

# Recent progress on stability of solitary wave solutions

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# Formulation of the problem

Hyperbolic two-dimensional NLS equation

$$i\psi_t + \psi_{xx} - \psi_{yy} + 2|\psi|^2\psi = 0,$$

where  $\psi : \mathbb{R}^2 \times \mathbb{R}_+ \mapsto \mathbb{C}$  is the envelope amplitude and  $\eta = \text{Re}(\psi e^{ik_0x - i\omega_0 t})$  is the elevation of the water wave surface.

A line soliton solution

$$\psi = e^{it} \text{sech}(x), \quad \eta = \cos(k_0x - (\omega_0 - 1)t) \text{sech}(x),$$

where four parameters (phase translation, coordinate translation, frequency, velocity) can be constructed by using the Lie point symmetries of the NLS equation.

**Question:** Is the line soliton stable with respect to transverse ( $y$ -dependent) perturbations?

# Linearized stability and instability

Consider a linear perturbation to the line soliton

$$\psi = (\operatorname{sech}(x) + \epsilon u(x, y, t) + i\epsilon v(x, y, t)) e^{it}$$

and find the linear PDEs:

$$-u_t = v_{xx} - v_{yy} + (2\operatorname{sech}^2 x - 1)v, \quad v_t = u_{xx} - u_{yy} + (6\operatorname{sech}^2 x - 1)u$$

Use the Fourier transform in  $y$  and Laplace transform in  $t$ , e.g.

$$u = U(x)e^{i\rho y}e^{\Omega t}, \quad v = V(x)e^{i\rho y}e^{\Omega t}, \quad \rho \in \mathbb{R}, \quad \Omega \in \mathbb{C}$$

where  $(U, W) \in L^2(\mathbb{R})$  is an eigenvector of Schrödinger operators

$$\Omega U = (L_- - \rho^2)V, \quad -\Omega V = (L_+ - \rho^2)U.$$

The line soliton is transversely unstable if  $\operatorname{Re}(\Omega) > 0$  for  $\rho \neq 0$ .

