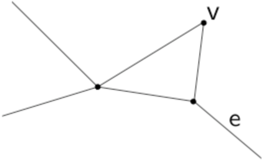
$$V(x;s) = \frac{1}{2}\left(V_0(x-s) + V_0(x+s)\right), \quad s \geq 0.$$

- Periodic potentials

$$V(x+L) = V(x), \quad L > 0,$$

  such as $V(x) = \sin^2(x)$.

Nonlinear Schrödinger equation on metric graphs



A metric graph $\Gamma = \{E, V\}$ is given by a set of edges E and vertices V, with a metric structure on each edge.

Nonlinear Schrödinger equation on a graph Γ:

$$i\Psi_t = -\Delta\Psi - |\Psi|^{2p}\Psi, \quad x \in \Gamma,$$

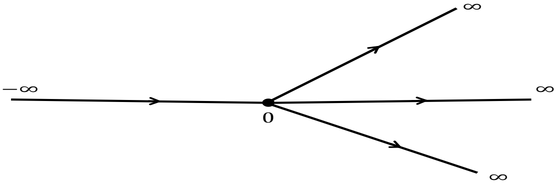
where Δ is the graph Laplacian and $\Psi(t,x)$ is defined componentwise on edges subject to Neumann–Kirchhoff boundary conditions at vertices:

$$\begin{cases} \Psi(v) \text{ is continuous} & \text{for every } v \in V, \\ \sum_{e\sim v} \partial\Psi_e(v) = 0, & \text{for every } v \in V, \end{cases}$$

where $e \sim v$ denotes all edges $e \in E$ adjacent to the vertex $v \in V$.

Example: a star graph

A star graph is the union of N half-lines connected at a single vertex. For $N = 2$, the graph is the line $\mathbb{R}$. For $N = 3$, the graph is a Y-junction.



Function spaces are defined componentwise:

$$L^2(\Gamma) = L^2(\mathbb{R}^-) \oplus \underbrace{L^2(\mathbb{R}^+) \oplus \cdots \oplus L^2(\mathbb{R}^+)}_{\text{(N-1) elements}},$$

subject to the Neumann–Kirchhoff conditions at a single vertex:

$$H^1_\Gamma := \{\Psi \in H^1(\Gamma) : \quad \psi_1(0) = \psi_2(0) = \cdots = \psi_N(0)\}$$

$$H^2_\Gamma := \{\Psi \in H^2(\Gamma) \cap H^1_\Gamma : \quad \psi_1'(0) = \sum_{j=2}^{N} \psi_j'(0)\},$$

NLS on the metric graph Γ

The Cauchy problem for the NLS flow:

$$\begin{cases} i\Psi_t = -\Delta\Psi - |\Psi|^{2p}\Psi, \\ \Psi|_{t=0} = \Psi_0. \end{cases}$$

Lemma. The Cauchy problem is locally well-posed for either $\Psi_0 \in H^1_\Gamma$ or for $\Psi_0 \in H^2_\Gamma$. Moreover, the mass

$$Q(\Psi) = \|\Psi\|^2_{L^2(\Gamma)}$$

and the energy

$$E(\Psi) = \|\nabla\Psi\|^2_{L^2(\Gamma)} - \frac{1}{p+1}\|\Psi\|^{2p+2}_{L^{2p+2}(\Gamma)},$$

are constants in time for $\Psi \in C(\mathbb{R}, H^1_\Gamma)$.

Ground state

Ground state is a standing wave of smallest energy E at fixed mass Q,

$$\mathcal{E}_\mu = \inf\{E(u) : \quad u \in H^1_\Gamma, \quad Q(u) = \mu\}.$$

All standing waves satisfy the Euler–Lagrange equation:

$$-\Delta\Phi - |\Phi|^{2p}\Phi = \omega\Phi,$$

where the Lagrange multiplier ω defines $\Psi(t,x) = \Phi(x)e^{-i\omega t}$.

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For $p \in (0,2)$, infimum $\mathcal{E}_\mu$ exists for every $\mu > 0$ thanks to Gagliardo–Nirenberg inequality:

$$\|\Psi\|^{2p+2}_{L^{2p+2}(\Gamma)} \leq C_{\Gamma,p}\|\nabla\Psi\|^p_{L^2(\Gamma)}\|\Psi\|^{p+2}_{L^2(\Gamma)},$$

where $C_{\Gamma,p} > 0$ depends on Γ and p only.

