Transverse instabilities of deep-water solitons

Dmitry Pelinovsky (McMaster University, Canada)

Joint work with Bernard Deconinck (University of Washington)

and John Carter (Seattle University)

International Conference on Nonlinear Science

Beijing, China, August 9, 2006

Formulation of the problem

Hyperbolic two-dimensional NLS equation

$$i\psi_t + \psi_{xx} - \psi_{yy} + 2|\psi|^2\psi = 0,$$

where $\psi : \mathbb{R}^2 \times \mathbb{R}_+ \mapsto \mathbb{C}$ is the envelope amplitude and $\eta = \mathrm{Re}(\psi e^{ik_0x - i\omega_0 t})$ is the elevation of the water wave surface.

Formulation of the problem

Hyperbolic two-dimensional NLS equation

$$i\psi_t + \psi_{xx} - \psi_{yy} + 2|\psi|^2\psi = 0,$$

where $\psi : \mathbb{R}^2 \times \mathbb{R}_+ \mapsto \mathbb{C}$ is the envelope amplitude and $\eta = \mathrm{Re}(\psi e^{ik_0 x - i\omega_0 t})$ is the elevation of the water wave surface.

A line soliton solution

$$\psi = e^{it}\mathrm{sech}(x), \qquad \eta = \cos(k_0 x - (\omega_0 - 1)t)\mathrm{sech}(x),$$

where four parameters (phase translation, coordinate translation, frequency, velocity) can be constructed by using the Lie point symmetries of the NLS equation.

Formulation of the problem

Hyperbolic two-dimensional NLS equation

$$i\psi_t + \psi_{xx} - \psi_{yy} + 2|\psi|^2\psi = 0,$$

where $\psi : \mathbb{R}^2 \times \mathbb{R}_+ \mapsto \mathbb{C}$ is the envelope amplitude and $\eta = \mathrm{Re}(\psi e^{ik_0 x - i\omega_0 t})$ is the elevation of the water wave surface.

A line soliton solution

$$\psi = e^{it}\mathrm{sech}(x), \qquad \eta = \cos(k_0 x - (\omega_0 - 1)t)\mathrm{sech}(x),$$

where four parameters (phase translation, coordinate translation, frequency, velocity) can be constructed by using the Lie point symmetries of the NLS equation.

Question: Is the line soliton stable with respect to transverse (y-dependent) perturbations?

Linearized stability and instability

Consider a linear perturbation to the line soliton

$$\psi = (\mathrm{sech}(x) + \epsilon u(x,y,t) + i\epsilon v(x,y,t))\, e^{it}$$

and find the linear PDEs:

$$-u_t = v_{xx} - v_{yy} + (2\mathrm{sech}^2 x - 1)v, \;\; v_t = u_{xx} - u_{yy} + (6\mathrm{sech}^2 x - 1)u$$

Use the Fourier transform in y and Laplace transform in t, e.g.

$$u = U(x)e^{i\rho y}e^{\Omega t}, \;\; v = V(x)e^{i\rho y}e^{\Omega t}, \qquad \rho \in \mathbb{R}, \;\; \Omega \in \mathbb{C}$$

where $(U,W) \in L^2(\mathbb{R})$ is an eigenvector of Schrödinger operators

$$\Omega U = (L_- - \rho^2)V, \qquad -\Omega V = (L_+ - \rho^2)U.$$

The line soliton is transversely unstable if $\mathrm{Re}(\Omega) > 0$ for $\rho \neq 0$.

History of analytical studies

- Zakharov–Rubenchik (1974), Yajima (1974): splitting of zero eigenvalues for small ρ into a pair of real eigenvalues $\pm\Omega_1$ and a pair of purely imaginary eigenvalues $\pm\Omega_2$:

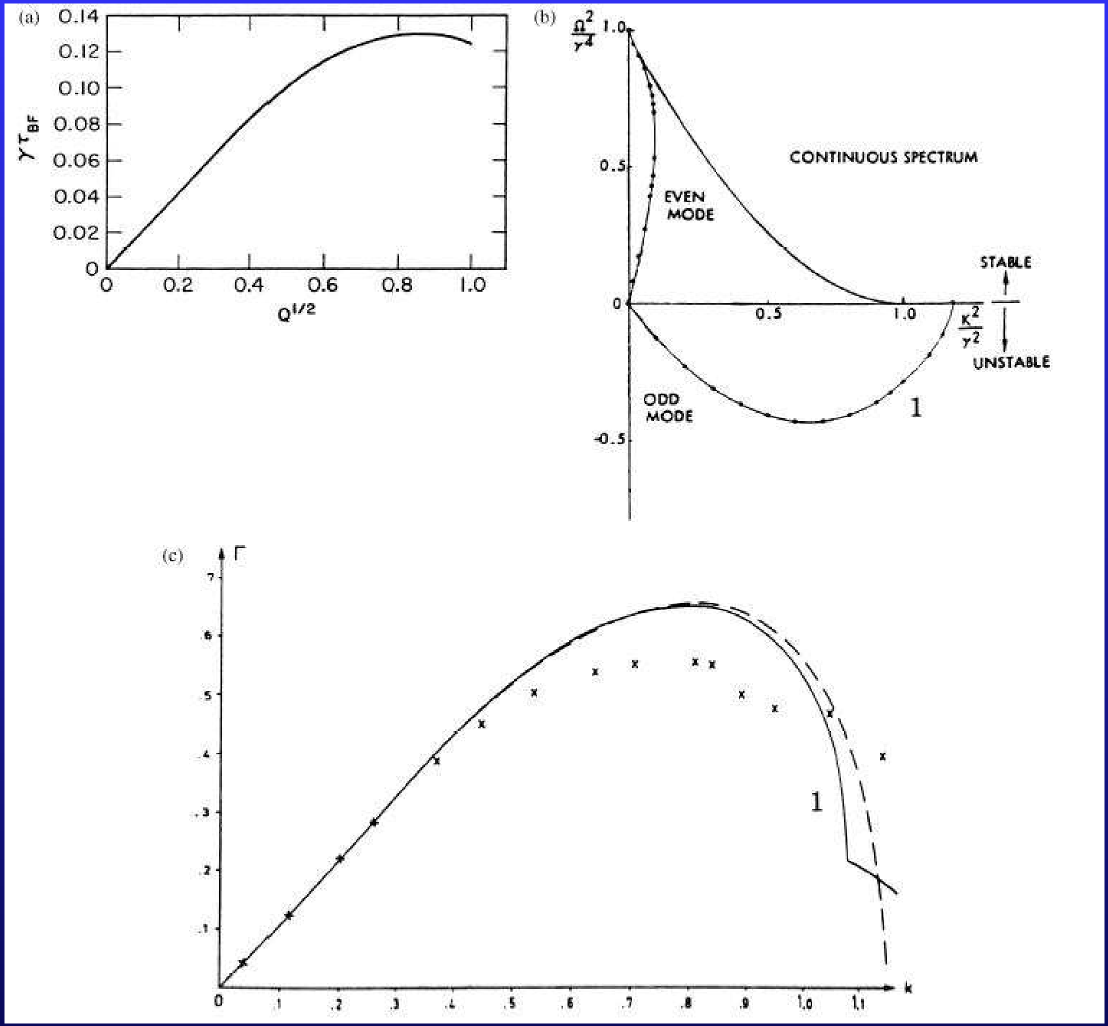
$$\Omega_1^2 = \frac{4}{3}\rho^2 + \mathrm{O}(\rho^4), \qquad \Omega_2^2 = -4\rho^2 + \mathrm{O}(\rho^4).$$

- Ablowitz–Segur (1979): no real eigenvalues exist for large ρ as $\Omega^2 = -\rho^4 + \mathrm{O}(\rho^2)$.

- Anderson et al. (1979): existence of complex eigenvalues for values of $\rho \sim 1$ in the Rayleigh–Ritz variational method.

- Kivshar–Pelinovsky (2000): nonlinear theory of break-up of line soliton into dispersive clusters.

History of numerical studies



Top left: Cohen et al. (1976). Top right: Saffman–Yuen (1978).
Bottom: Anderson et al. (1979).

Summary of our new results

- Numerical analysis of **all** isolated eigenvalues and their bifurcations
- Prediction of instabilities for **any** $\rho > 0$.
- Analytical **proof** of existence of unstable eigenvalues for $0 < \rho < 1$
- Rigorous **analysis** of bifurcations of unstable eigenvalues from $\rho = 1$ to $\rho > 1$.