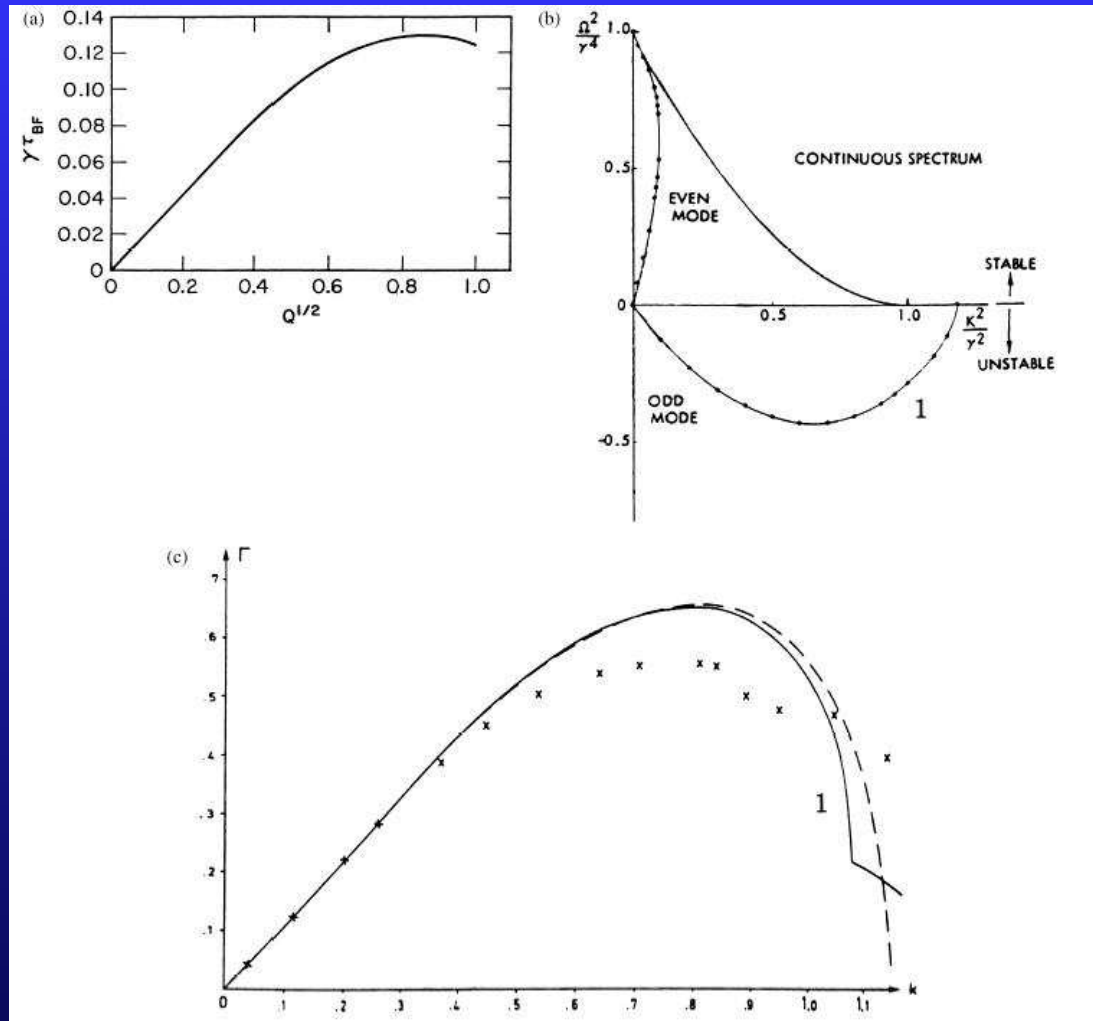
# History of analytical studies

- Zakharov–Rubenchik (1974), Yajima (1974): splitting of zero eigenvalues for small  $\rho$  into a pair of real eigenvalues  $\pm\Omega_1$  and a pair of purely imaginary eigenvalues  $\pm\Omega_2$ :

$$\Omega_1^2 = \frac{4}{3}\rho^2 + O(\rho^4), \quad \Omega_2^2 = -4\rho^2 + O(\rho^4).$$

- Ablowitz–Segur (1979): no real eigenvalues exist for large  $\rho$  as  $\Omega^2 = -\rho^4 + O(\rho^2)$ .
- Anderson et al. (1979): existence of complex eigenvalues for values of  $\rho \sim 1$  in the Rayleigh–Ritz variational method.
- Kivshar–Pelinovsky (2000): nonlinear theory of break-up of line soliton into dispersive clusters.

# History of numerical studies



Top left: Cohen et al. (1976). Top right: Saffman–Yuen (1978).  
Bottom: Anderson et al. (1979).

# Summary of our new results

- Numerical analysis of **all** isolated eigenvalues and their bifurcations for **any**  $\rho > 0$ .
- Analytical **proof** of existence of unstable eigenvalues for  $0 < \rho < 1$
- Rigorous **analysis** of bifurcations of unstable eigenvalues from  $\rho = 1$  to  $\rho > 1$ .

## References:

D.P., Proc. Roy. Soc. A **461**, 783-812 (2005)

B. Deconinck, D.P, J. Carter, Proc. Roy. Soc. A **462**, 2039 (2006)

# Numerical Hill's method

Consider the spectral problem

$$\phi_x = (A(x) + \lambda) \phi, \quad \phi \in \mathbb{C}^n$$

where  $A(x + L) = A(x)$  and  $\lambda \in \mathbb{C}$ . We are looking for eigenvalues  $\lambda$  when eigenvector  $\phi$  is in  $L^\infty([0, L])$  space.