Theorem. (Adami–Serra–Tilli, 2015) If Γ is unbounded and contains at least one half-line, then for $p \in (0,2)$,

$$\min_{u \in H^1(\mathbb{R}^+)} E(u;\mathbb{R}^+) \leq \mathcal{E}_\mu \leq \min_{u \in H^1(\mathbb{R})} E(u;\mathbb{R}) \quad \text{for fixed } \mu,$$

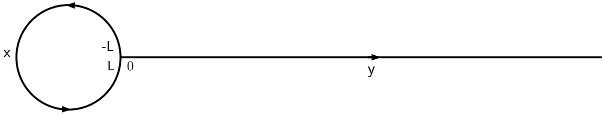
Infimum may not be attained by any of the standing waves Φ.

Ground state in the subcritical case $p \in (0, 2)$

Theorem. (Adami–Serra–Tilli, 2016) If Γ consists of only one half-line, then

$$\mathcal{E}_\mu < \min_{u \in H^1(\mathbb{R})} E(u; \mathbb{R})$$

and the infimum is attained for every $\mu > 0$.



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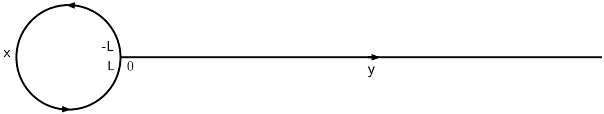


If Γ consists of more than two half-lines and is *connective to infinity*, then

$$\mathcal{E}_\mu = \min_{u \in H^1(\mathbb{R})} E(u; \mathbb{R})$$

and the infimum is not attained because a minimizing sequence escapes to infinity along an unbounded edge.

Ground state in the critical case $p = 2$

Recall the fixed mass

$$Q(\Psi) = \|\Psi\|^2_{L^2(\Gamma)} = \mu$$

and the energy

$$E(\Psi) = \|\nabla\Psi\|^2_{L^2(\Gamma)} - \|\Psi\|^6_{L^6(\Gamma)}$$

Gagliardo–Nirenberg inequality is now

$$\|\Psi\|^6_{L^6(\Gamma)} \leq C_\Gamma \|\nabla\Psi\|^2_{L^2(\Gamma)} \|\Psi\|^4_{L^2(\Gamma)} = C_\Gamma \mu^2 \|\nabla\Psi\|^2_{L^2(\Gamma)}$$

Theorem. (Adami–Serra–Tilli, 2017) If Γ consists of only one half-line, then the ground state is attained if and only if $\mu \in (\mu_{\mathbb{R}+}, \mu_{\mathbb{R}}]$, where $\mu_{\mathbb{R}}$ is the fixed mass of the NLS soliton and $\mu_{\mathbb{R}+}$ is the fixed mass of the half-soliton. Moreover,

$$\mathcal{E}_\mu = \begin{cases} 0, & \mu \in [0, \mu_{\mathbb{R}+}], \\ < 0, & \mu \in (\mu_{\mathbb{R}+}, \mu_{\mathbb{R}}], \\ -\infty, & \mu \in (\mu_{\mathbb{R}}, \infty). \end{cases}$$

Uniqueness is proven for almost all μ (Dovetta-Serra-Tilli, 2020).

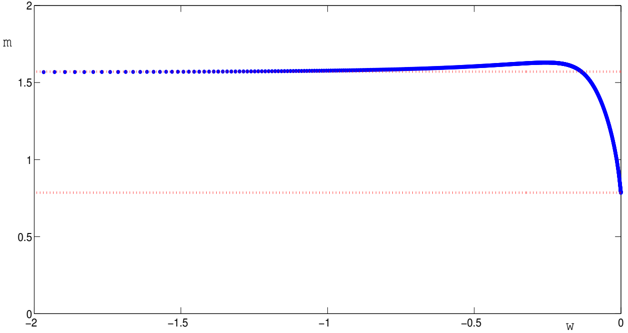
Main goal

Recall the standing wave solutions $\Psi(t,x) = \Phi(x)e^{-i\omega t}$ with

$$-\Delta\Phi - 3|\Phi|^4\Phi = \omega\Phi.$$

Main question: What is the range of frequencies ω for the ground states?

