Periodic oscillations in the Gross-Pitaevskii equation with a parabolic potential

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Introduction

Density waves in Bose–Einstein condensates are modeled by the Gross-Pitaevskii equation

$$iU_T = -\frac{1}{2}U_{XX} + \gamma^2 X^2 U + \nu V(X)U + \sigma|U|^2 U,$$

where $V(X)$ is a bounded potential on $\mathbb{R}$, γ and ν are real-valued strength constants for the parabolic and bounded potentials, and $\sigma = \pm 1$.

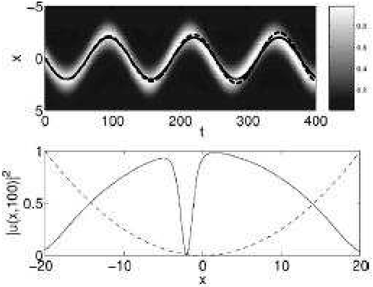
Examples of $V(X)$:

- $V(X+L) = V(X)$ for optical lattice with period L
- $|V(x)| \leq Ce^{-\kappa|x|}$ for red-detuned laser beam or all-optical trappings

If $\gamma = \nu = 0$ and $\sigma = +1$, the Gross–Pitaevskii equation becomes the defocusing NLS equation with a dark soliton $U(X,T) = e^{-iT}\tanh(X)$.

The problem

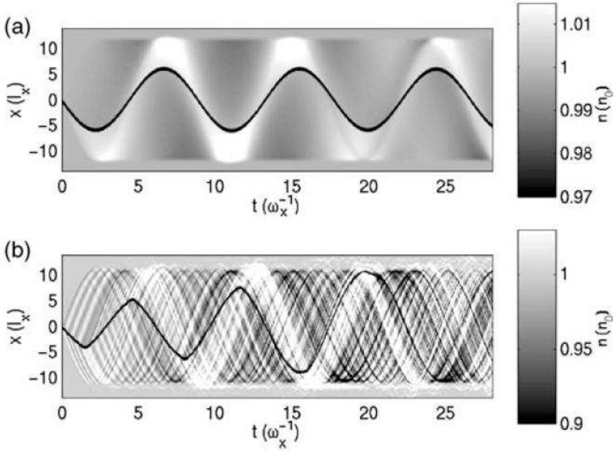
Numerical pictures (D.P., P.K., D. Franzeskakis, Phys. Rev. E **72** 016615 2005):



Main question is to find the frequency of oscillations and the change in the amplitude of oscillations if the oscillations are not periodic.

Numerical results

Numerical pictures (N. Parker, N. Proukakis, et al., 2004):



Top picture : periodic oscillations for $\gamma \neq 0$ and $\nu = 0$
Bottom picture : oscillations of increasing amplitude for $\gamma, \nu \neq 0$

Background

Let us consider the normalized Gross–Pitaevskii equation in the form

$$iu_t = -\frac{1}{2}u_{xx} + \frac{1}{2}x^2 u + \delta W(x)u + \sigma|u|^2 u,$$

where δ is small and $W(x)$ is an external potential.

Theorem (Carles, 2002): If $W \in L^2(\mathbb{R})$, there exists a global solution $u \in C^1(\mathbb{R}, \mathcal{H}_1(\mathbb{R}))$ of the GP equation in space

$$\mathcal{H}_1(\mathbb{R}) = \{u \in H^1(\mathbb{R}) : \quad xu \in L^2(\mathbb{R})\}$$

Stationary solutions have the form

$$u(x,t) = e^{-\frac{i}{2}t - i\mu t}\phi(x),$$

where $\phi : \mathbb{R} \mapsto \mathbb{R}$ solves

$$\mathcal{L}\phi(x) + \delta W(x)\phi(x) + \sigma\phi^3(x) = \mu\phi(x),$$

and $\mathcal{L} = (-\partial_x^2 + x^2 - 1)/2$.

Stationary solutions

We consider localized solutions $\phi(x)$ with a single zero on $\mathbb{R}$.