B. Deconinck, D.P, J. Carter, Proc. Roy. Soc. A **462**, 2039 (2006)

D.P., Math. Comp. Simul. **55**, 585 (2001)

Numerical Hill's method

Consider the spectral problem

$$\phi_x = (A(x) + \lambda)\,\phi, \qquad \phi \in \mathbb{C}^n$$

where $A(x+L) = A(x)$ and $\lambda \in \mathbb{C}$. We are looking for eigenvalues λ when eigenvector ϕ is in $L^\infty([0,L])$ space.

Numerical Hill's method

Consider the spectral problem

$$\phi_x = (A(x) + \lambda)\,\phi, \qquad \phi \in \mathbb{C}^n$$

where $A(x+L) = A(x)$ and $\lambda \in \mathbb{C}$. We are looking for eigenvalues λ when eigenvector ϕ is in $L^\infty([0, L])$ space.

Floquet's Theorem: There exists a constant $n \times n$ matrix R and L-periodic $n \times n$ matrix P, such that the fundamental matrix solution $\phi(x)$ is $\phi(x) = P(x)e^{Rx}$.

Numerical Hill's method

Consider the spectral problem

$$\phi_x = (A(x) + \lambda)\,\phi, \qquad \phi \in \mathbb{C}^n$$

where $A(x+L) = A(x)$ and $\lambda \in \mathbb{C}$. We are looking for eigenvalues λ when eigenvector ϕ is in $L^\infty([0, L])$ space.

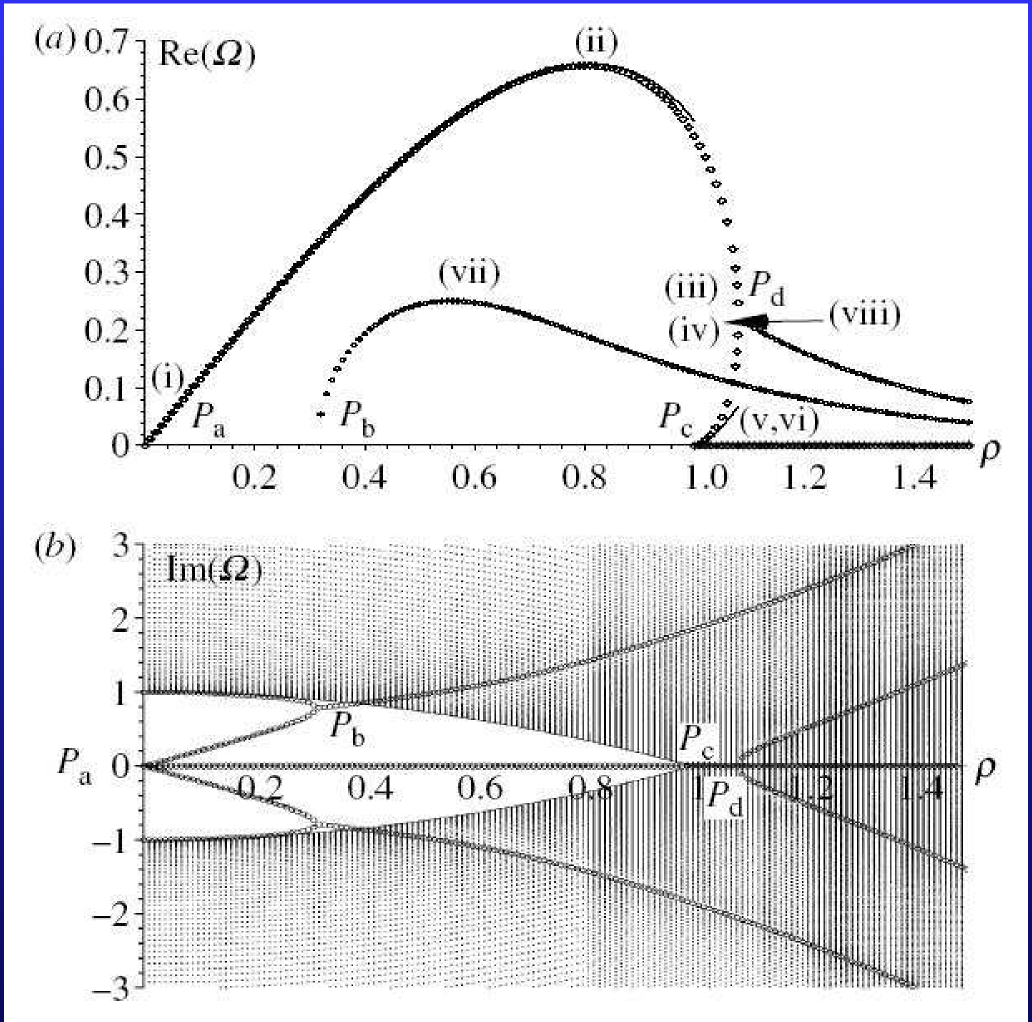
Floquet's Theorem: There exists a constant $n \times n$ matrix R and L-periodic $n \times n$ matrix P, such that the fundamental matrix solution $\phi(x)$ is $\phi(x) = P(x)e^{Rx}$.