For the tadpole graph, the answer is suggested by the following figure:



New variational formulation

We explore the following constrained minimization problem:

$$\mathcal{B}(\omega) = \inf_{u \in H^1_\Gamma} \left\{ B_\omega(u) : \quad \|u\|_{L^6(\Gamma)} = 1 \right\}, \quad \omega < 0,$$

where

$$B_\omega(u) := \|\nabla u\|^2_{L^2(\Gamma)} + |\omega| \|u\|^2_{L^2(\Gamma)}.$$

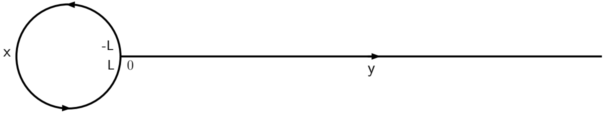
It generates the same Euler–Lagrange equation

$$-\Delta \Phi - 3|\Phi|^4 \Phi = \omega \Phi$$

after the Lagrange multiplier is scaled out by a simple transformation.

Theorem (Noja–Pelinovsky, Calc Var PDE, 2020)

For every $\omega < 0$, there exists a global minimizer $\Psi(\cdot, \omega) \in H^1_\Gamma$ which yields a strong solution $\Phi(\cdot, \omega) \in H^2_\Gamma$ to the stationary NLS equation. The standing wave Φ is real up to the phase rotation, positive up to the sign choice, symmetric on $[-L, L]$ and monotonically decreasing on $[0, L]$ and $[0, \infty)$.



- $B_\omega(u) = \|\nabla u\|^2_{L^2(\Gamma)} + |\omega|\|u\|^2_{L^2(\Gamma)}$ is equivalent to $\|u\|^2_{H^1(\Gamma)}$.

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- Constraint $\|u\|_{L^6(\Gamma)} = 1$ ensures that $\mathcal{B}(\omega) = \inf_{u \in H^1_\Gamma}\{B_\omega(u)\} > 0$ due to Sobolev's embedding $\|u\|_{L^6} \le C\|u\|_{H^1}$.

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- A minimizing sequence $\{u_n\}_n$ in $H^1(\Gamma)$ satisfying the constraint $\|u_n\|_{L^6} = 1$ such that $B_\omega(u_n) \to \mathcal{B}(\omega)$ has a weak limit u_*. By Fatou's lemma, $0 \leq \|u_*\|_{L^6} \leq \lim_{n\to\infty} \|u_n\|_{L^6} = 1$. Let $\gamma := \|u_*\|_{L^6}$.

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- If $\gamma = 0$, the minimizing sequence vanishes. It would mean that $\mathcal{B}(\omega) = \min_{u \in H^1(\mathbb{R})} B_\omega(u; \mathbb{R})$. This is ruled out by an example of $u_0 \in H^1_\Gamma$ such that $\|u_0\|_{L^6} = 1$ and $B_\omega(u_0) < \min_{u \in H^1(\mathbb{R})} B_\omega(u; \mathbb{R})$.

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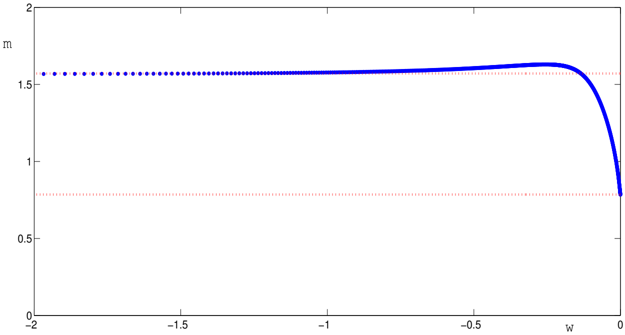
- Hence, $\gamma = 1$ and u_* is a strong limit of $\{u_n\}_n$ (minimizer).

- Symmetry of u_* follows from the Polya–Szegö inequality on graphs.