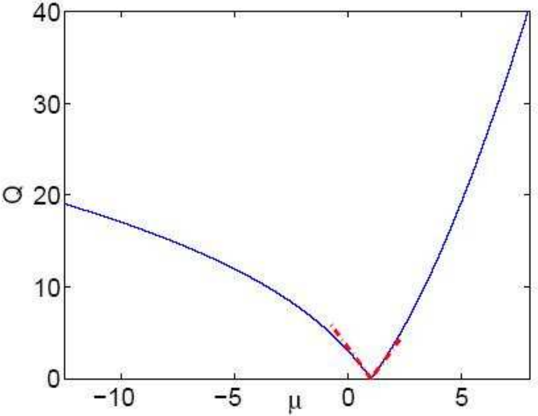
Since the Schrödinger operator $\mathcal{L}$ has an eigenvalue $\mu = 1$ with the eigenfunction $\phi = \varepsilon x e^{-x^2/2}$, the local bifurcation analysis gives the existence result.

Theorem: There exists $\varepsilon_0 > 0$ and $\delta_0 > 0$, such that the ODE for $\phi(x)$ admits a unique family of solutions for any $\varepsilon \in [0, \varepsilon_0)$ and $\delta \in [0, \delta_0)$ with the property

$$\|\phi - \varepsilon x e^{-x^2/2}\|_{\mathcal{H}_1} \leq C_1 \varepsilon \left(\delta + \varepsilon^2\right), \quad |\mu - 1| \leq C_2 \left(\delta + \varepsilon^2\right),$$

for some (ε,δ)-independent constants $C_1, C_2 > 0$.

Approximation of stationary solutions for $\delta = 0$



If $\sigma = 1$ (defocusing case), then $\mu > 1$.
If $\sigma = -1$ (focusing case), then $\mu < 1$.

Linearization of stationary solutions

If

$$u(x,t) = e^{-\frac{i}{2}t - i\mu t}\left(\phi(x) + (v(x) - w(x))\,e^{i\Omega t} + (\bar{v}(x) + \bar{w}(x))\,e^{-i\bar{\Omega} t}\right),$$

then $u(x)$ and $w(x)$ satisfy the linearized problem

$$\begin{aligned} \left(\mathcal{L} + \delta W(x) + 3\sigma\phi^2(x) - \mu\right) v(x) &= \Omega w(x), \\ \left(\mathcal{L} + \delta W(x) + \sigma\phi^2(x) - \mu\right) w(x) &= \Omega v(x). \end{aligned}$$

When $\varepsilon = 0$ and $\delta = 0$, the spectrum of the linearized problem consists of the double eigenvalue $\Omega = 0$, the pair of double eigenvalues $\Omega = \pm 1$, and the pairs of simple eigenvalues $\Omega = \pm m$, $m \geq 2$.

We shall prove for $\sigma = 1$ that the double eigenvalue $\Omega = 0$ is preserved, the pair $\Omega = \pm 1$ split into the eigenvalue $\Omega_0 = 1$ and $\Omega_1 < 1$ and the pairs $\Omega = \pm m$ shift to $\Omega_m < m$. As a result, the Lyapunov theorem on persistence of periodic orbits implies the following result.

Main result

Theorem: If $\delta = 0$ or if $\delta = \delta_*(\varepsilon)$ near $(\varepsilon, \delta) = (0,0)$, then there exists a family of time-periodic space-localized solutions in the form

$$u(x,t) = e^{-\frac{i}{2}t - i\mu t - i\theta_0} v(x,t)$$

with the properties:
(1) $v \in \mathcal{H}_1(\mathbb{R})$ for any $t \in \mathbb{R}$,
(2) $v\left(x, t + \frac{2\pi}{\Omega}\right) = v(x,t)$ for all $(x,t) \in \mathbb{R}^2$,
(3) $|\Omega - 1| \leq C_0 \varepsilon^2 s^2$, and
(4) $\|v(\cdot,t) - \phi(x) - s\phi'(x)\cos(\Omega t + \varphi_0) - isx\phi(x)\sin(\Omega t + \varphi_0)\|_{\mathcal{H}_1} \leq C\varepsilon s^2$,
where $s \in [0, s_0)$ for some $s_0 > 0$, θ_0 and φ_0 are arbitrary parameters, and C_0, C are (ε, s)-independent positive constants.