The bounded eigenvector can be decomposed as

$$\phi(x) = e^{i\mu x} \sum_{k \in \mathbb{Z}} \phi_k e^{2\pi i x/L},$$

where $\mu \in \left[-\frac{\pi}{L}, \frac{\pi}{L}\right]$. For each value of μ, the spectrum of λ can be found by truncation of Fourier series.

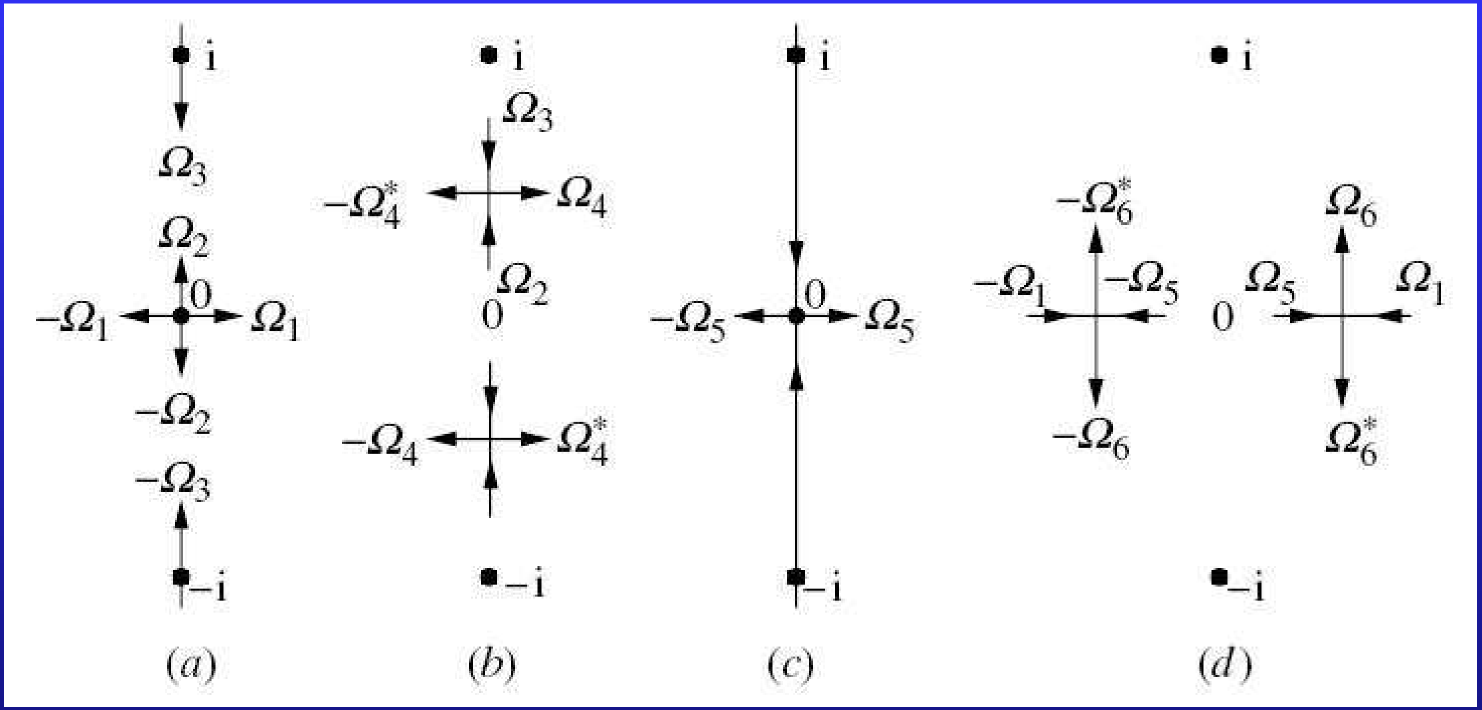
Numerical analysis of all eigenvalues



Conclusion: Line soliton is unstable for any $\rho > 0$

Bifurcations: $\rho = 0$, $\rho \approx 0.31$, $\rho = 1$, $\rho \approx 1.08$.

Numerical analysis of all bifurcations



Bifurcations:

(a) $\rho = 0$ - two pairs arise from multiple eigenvalue $\Omega = 0$
one pair arises from the end point of the continuous spectrum

(b) $\rho \approx 0.31$ - Hamiltonian-Hopf bifurcation

(c) $\rho = 1$ - collision of end points of the continuous spectrum

(d) $\rho \approx 1.08$ - double real eigenvalue bifurcation

Proof of existence of unstable eigenvalues

Consider the spectral problem

$$(L_+ - \rho^2)U = -\Omega V, \qquad (L_- - \rho^2)V = \Omega U$$

$$L_+ = -\partial_x^2 + 1 - 6\mathrm{sech}^2 x, \qquad L_- = -\partial_x^2 + 1 - 2\mathrm{sech}^2 x.$$

Proof of existence of unstable eigenvalues

Consider the spectral problem

$$(L_+ - \rho^2)U = -\Omega V, \qquad (L_- - \rho^2)V = \Omega U$$

$$L_+ = -\partial_x^2 + 1 - 6\,\mathrm{sech}^2 x, \qquad L_- = -\partial_x^2 + 1 - 2\,\mathrm{sech}^2 x.$$