The standing wave solutions $\Psi(t,x) = \Phi(x)e^{-i\omega t}$ with

$$-\Delta\Phi - 3|\Phi|^4\Phi = \omega\Phi.$$

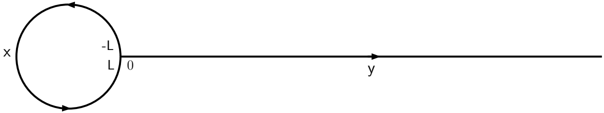


Dynamical formulation

Consider the stationary NLS equation

$$-\Delta\Phi - 3|\Phi|^4\Phi = \omega\Phi$$

and split $\Phi = (u, v)$ on the tadpole graph.



Use the scaling transformation $\omega = -\varepsilon^4$ and

$$\begin{cases} u(x) = \varepsilon U(\varepsilon^2 x), & x \in [-L, L], \\ v(x) = \varepsilon V(\varepsilon^2 x), & x \in [0, \infty). \end{cases}$$

Then, we obtain the boundary-value problem:

$$\begin{cases} -U'' + U - 3U^5 = 0, & z \in (-L\varepsilon^2, L\varepsilon^2), \\ -V'' + V - 3V^5 = 0, & z \in (0, \infty), \\ U(L\varepsilon^2) = U(-L\varepsilon^2) = V(0), & \\ U'(L\varepsilon^2) - U'(-L\varepsilon^2) = V'(0). & \end{cases}$$

Orbits of $-U'' + U - 3U^5 = 0$ are level curves of the energy function

$$E(U, U') = (U')^2 - U^2 + U^6.$$

The solution in the tail $V \in H^2(0, \infty)$ is a part of the homoclinic orbit.

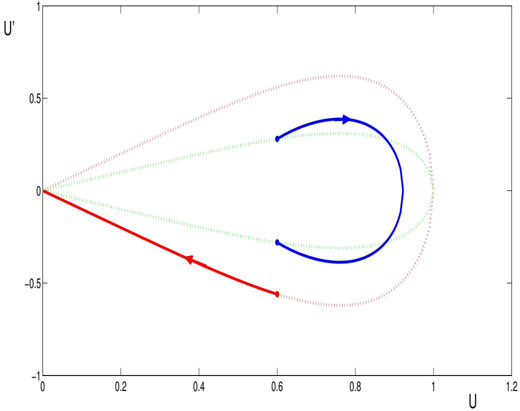


Figure: Representation of the solutions on the phase plane (U, U').

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the solution in the tail is determined uniquely

$$V(z) = \varphi(z+a), \quad \text{where } \varphi(z) := \operatorname{sech}^{1/2}(2z) \text{ is the soliton,}$$

up to the parameter $U_0 = V(0) = \varphi(a) \in (0,1)$, equivalently, by $a > 0$.

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up to the parameter $U_0 = V(0) = \varphi(a) \in (0, 1)$, equivalently, by $a > 0$.

- $V'(0)$ is determined uniquely from U_0.
- This determines uniquely $U(L\varepsilon^2) = U_0$ and $U'(L\varepsilon^2) = \frac{1}{2}V'(0)$, hence the energy level E_0.
- The existence problem then reduces to the study of the period function

$$L\varepsilon^2 = T(U_0) := \int_{U_0}^{U_+} \frac{du}{\sqrt{E_0 + u^2 - u^6}},$$

where U_+ is the right turning point from $E_0 + U_+^2 - U_+^6 = 0$.

Main result

Lemma (Noja–Pelinovsky, Calc Var PDE, 2020)

For every $U_0 \in (0,1)$ there exists a unique value of $\varepsilon > 0$ for which there exists a unique solution $U \in C^2(0, L\varepsilon^2)$ to the boundary-value problem such that U is monotonically decreasing on $[0, L\varepsilon^2]$. Moreover, the map $(0,1) \ni U_0 \mapsto \varepsilon(U_0) \in (0,\infty)$ is C^1, onto, and monotonically decreasing.

From the period function

$$L\varepsilon^2 = T(U_0) := \int_{U_0}^{U_+} \frac{du}{\sqrt{E_0 + u^2 - u^6}},$$

we only need to prove that $T'(U_0) < 0$, where U_+ and E_0 depend on U_0.