Remark: Parameters θ_0 and φ_0 can be set to zero because of the symmetries of the GP equation. Parameter s measures a small amplitude of periodic oscillations.

Exact periodic solution

The result for $\delta = 0$ is trivial because of the existence of exact periodic solutions for any $\varepsilon \in \mathbb{R}$ and any $s \in \mathbb{R}$. Moreover, $C_0 = 0$ in the bound $|\Omega - 1| \leq C_0 \varepsilon^2 s^2$ such that $\Omega = 1$ in the exact periodic solution.

The exact solution is constructed with an explicit transformation for the GP equation for $\delta = 0$:

$$u(x,t) = e^{ip(t)x - \frac{i}{2}p(t)s(t) - \frac{i}{2}t - i\mu t} \phi(x - s(t)),$$

where $\dot{s} = p$, $\dot{p} = -s$, such that $\ddot{s} + s = 0$ and

$$s(t) = s_0 \cos(t + \varphi_0), \qquad p(t) = -s_0 \sin(t + \varphi_0),$$

for any $s_0 \in \mathbb{R}$ and $\varphi_0 \in \mathbb{R}$.

The exact periodic solution does not exist for $\delta \neq 0$. Our result shows that the same family of periodic solutions bifurcates at $\delta = \delta_*(\varepsilon) \neq 0$.

Background history

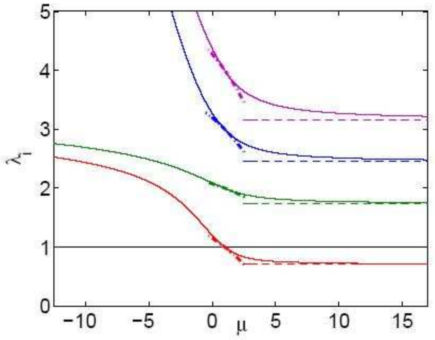
Oscillations of the GP equation

$$i\psi_t = -\frac{1}{2}\psi_{xx} + \epsilon^2 x^2 \psi + |\psi|^2 \psi,$$

have been studied in physics literature in the past ten years for small ϵ.

- $\Omega = 1$ is obtained with the Ehrenfest Theorem (Reinhardt and Clark, 1997; Morgan et al., 1997)
- $\Omega = \frac{1}{\sqrt{2}}$ is obtained with boundary-layer integrals (Busch and Anglin, 2000); small-wave expansions (Huang, 2002); perturbation theory for dark solitons (Brazhnyi and Konotop, 2003)
- Both frequencies are present in the spectrum of the limiting problem with $\epsilon \to 0$ ($\mu \to \infty$).

Numerical approximation of eigenvalues for $\delta = 0$



$$\sigma = 1: \quad \Omega_0 = 1, \quad \lim_{\mu\to\infty} \Omega_1 = \frac{1}{\sqrt{2}}, \quad \lim_{\mu\to\infty} \Omega_m = \frac{\sqrt{m(m+1)}}{\sqrt{2}}, \qquad \forall m \geq 2$$

Non-resonance condition $n\Omega_1 \neq \Omega_m$ is not satisfied in the limit $n = m \to \infty$.

Hamiltonian lattice

Schrödinger operator $\mathcal{L} = \frac{1}{2}\left(-\partial_x^2 + x^2 - 1\right)$ has a complete set of eigenfunctions called Hermite functions

$$\phi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-x^2/2}, \qquad \forall n = 0, 1, 2, 3, ...,$$

where $H_n(x)$ are the Hermite polynomials.

Let $u(x,t) = e^{-\frac{i}{2}t} \sum_{n=0}^{\infty} a_n(t)\phi_n(x)$ and convert the PDE problem to the discrete Hamiltonian system

$$i\dot{a}_n = n a_n + \delta \sum_{m=0}^{\infty} W_{n,m} a_m + \sigma \sum_{n_1,n_2,n_3=0}^{\infty} K_{n,n_1,n_2,n_3} a_{n_1} \bar{a}_{n_2} a_{n_3},$$

where $W_{n,m} = (\phi_n, W\phi_m)$ and $K_{n,n_1,n_2,n_3} = (\phi_n, \phi_{n_1}\phi_{n_2}\phi_{n_3})$.