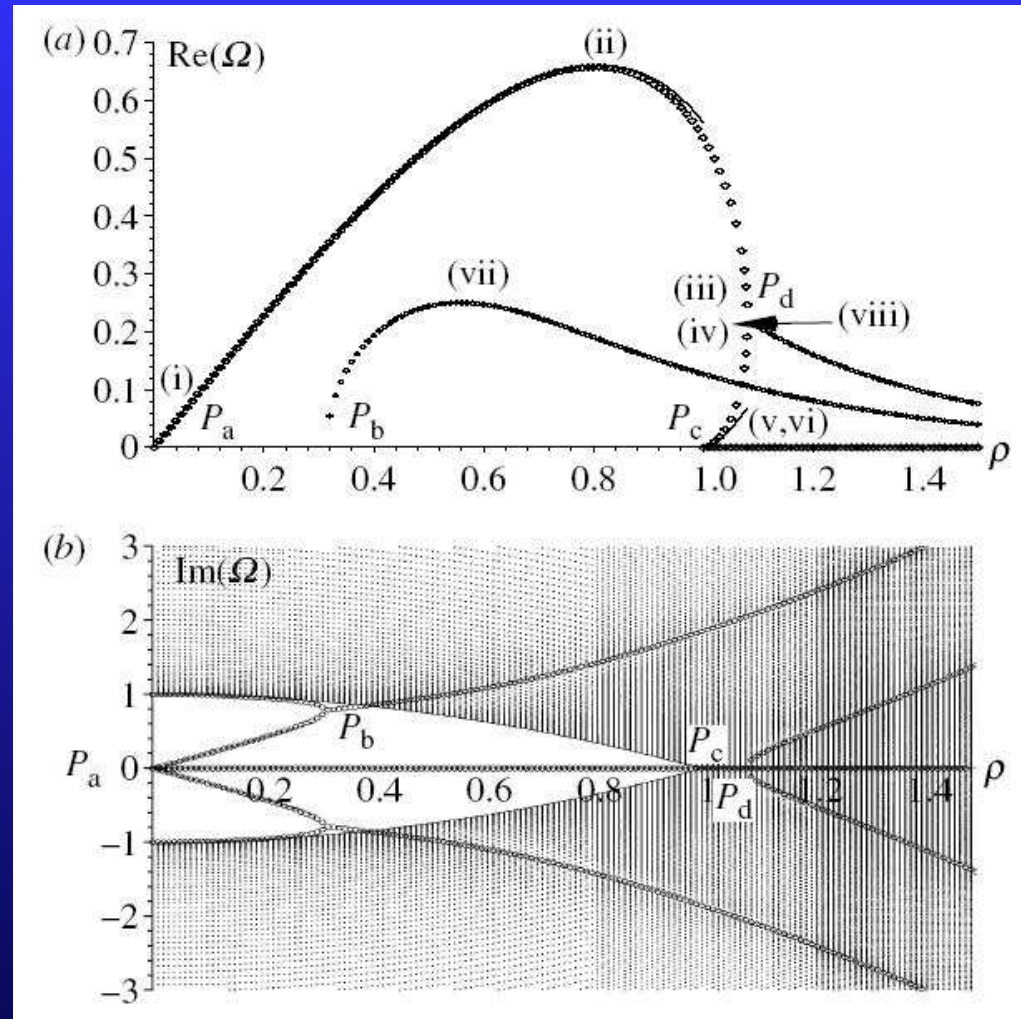
**Floquet's Theorem:** There exists a constant  $n \times n$  matrix  $R$  and  $L$ -periodic  $n \times n$  matrix  $P$ , such that the fundamental matrix solution  $\phi(x)$  is  $\phi(x) = P(x)e^{Rx}$ .

The bounded eigenvector can be decomposed as

$$\phi(x) = e^{i\mu x} \sum_{k \in \mathbb{Z}} \phi_k e^{2\pi i k x / L},$$

where  $\mu \in \left[-\frac{\pi}{L}, \frac{\pi}{L}\right]$ . For each value of  $\mu$ , the spectrum of  $\lambda$  can be found by truncation of Fourier series.