Main tool : potential function on the plane

If $W(u,v)$ is a C^1 function in an open region of $\mathbb{R}^2$, then the differential of W is defined by

$$dW(u,v) = \frac{\partial W}{\partial u} du + \frac{\partial W}{\partial v} dv$$

and the line integral of $dW(u,v)$ along any C^1 contour γ connecting two points (u_0, v_0) and (u_1, v_1) does not depend on γ and is evaluated as

$$\int_\gamma dW(u,v) = W(u_1, v_1) - W(u_0, v_0).$$

The period function can be expressed as

$$T(U_0) := \int_{U_0}^{U_+} \frac{du}{v}, \quad v := \sqrt{E_0 + u^2 - u^6}.$$

so that with $A(u) = u^2 - u^6$,

$$[E_0 + A(u_*)]T(U_0) = \int_{U_0}^{U_+} v du - \int_{U_0}^{U_+} \frac{A(u) - A(u_*)}{v} du,$$

where $u_* = \max_{u \in [0,1]} A(u)$ and $E_0 + A(u_*) > 0$.

Using

$$d\left(\frac{2v[A(u) - A(u_*)]}{A'(u)}\right) = 2\left[1 - \frac{A''(u)[A(u) - A(U_*)]}{[A'(u)]^2}\right] v du + \frac{2[A(u) - A(u_*)]}{A'(u)} dv$$

we eliminate the singular term in $T(U_0)$:

$$\frac{2[A(u) - A(u_*)]}{A'(u)} dv = \frac{A(u) - A(u_*)}{v} du.$$

Characterization of the ground state

The ground state $\Psi(\cdot,\omega) \in H^1_\Gamma$ of the stationary NLS equation

$$-\Delta\Phi - 3|\Phi|^4\Phi = \omega\Phi$$

is represented dynamically as a family of orbits with parameter $U_0 \in (0,1)$ such that $(0,1) \ni U_0 \mapsto \omega = -\varepsilon^4 \in (-\infty,0)$ is one-to-one and onto.

Consider the linearized operator

$$\mathcal{L} = -\Delta - 15\Phi^4 - \omega.$$

Then,

$$\langle \mathcal{L}\Psi, \Psi\rangle_{L^2(\Gamma)} = -12\|\Psi\|^6_{L^6(\Gamma)} < 0,$$

hence $\mathcal{L}$ has exactly one simple negative eigenvalue.
(Morse index $n(\mathcal{L}) = 1$.)
Moreover, $\mathrm{Ker}(\mathcal{L}) = \{0\}$ follows from the same dynamical representation.

It remains to consider the mass $\mu(\omega) = \|\Psi(\cdot,\omega)\|^2_{L^2(\Gamma)}$ relatively to $\mu_{\mathbb{R}_+}$, $\mu_{\mathbb{R}}$.

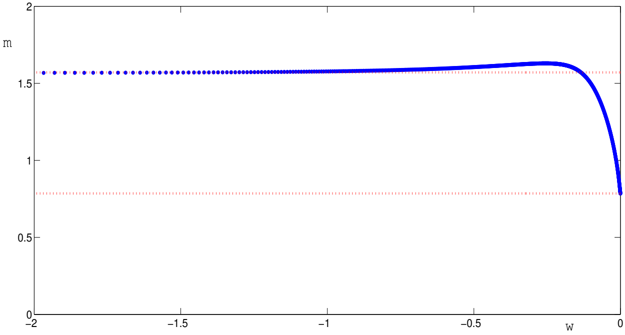
Theorem (Noja–Pelinovsky, Calc Var PDE, 2020)

The mapping $\omega \mapsto \mu(\omega) = Q(\Phi(\cdot,\omega))$ is C^1 for every $\omega < 0$ and satisfies

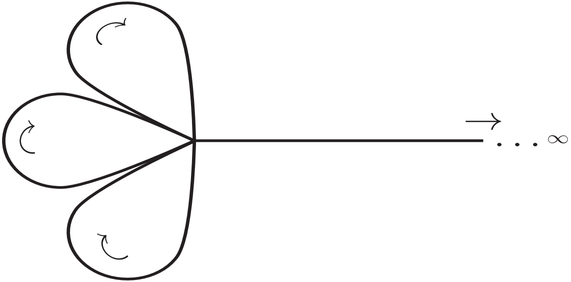
$$\mu'(\omega) > 0 \;\; \text{for} \;\; \omega \in (-\infty, \omega_1) \quad \text{and} \quad \mu'(\omega) < 0 \;\; \text{for} \;\; \omega \in (\omega_1, 0)$$

and

$$\mu(\omega) \notin (\mu_{\mathbb{R}+}, \mu_{\mathbb{R}}] \;\; \text{for} \;\; \omega \in (-\infty, \omega_0) \quad \text{and} \quad \mu(\omega) \in (\mu_{\mathbb{R}+}, \mu_{\mathbb{R}}] \;\; \text{for} \;\; \omega \in [\omega_0, 0).$$