Spectra $\sigma(L_\pm)$ in $L^2(\mathbb{R})$:

- $\sigma(L_+)$ - two isolated eigenvalues at $\sigma = -3$ and $\sigma = 0$ and continuous spectrum for $\sigma \geq 1$
- $\sigma(L_-)$ - one isolated eigenvalue at $\sigma = 0$ and continuous spectrum for $\sigma \geq 1$

Proof of existence of unstable eigenvalues

Consider the spectral problem

$$(L_+ - \rho^2)U = -\Omega V, \qquad (L_- - \rho^2)V = \Omega U$$

$$L_+ = -\partial_x^2 + 1 - 6\text{sech}^2 x, \qquad L_- = -\partial_x^2 + 1 - 2\text{sech}^2 x.$$

Spectra $\sigma(L_\pm)$ in $L^2(\mathbb{R})$:

- $\sigma(L_+)$ - two isolated eigenvalues at $\sigma = -3$ and $\sigma = 0$ and continuous spectrum for $\sigma \geq 1$
- $\sigma(L_-)$ - one isolated eigenvalue at $\sigma = 0$ and continuous spectrum for $\sigma \geq 1$

For $0 < \rho < 1$, the spectral problem is equivalent to the generalized eigenvalue problem

$$\left(L_+ - \rho^2\right) U = \gamma \left(L_- - \rho^2\right)^{-1} U, \qquad \gamma = -\Omega^2$$

Sylvester–Pontryagin–Grillakis Theorem

Theorem: Let L and M be a self-adjoint operators in $L^2(\mathbb{R})$ with finitely many negative eigenvalues $n(L)$ and $n(M)$ and empty kernels. Then, there are exactly $n(L)$ and $n(M)$ eigenvalues γ of $Lu = \gamma M u$ in $L^2(\mathbb{R})$ such that $(u, Lu) \leq 0$ and $(u, Mu) \leq 0$.