# Numerical analysis of all eigenvalues

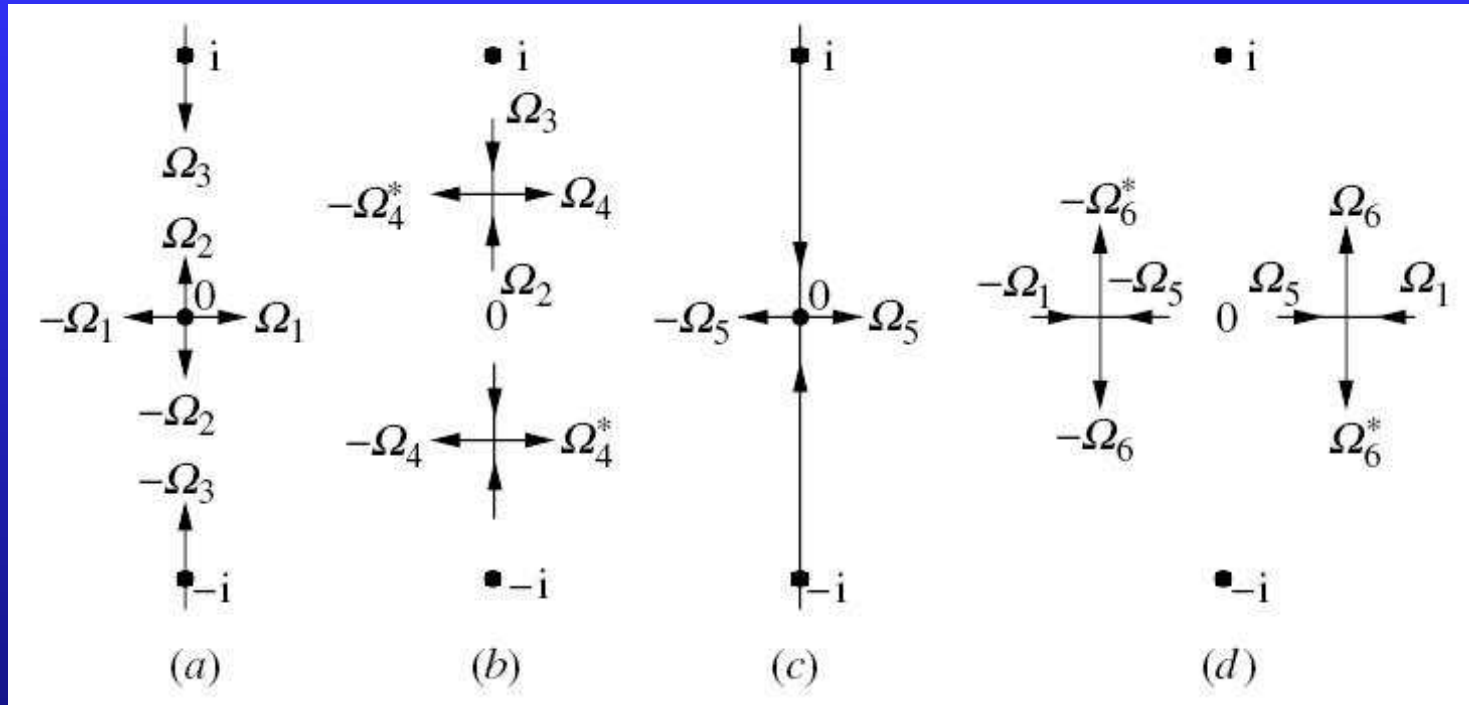


**Conclusion:** Line soliton is unstable for any  $\rho > 0$

**Bifurcations:**  $\rho = 0$ ,  $\rho \approx 0.31$ ,  $\rho = 1$ ,  $\rho \approx 1.08$ .



# Numerical analysis of all bifurcations



## Bifurcations:

- (a)  $\rho = 0$  - two pairs arise from multiple eigenvalue  $\Omega = 0$   
 one pair arises from the end point of the continuous spectrum
- (b)  $\rho \approx 0.31$  - Hamiltonian-Hopf bifurcation
- (c)  $\rho = 1$  - collision of end points of the continuous spectrum
- (d)  $\rho \approx 1.08$  - double real eigenvalue bifurcation

# Proof of existence of unstable eigenvalues

Consider the spectral problem

$$\begin{aligned}(L_+ - \rho^2)U &= -\Omega V, & (L_- - \rho^2)V &= \Omega U \\ L_+ &= -\partial_x^2 + 1 - 6\operatorname{sech}^2 x, & L_- &= -\partial_x^2 + 1 - 2\operatorname{sech}^2 x.\end{aligned}$$

Spectra  $\sigma(L_\pm)$  in  $L^2(\mathbb{R})$ :

- $\sigma(L_+)$  - two isolated eigenvalues at  $\sigma = -3$  and  $\sigma = 0$  and continuous spectrum for  $\sigma \geq 1$
- $\sigma(L_-)$  - one isolated eigenvalue at  $\sigma = 0$  and continuous spectrum for  $\sigma \geq 1$

For  $0 < \rho < 1$ , the spectral problem is equivalent to the generalized eigenvalue problem

$$(L_+ - \rho^2)U = \gamma(L_- - \rho^2)^{-1}U, \quad \gamma = -\Omega^2$$

# Sylvester–Pontryagin–Grillakis Theorem

**Theorem:** Let  $L$  and  $M$  be self-adjoint operators in  $L^2(\mathbb{R})$  with finitely many negative eigenvalues  $n(L)$  and  $n(M)$  and empty kernels. Then, there are exactly  $n(L)$  and  $n(M)$  eigenvalues  $\gamma$  of  $Lu = \gamma Mu$  in  $L^2(\mathbb{R})$  such that  $(u, Lu) \leq 0$  and  $(u, Mu) \leq 0$ .