Extension: flower graph with N loops



Theorem (Kairzhan–Marangell–Pelinovsky–Xiao, JDE, 2021)

For every $\omega < 0$, there exists only one positive symmetric state $\Phi \in H^2_\Gamma$ which satisfies the stationary NLS equation (cubic case). Moreover,

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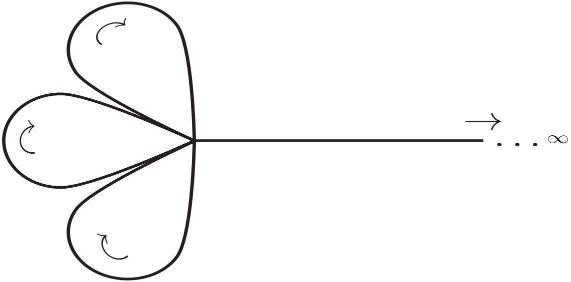

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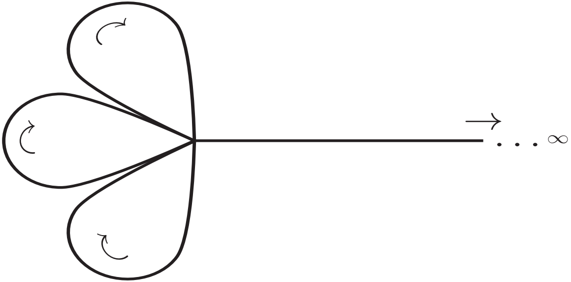
For every $\omega < 0$, there exists only one positive symmetric state $\Phi \in H^2_\Gamma$ which satisfies the stationary NLS equation (cubic case). Moreover,

- *The map $(-\infty, 0) \ni \omega \mapsto \Phi(\cdot, \omega) \in H^2_\Gamma$ is C^1 and the map $(-\infty, 0) \ni \omega \mapsto \mu(\omega) \in (0, \infty)$ is one-to-one, onto, and decreasing.*

Extension: flower graph with N loops



Theorem (Kairzhan–Marangell–Pelinovsky–Xiao, JDE, 2021)

For every $\omega < 0$, there exists only one positive symmetric state $\Phi \in H^2_\Gamma$ which satisfies the stationary NLS equation (cubic case). Moreover,

- *The map $(-\infty, 0) \ni \omega \mapsto \Phi(\cdot, \omega) \in H^2_\Gamma$ is C^1 and the map $(-\infty, 0) \ni \omega \mapsto \mu(\omega) \in (0, \infty)$ is one-to-one, onto, and decreasing.*
- *There exists $\omega_* \in (-\infty, 0)$ such that $\dim\mathrm{Ker}(\mathcal{L}) = N - 1$ for $\omega = \omega_*$. Morse index $n(\mathcal{L}) = N$ for $\omega \in (-\infty, \omega_*)$; $n(\mathcal{L}) = 1$ for $\omega \in [\omega_*, 0)$.*

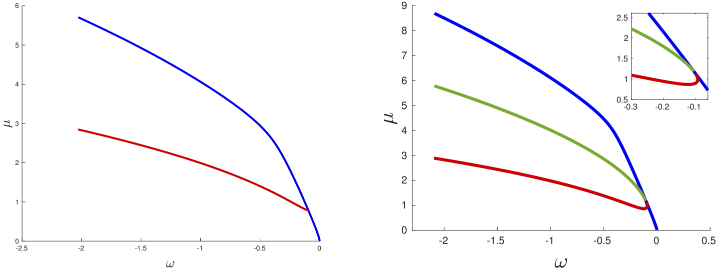


Figure: The bifurcation diagram of positive states on the parameter plane (ω, μ) for $N = 2$ (left) and $N = 3$ (right).