Sylvester–Pontryagin–Grillakis Theorem

Theorem: Let L and M be a self-adjoint operators in $L^2(\mathbb{R})$ with finitely many negative eigenvalues $n(L)$ and $n(M)$ and empty kernels. Then, there are exactly $n(L)$ and $n(M)$ eigenvalues γ of $Lu = \gamma M u$ in $L^2(\mathbb{R})$ such that $(u, Lu) \leq 0$ and $(u, Mu) \leq 0$.

Application:

$$\left(L_+ - \rho^2\right) U = \gamma \left(L_- - \rho^2\right)^{-1} U, \qquad \gamma = -\Omega^2$$

For $0 < \rho < 1$, there exist two eigenvalues γ such that $(U, (L_+ - \rho^2) U) \leq 0$ and one eigenvalue γ such that $(U, (L_- - \rho^2)^{-1} U) \leq 0$.

Sylvester–Pontryagin–Grillakis Theorem

Theorem: Let L and M be a self-adjoint operators in $L^2(\mathbb{R})$ with finitely many negative eigenvalues $n(L)$ and $n(M)$ and empty kernels. Then, there are exactly $n(L)$ and $n(M)$ eigenvalues γ of $Lu = \gamma Mu$ in $L^2(\mathbb{R})$ such that $(u, Lu) \leq 0$ and $(u, Mu) \leq 0$.

Application:

$$\left(L_+ - \rho^2\right) U = \gamma \left(L_- - \rho^2\right)^{-1} U, \qquad \gamma = -\Omega^2$$

For $0 < \rho < 1$, there exist two eigenvalues γ such that $(U, (L_+ - \rho^2)U) \leq 0$ and one eigenvalue γ such that $(U, (L_- - \rho^2)^{-1}U) \leq 0$.

Conclusion: One eigenvalue $\Omega \in \mathbb{R}_+$ is always unstable for $0 < \rho < 1$, and two eigenvalues are either purely imaginary for $0 < \rho < 0.31$ or complex for $0.31 < \rho < 1$.

Bifurcations of unstable eigenvalues

Let $\rho^2 = 1 - \frac{\kappa_+^2 + \kappa_-^2}{2}$, $\Omega = \frac{\kappa_+^2 - \kappa_-^2}{2\mathrm{i}}$ and rewrite the eigenvalue problem

$$\begin{aligned}\left(-\partial_x^2 - 6\mathrm{sech}^2 x\right) U &= -\frac{1}{2}(\kappa_+^2 + \kappa_-^2)U + \frac{i}{2}(\kappa_+^2 - \kappa_-^2)V, \\ \left(-\partial_x^2 - 2\mathrm{sech}^2 x\right) V &= -\frac{i}{2}(\kappa_+^2 - \kappa_-^2)U - \frac{1}{2}(\kappa_+^2 + \kappa_-^2)V,\end{aligned}$$

Bifurcations of unstable eigenvalues

Let $\rho^2 = 1 - \frac{\kappa_+^2 + \kappa_-^2}{2}$, $\Omega = \frac{\kappa_+^2 - \kappa_-^2}{2\mathrm{i}}$ and rewrite the eigenvalue problem

$$\begin{aligned} \left(-\partial_x^2 - 6\mathrm{sech}^2 x\right) U &= -\frac{1}{2}(\kappa_+^2 + \kappa_-^2)U + \frac{i}{2}(\kappa_+^2 - \kappa_-^2)V, \\ \left(-\partial_x^2 - 2\mathrm{sech}^2 x\right) V &= -\frac{i}{2}(\kappa_+^2 - \kappa_-^2)U - \frac{1}{2}(\kappa_+^2 + \kappa_-^2)V, \end{aligned}$$

- Bifurcation point $\rho = 1$, $\Omega = 0$ corresponds to $(\kappa_+, \kappa_-) = (0, 0)$
- Two solutions $\mathbf{u}_\pm(x)$ decay like $e^{\kappa_\pm x}$ as $x \to -\infty$ and two solutions $\mathbf{v}_\pm(x)$ decays like $e^{-\kappa_\pm x}$ as $x \to +\infty$ in the domain $\mathrm{Re}(\kappa_\pm) > 0$
- The coordinates $(\kappa_+, \kappa_-) \in \mathbb{C}^2$ unfold the branch point singularity in coordinates $\Omega \in \mathbb{C}$ and $\rho^2 - 1 \in \mathbb{R}$.