## References:

L. Pontryagin, *Izv. Acad. Nauk SSSR* **8**, 243–280 (1944)

M. Grillakis, *Comm. Pure Appl. Math.* **43**, 299–333 (1990)

T. Kapitula, P. Kevrekidis, and B. Sandstede, *Physica D* **195**, 263–282 (2004)

M. Chugunova and D.P., preprint (2006)

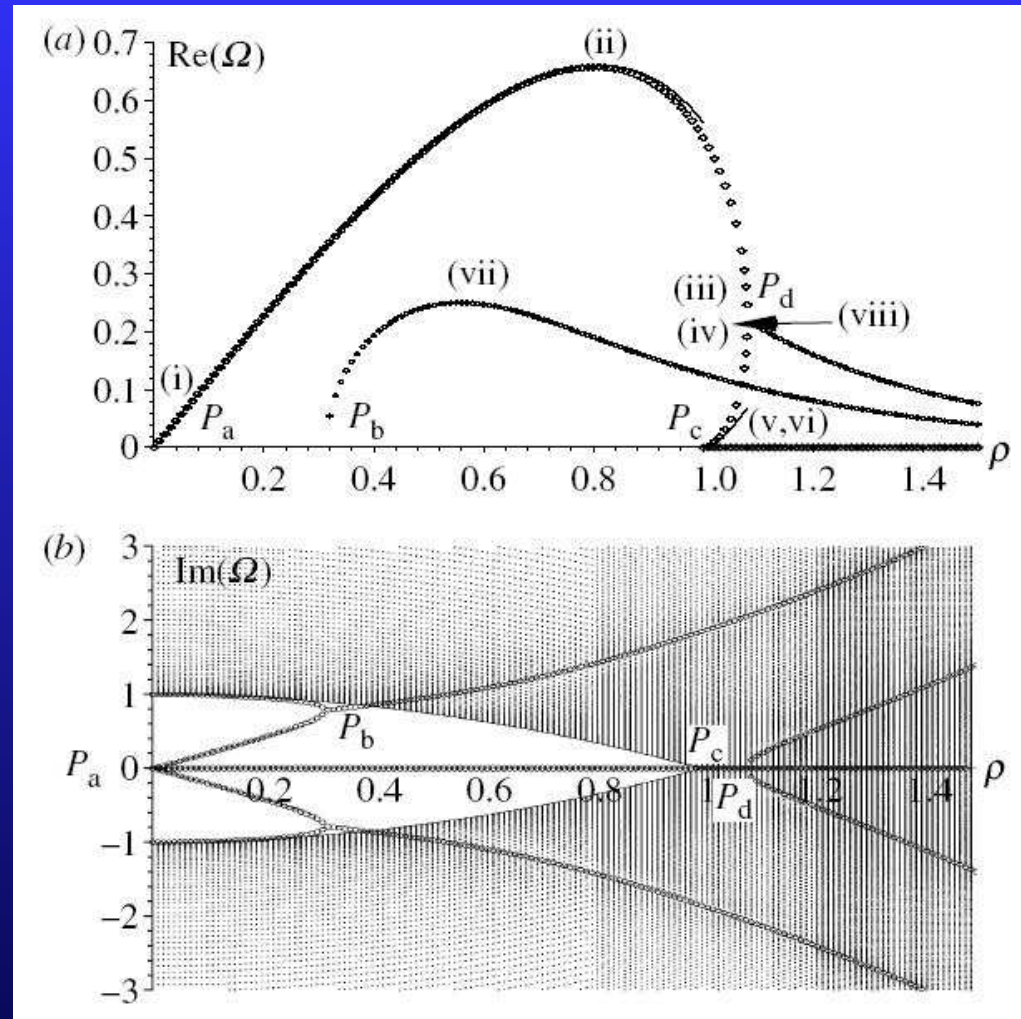
# Application of the Theorem

$$(L_+ - \rho^2)U = \gamma (L_- - \rho^2)^{-1}U, \quad \gamma = -\Omega^2$$

For  $0 < \rho < 1$ , there exist two eigenvalues  $\gamma$  such that  $(U, (L_+ - \rho^2)U) \leq 0$  and one eigenvalue  $\gamma$  such that  $(U, (L_- - \rho^2)^{-1}U) \leq 0$ .