Phase space of the dynamical system

Lemma: Let $u(x) = \sum_{m=0}^{\infty} a_n \phi_n(x)$. Then $u \in \mathcal{H}_1(\mathbb{R})$ if and only if $\mathbf{a} \in l^2_{1/2}(\mathbb{N})$.

Lemma: The vector field $\mathbf{F}(\mathbf{a})$ of the discrete system maps $l^2_{1/2}(\mathbb{N})$ to $l^2_{-1/2}(\mathbb{N})$.

Theorem: The discrete system $i\dot{\mathbf{a}} = \mathbf{F}(\mathbf{a})$ is globally well-posed in $l^1_{1/2}(\mathbb{N})$.

Decomposition: Let $\mathbf{a}(t) = e^{-i\mu t}\left[\mathbf{A} + \mathbf{B}(t) + i\mathbf{C}(t)\right]$ and rewrite the system in the form

$$\dot{\mathbf{B}} = L_-\mathbf{C} + \sigma\mathbf{N}_-(\mathbf{B}, \mathbf{C}), \qquad -\dot{\mathbf{C}} = L_+\mathbf{B} + \sigma\mathbf{N}_+(\mathbf{B}, \mathbf{C}),$$

where $\mathbf{N}_\pm(\mathbf{B}, \mathbf{C})$ contains quadratic and cubic terms with respect to $\mathbf{B}$ and $\mathbf{C}$. If $\|\mathbf{B}(t)\|_{l^2_{1/2}} + \|\mathbf{C}(t)\|_{l^2_{1/2}} \leq C\varepsilon s$, then

$$\|\mathbf{N}_\pm(\mathbf{B}(t), \mathbf{C}(t))\|_{l^2_{-1/2}} \leq C_\pm \varepsilon^3 s^2$$

for some $C, C_\pm > 0$.

Using the series of eigenvectors (if all but zero eigenvalues are simple),

$$\left\{ \begin{array}{rcl} \mathbf{B}(t) & = & \sum_{m=0}^{\infty} b_m(t)\mathbf{B}_m + \sum_{m=0}^{\infty} \bar{b}_m(t)\mathbf{B}_m + \alpha(t)\partial_\mu \mathbf{A}, \\ \mathbf{C}(t) & = & i\sum_{m=0}^{\infty} b_m(t)\mathbf{C}_m - i\sum_{m=0}^{\infty} \bar{b}_m(t)\mathbf{C}_m + \beta(t)\mathbf{A}, \end{array} \right.$$

we block-diagonalize the system in the form

$$\dot{b}_m - i\Omega_m b_m = \sigma N_m(b_0, \mathbf{b}, \alpha, \beta), \quad m \geq 0$$

$$\dot{\alpha} = \sigma S_0(b_0, \mathbf{b}, \alpha, \beta), \quad \dot{\beta} + \alpha = \sigma S_1(b_0, \mathbf{b}, \alpha, \beta).$$

We are looking for T-periodic C^1 functions $b_0(t)$, $\mathbf{b}(t)$, $\alpha(t)$ and $\beta(t)$.

If constant $Q_A = Q - \|\mathbf{A}\|_{l^2}^2$ is found from

$$Q_A = \frac{1}{T}\int_0^T \left(\|\mathbf{B}\|_{l^2}^2 + \|\mathbf{C}\|_{l^2}^2 - 2\sigma\langle \partial_\mu \mathbf{A}, \mathbf{N}_+(\mathbf{B},\mathbf{C})\rangle \right) dt,$$

then there exists a unique T-periodic solution for $\alpha(t)$ and $\beta(t)$ such that

$$|\alpha(t)| \leq \varepsilon^2 s^2 C_\alpha, \quad |\beta(t)| \leq \varepsilon^2 s^2 C_\beta, \quad |Q_A| \leq C_Q \varepsilon^2 s^2,$$

for some $C_\alpha, C_\beta, C_Q > 0$.