Evans function

Let $\mathbf{u}_\pm(x)$ and $\mathbf{v}_\pm(x)$ be fundamental solutions in the domain $\mathrm{Re}(\kappa_\pm) > 0$ and define the Wronskian determinant:

$$E(\kappa_+, \kappa_-) = \det\left(\mathbf{u}_+(x), \mathbf{u}_-(x), \mathbf{v}_+(x), \mathbf{v}_-(x)\right)\Big|_{x=0}.$$

Zeros of $E(\kappa_+, \kappa_-)$ coincide with eigenvalues of the original problem in the domain $\mathrm{Re}(\kappa_\pm) > 0$.

Evans function

Let $\mathbf{u}_\pm(x)$ and $\mathbf{v}_\pm(x)$ be fundamental solutions in the domain $\mathrm{Re}(\kappa_\pm) > 0$ and define the Wronskian determinant:

$$E(\kappa_+, \kappa_-) = \det\left(\mathbf{u}_+(x), \mathbf{u}_-(x), \mathbf{v}_+(x), \mathbf{v}_-(x)\right)|_{x=0}.$$

Zeros of $E(\kappa_+, \kappa_-)$ coincide with eigenvalues of the original problem in the domain $\mathrm{Re}(\kappa_\pm) > 0$.

- Fundamental solutions $\mathbf{u}_\pm(x)$, $\mathbf{v}_\pm(x)$, and their determinant $E(\kappa_+, \kappa_-)$ are analytic functions near $(\kappa_+, \kappa_-) = (0, 0)$.
- The Taylor series expansion holds near $(0, 0)$:

$$E(\kappa_+, \kappa_-) = -4(\kappa_+ + \kappa_-)^2 + 10(\kappa_+ + \kappa_-)^3 - 13(\kappa_+^4 + \kappa_-^4)$$
$$-51(\kappa_+^2 + \kappa_-^2)\kappa_+\kappa_- - 72\kappa_+^2\kappa_-^2 - \alpha_0(\kappa_+^2 - \kappa_-^2)^2 + \mathrm{O}(5),$$

where α_0 is a numerical coefficient.

Asymptotic result

Let $\alpha = \kappa_+ + \kappa_-$ and $\beta = \kappa_+ - \kappa_-$, such that

$$E(\alpha, \beta) = -4\alpha^2 + 10\alpha^3 - \frac{25}{2}\alpha^4 + \frac{1}{4}\beta^4 - \left(\alpha_0 + \frac{3}{4}\right)\alpha^2\beta^2 + \mathrm{O}(5).$$

Asymptotic result

Let $\alpha = \kappa_+ + \kappa_-$ and $\beta = \kappa_+ - \kappa_-$, such that

$$E(\alpha, \beta) = -4\alpha^2 + 10\alpha^3 - \frac{25}{2}\alpha^4 + \frac{1}{4}\beta^4 - \left(\alpha_0 + \frac{3}{4}\right)\alpha^2\beta^2 + \mathrm{O}(5).$$

By the Newton polygon technique, there exists only one family of zeros of $E(\alpha, \beta) = 0$ in a neighborhood of $(\alpha, \beta) = (0, 0)$ such that

$$\alpha^2 = \frac{1}{16}\beta^4 + \mathrm{O}(\beta^6).$$

Asymptotic result

Let $\alpha = \kappa_+ + \kappa_-$ and $\beta = \kappa_+ - \kappa_-$, such that

$$E(\alpha, \beta) = -4\alpha^2 + 10\alpha^3 - \frac{25}{2}\alpha^4 + \frac{1}{4}\beta^4 - \left(\alpha_0 + \frac{3}{4}\right)\alpha^2\beta^2 + \mathrm{O}(5).$$