**Conclusion:** One eigenvalue  $\Omega \in \mathbb{R}_+$  is always unstable for  $0 < \rho < 1$ , and two eigenvalues are either purely imaginary for  $0 < \rho < 0.31$  or complex for  $0.31 < \rho < 1$ .

# Numerical analysis of all eigenvalues



**Conclusion:** Line soliton is unstable for any  $\rho > 0$

**Bifurcations:**  $\rho = 0$ ,  $\rho \approx 0.31$ ,  $\rho = 1$ ,  $\rho \approx 1.08$ .

# Bifurcations of unstable eigenvalues

Let  $\rho^2 = 1 - \frac{\kappa_+^2 + \kappa_-^2}{2}$ ,  $\Omega = \frac{\kappa_+^2 - \kappa_-^2}{2i}$  and rewrite the eigenvalue problem

$$(-\partial_x^2 - 6\operatorname{sech}^2 x) U = -\frac{1}{2}(\kappa_+^2 + \kappa_-^2)U + \frac{i}{2}(\kappa_+^2 - \kappa_-^2)V,$$

$$(-\partial_x^2 - 2\operatorname{sech}^2 x) V = -\frac{i}{2}(\kappa_+^2 - \kappa_-^2)U - \frac{1}{2}(\kappa_+^2 + \kappa_-^2)V,$$

- Bifurcation point  $\rho = 1$ ,  $\Omega = 0$  corresponds to  $(\kappa_+, \kappa_-) = (0, 0)$
- Two solutions  $\mathbf{u}_\pm(x)$  decay like  $e^{\kappa_\pm x}$  as  $x \rightarrow -\infty$  and two solutions  $\mathbf{v}_\pm(x)$  decays like  $e^{-\kappa_\pm x}$  as  $x \rightarrow +\infty$  in the domain  $\operatorname{Re}(\kappa_\pm) > 0$
- The coordinates  $(\kappa_+, \kappa_-) \in \mathbb{C}^2$  unfold the branch point singularity in coordinates  $\Omega \in \mathbb{C}$  and  $\rho^2 - 1 \in \mathbb{R}$ .

# Evans function

Let  $\mathbf{u}_\pm(x)$  and  $\mathbf{v}_\pm(x)$  be fundamental solutions in the domain  $\operatorname{Re}(\kappa_\pm) > 0$  and define the Wronskian determinant:

$$E(\kappa_+, \kappa_-) = \det(\mathbf{u}_+(x), \mathbf{u}_-(x), \mathbf{v}_+(x), \mathbf{v}_-(x)) \Big|_{x=0}.$$

Zeros of  $E(\kappa_+, \kappa_-)$  coincide with eigenvalues of the original problem in the domain  $\operatorname{Re}(\kappa_\pm) > 0$ .

## References:

T. Kapitula and B. Sandstede, *Physica D* **124**, 58–103 (1998); *SIAM J. Math. Anal.* **33**, 1117–1143 (2002)

T. Bridges, G. Derks and G. Gottwald, *Physica D* **172** 190–216 (2002)

# Extensions of the Evans function

- Fundamental solutions  $\mathbf{u}_\pm(x)$ ,  $\mathbf{v}_\pm(x)$ , and their determinant  $E(\kappa_+, \kappa_-)$  are analytic functions near  $(\kappa_+, \kappa_-) = (0, 0)$ .
- The Taylor series expansion holds near  $(0, 0)$ :

$$E(\kappa_+, \kappa_-) = -4(\kappa_+ + \kappa_-)^2 + 10(\kappa_+ + \kappa_-)^3 - 13(\kappa_+^4 + \kappa_-^4) \\ - 51(\kappa_+^2 + \kappa_-^2)\kappa_+\kappa_- - 72\kappa_+^2\kappa_-^2 - \alpha_0(\kappa_+^2 - \kappa_-^2)^2 + O(5),$$

where  $\alpha_0$  is a numerical coefficient.