- Blue line is the positive symmetric state Φ.
- Red line is the positive state with one component having larger amplitude than the other components.
- Green line (for $N = 3$) is the positive state with two components having larger amplitudes than the third one.

Dynamical characterization: symmetric state

Recall the period function

$$L\varepsilon = T(U_0) := \int_{U_0}^{U_+} \frac{du}{\sqrt{E_0 + u^2 - u^4}},$$

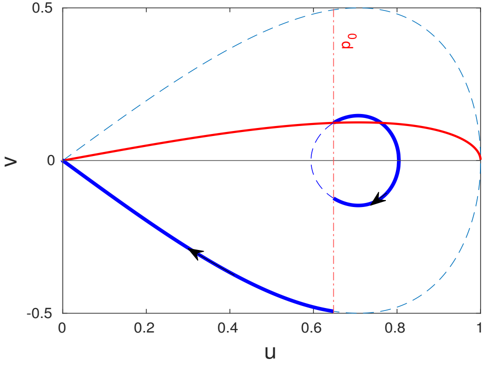
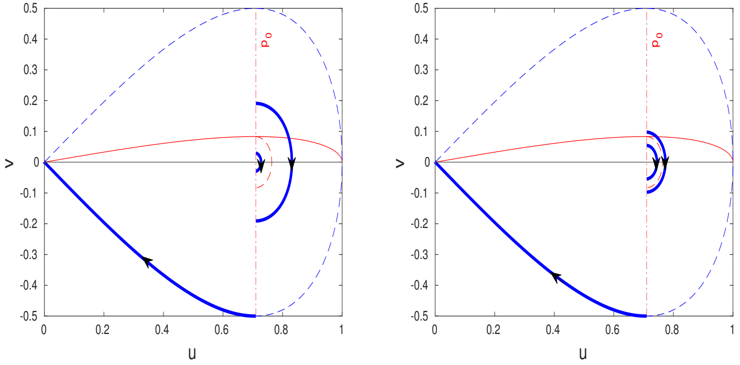


Figure: Geometric construction of the positive symmetric state on the phase plane.

Dynamical characterization: bifurcating states

If $U_0 > U_*$, where $(U_*, 0)$ is the center point, the symmetric state splits into bifurcating states. Here $N = 3$ and the left figure corresponds to the state with one large component and the right figure corresponds to the state with two large components.

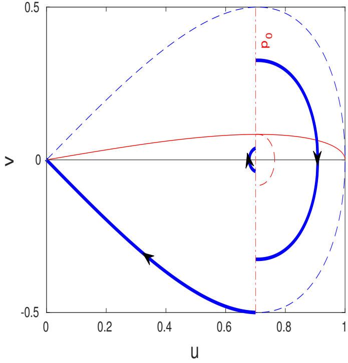


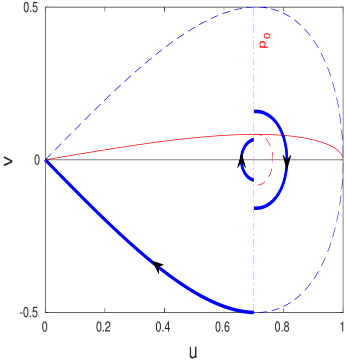
Dynamical characterization: bifurcating states

If $U_0 < U_*$, where $(U_*, 0)$ is the center point, then the smaller components flip. This can be characterized with two period functions

$$T_+(U_0, V_0) := \int_{U_0}^{U_+} \frac{du}{\sqrt{E_0 + u^2 - u^4}}, \quad T_-(U_0, V_0) := \int_{U_-}^{U_0} \frac{du}{\sqrt{E_0 + u^2 - u^4}},$$

where the turning points $U_\pm$ solves $E_0 + U_\pm^2 - U_\pm^4 = 0$ and (U_0, V_0) determines the energy level $E_0 = V_0^2 - U_0^2 + U_0^4$.





Summary

Dynamical construction of positive stationary states is based on:

- Periodic and homoclinic orbits on the phase plane connected together according to the Neumann-Kirchhoff boundary conditions;
- Parameterization is provided from the period function;
- Characterization of the Morse index and local stability properties follow from analysis of the period function.

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Thank you! Questions ???