By the Newton polygon technique, there exists only one family of zeros of $E(\alpha, \beta) = 0$ in a neighborhood of $(\alpha, \beta) = (0, 0)$ such that

$$\alpha^2 = \frac{1}{16}\beta^4 + \mathrm{O}(\beta^6).$$

Restoring the original variables

$$\kappa_\pm = \sqrt{1 - \rho^2 \pm i\Omega},$$

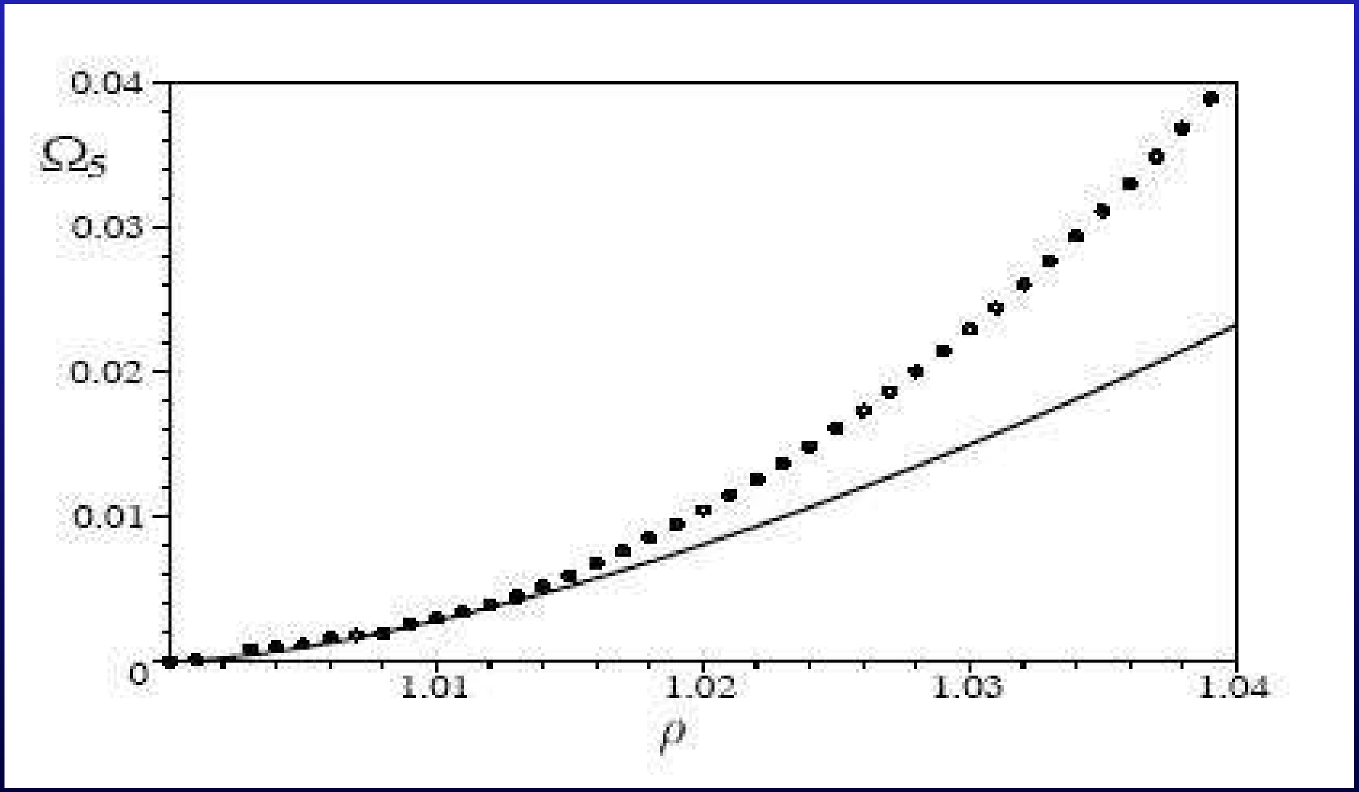
and performing a careful analysis of branches of the square root function, we obtain the final asymptotic result.

Bifurcation of eigenvalue Ω_5

Asymptotic result

$$\Omega_5 = 2\sqrt{2}\,(\rho - 1)^{3/2} + \mathrm{O}((\rho - 1)^{5/2})$$

Numerical result



Conclusions

- Earlier analytical results on $\rho \ll 1$, $\rho \gg 1$, and $\rho \sim 1$ are confirmed numerically by using the Hill's method

- Instabilities are rigorously proved for $0 < \rho < 1$ with the count of eigenvalues in Pontryagin spaces

- Bifurcations of collision of end points of the continuous spectrum is rigorously analyzed with the use of the Evans function and a new unfolding technique.