- Let  $\alpha = \kappa_+ + \kappa_-$  and  $\beta = \kappa_+ - \kappa_-$ , such that

$$E(\alpha, \beta) = -4\alpha^2 + 10\alpha^3 - \frac{25}{2}\alpha^4 + \frac{1}{4}\beta^4 - \left(\alpha_0 + \frac{3}{4}\right)\alpha^2\beta^2 + O(5)$$



# Asymptotic result

$$E(\alpha, \beta) = -4\alpha^2 + 10\alpha^3 - \frac{25}{2}\alpha^4 + \frac{1}{4}\beta^4 - \left(\alpha_0 + \frac{3}{4}\right)\alpha^2\beta^2 + O(5).$$

By the Newton polygon technique, there exists only one family of zeros of  $E(\alpha, \beta) = 0$  in a neighborhood of  $(\alpha, \beta) = (0, 0)$  such that

$$\alpha^2 = \frac{1}{16}\beta^4 + O(\beta^6).$$

Restoring the original variables

$$\kappa_{\pm} = \sqrt{1 - \rho^2 \pm i\Omega},$$

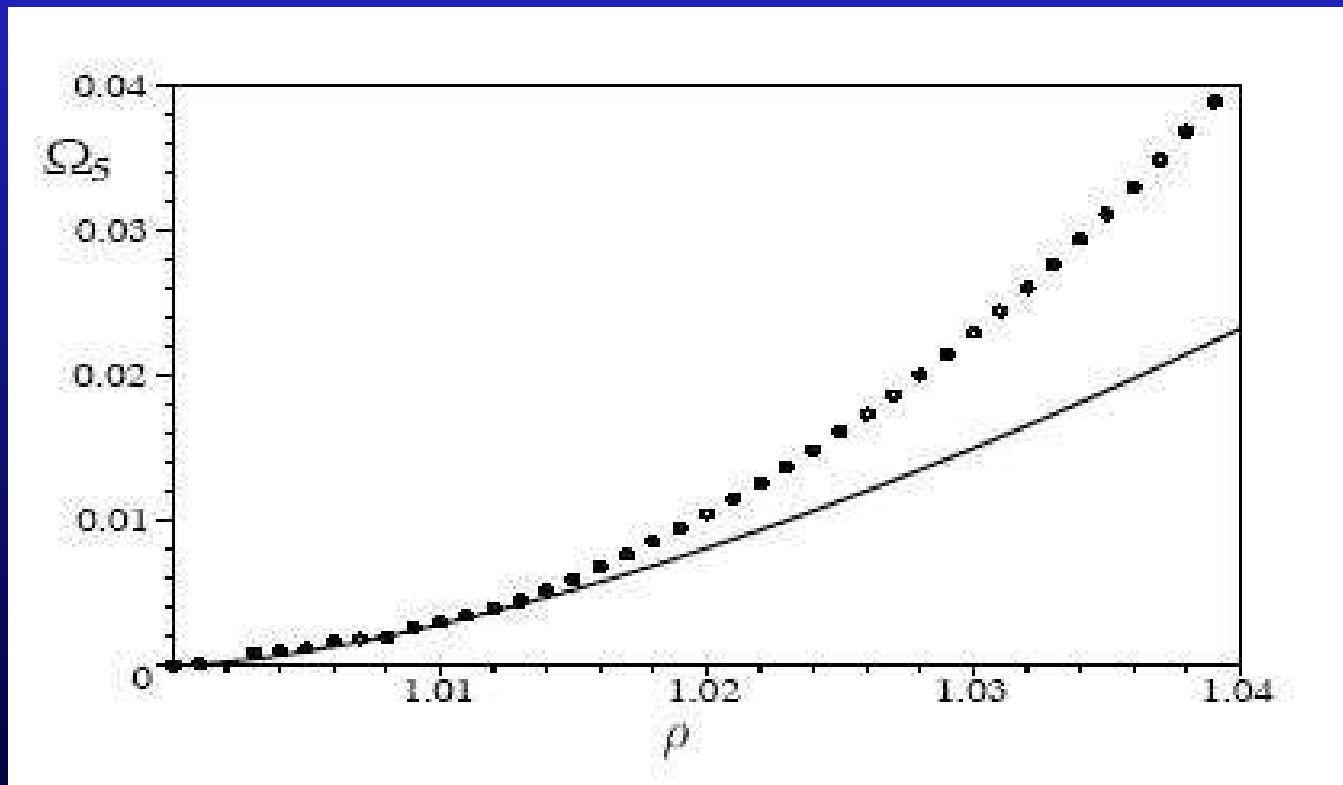
and performing a careful analysis of branches of the square root function, we obtain the final asymptotic result.