Oscillatory components of the solution

Since $\Omega_m - m = \mathrm{O}(\varepsilon^2)$ uniformly in $m \in \mathbb{N}$, the Implicit Function Theorem in space $C^1_{\rm per}(\mathbb{R}, l^2_{1/2}(\mathbb{N})) \times C^1_{\rm per}(\mathbb{R})$ implies that there exists a unique T-periodic solution $\mathbf{b}(t) \in l^2_{1/2}(\mathbb{N})$ for any T-periodic function $b_0(t)$ such that if $|b_0(t)| \leq \varepsilon s C_0$, then

$$\|\mathbf{b}(t)\|_{l^2_{1/2}} \leq \varepsilon s^2 C_b$$

for some $C_0, C_b > 0$.

We are left with a reduced evolution equation

$$\dot{b}_0 = i b_0 + R(b_0),$$

where

$$R(b_0) = \varepsilon \left[i K_1(\varepsilon) b_0^2 + i K_2(\varepsilon) \bar{b}_0^2 + i K_3(\varepsilon) |b_0|^2 \right] + \mathrm{O}\left(|b_0|^3, \varepsilon |b_0| \|\mathbf{b}\| \right).$$

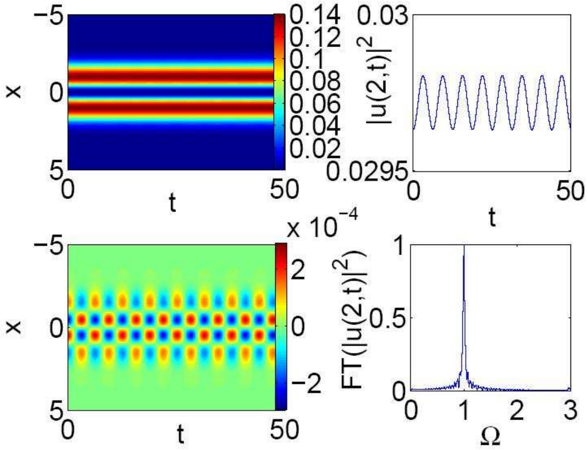
Persistence of the T-periodic solution $b_0(t) \sim \varepsilon s e^{it + i\varphi_0}$ is proved with the normal form analysis, which gives

$$|\Omega - 1| \leq C_\Omega \varepsilon^2 s^2$$

for some $C_\Omega > 0$.

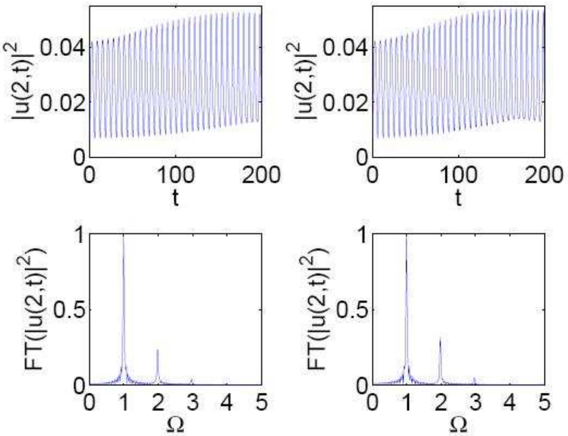
Numerical simulations for $\delta = 0$, $\sigma = 1$

$u(x,0) = \phi(x) + s\phi'(x)$,



Numerical simulations for $\delta \neq 0$, $\sigma = 1$

$u(x,0) = \phi(x) + sw(x)$ for $\delta = 0.05$ and $\delta = 0.15$:



Conclusion

Summary: Two-period quasi-periodic oscillations exist typically along a Cantor set of parameter values. We have proven persistence of the two-periodic solutions along a continuous set of parameter values. These solutions are spectrally stable with respect to the linearization but are structurally stable with respect to perturbations of the external potential potential.

Other projects:

- Well-posedness of time evolution and Birkhoff normal forms for *n*-tori in fractional spaces $\mathcal{H}_s$ and $l^2_{s/2}$ (W. Craig, Z. Yan)
- Rigorous analysis of eigenvalues in the Thomas–Fermi asymptotic limit $\mu \to \infty$ (C. Gallo, D. P.)
- Persistence of oscillations with $\Omega = \frac{1}{\sqrt{2}}$ or quasi-periodic oscillations with $\Omega_0 = 1$ and $\Omega_1 = \frac{1}{\sqrt{2}}$