# Bifurcation of eigenvalue $\Omega_5$

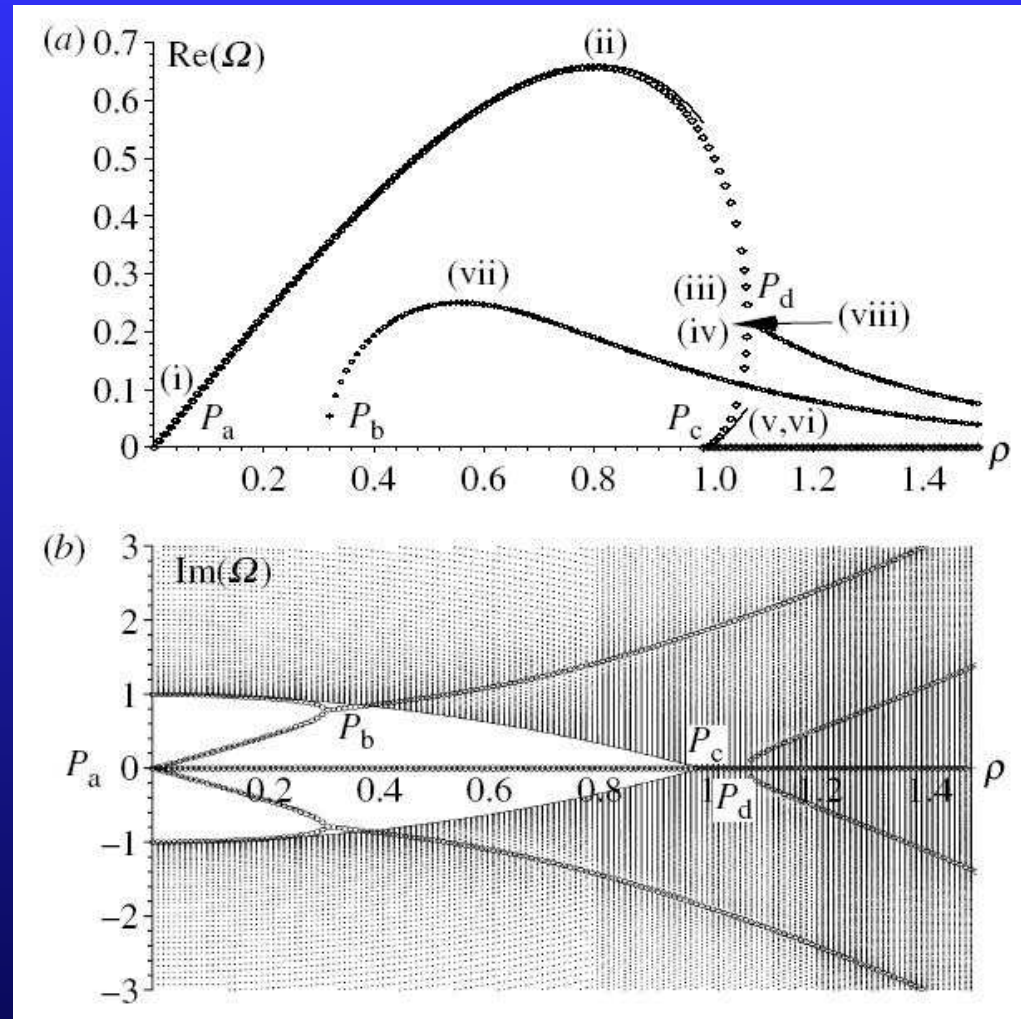
Asymptotic result

$$\Omega_5 = 2\sqrt{2}(\rho - 1)^{3/2} + O((\rho - 1)^{5/2})$$

Numerical result



# Numerical analysis of all eigenvalues



**Conclusion:** Line soliton is unstable for any  $\rho > 0$

**Bifurcations:**  $\rho = 0$ ,  $\rho \approx 0.31$ ,  $\rho = 1$ ,  $\rho \approx 1.08$ .

# Conclusions

- Earlier analytical results on  $\rho \ll 1$ ,  $\rho \gg 1$ , and  $\rho \sim 1$  are confirmed numerically by using the Hill's method
- Instabilities are rigorously proved for  $0 < \rho < 1$  with the count of eigenvalues in Pontryagin spaces
- Bifurcations of collision of end points of the continuous spectrum is rigorously analyzed with the use of the Evans function and a new unfolding technique.
- Similar analytical and computational methods can be applied to other problems on stability of solitary wave solutions.