Breathers and rogue waves on the background of periodic and double-periodic waves

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The focusing NLS equation

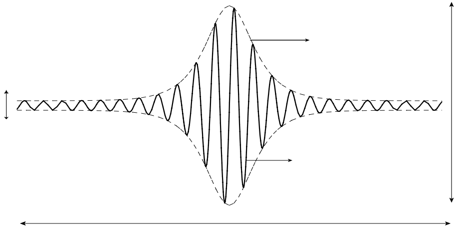
The focusing nonlinear Schrödinger (NLS) equation

$$i\partial_t \psi + \partial_x^2 \psi + |\psi|^2 \psi = 0$$

has been derived as the main model for modulating quasi-harmonic waves

$$\epsilon\psi(\epsilon(x-ct),\epsilon^2 t)e^{i(k_0 x-\omega_0 t)} + \epsilon\bar{\psi}(\epsilon(x-ct),\epsilon^2 t)e^{-i(k_0 x-\omega_0 t)} + \text{higher-order terms}$$

from water wave equations, Maxwell equations, and the like.



$\psi = e^{it}$ is the constant-amplitude wave, $\psi = \mathrm{sech}(x)e^{it/2}$ is a solitary wave.

The rogue wave of the cubic NLS equation

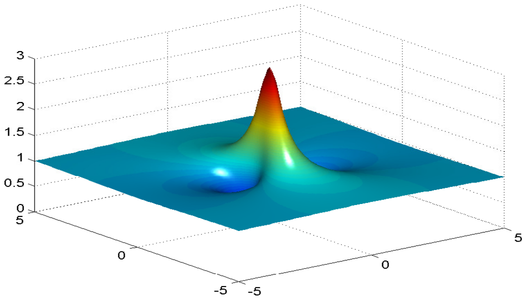
The focusing nonlinear Schrödinger (NLS) equation

$$i\partial_t \psi + \partial_x^2 \psi + |\psi|^2 \psi = 0$$

admits the exact solution

$$\psi(x,t) = \left[1 - \frac{4(1+2it)}{1+4x^2+4t^2}\right] e^{it}.$$

It was discovered by H. Peregrine (1983) and was labeled as *the rogue wave*.



Modulational instability of the constant-amplitude wave

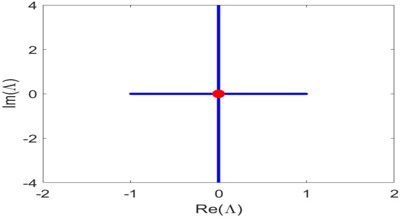
The rogue wave solution is related to the modulational instability of the constant-amplitude wave:

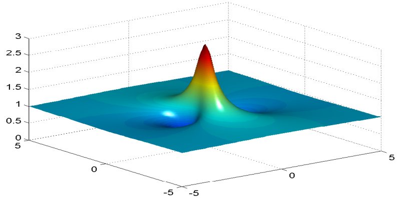
$$\psi(x,t) = e^{it}\left[1 + (k^2 + 2i\Lambda)e^{\Lambda t + ikx} + (k^2 + 2i\bar{\Lambda})e^{\bar{\Lambda} t - ikx}\right],$$

where $k \in \mathbb{R}$ is the wave number and Λ is given by

$$\Lambda^2 = k^2\left(1 - \frac{1}{4}k^2\right).$$

The wave is unstable for $k \in (0,2)$.



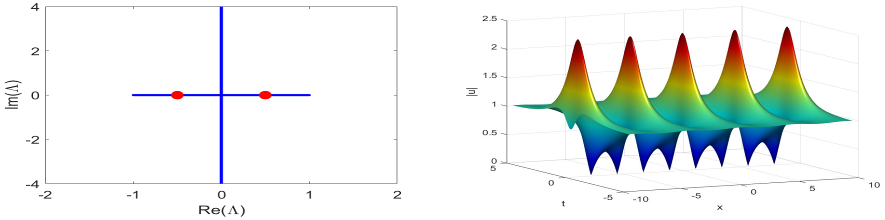


Other rogue waves - Akhmediev breathers (AB)

Spatially periodic homoclinic solution was constructed by N.N. Akhmediev, V.M. Eleonsky, and N.E. Kulagin (1985):

$$\psi(x,t) = e^{it}\left[1 - \frac{2(1-\lambda^2)\cosh(k\lambda t) + ik\lambda\sinh(k\lambda t)}{\cosh(k\lambda t) - \lambda\cos(kx)}\right],$$

where $k = 2\sqrt{1-\lambda^2} \in (0,2)$ and $\lambda \in (0,1)$ is the only free parameter. There is a unique solution for each spatial period $L = \frac{2\pi}{k} = \frac{\pi}{\sqrt{1-\lambda^2}} > \pi$.

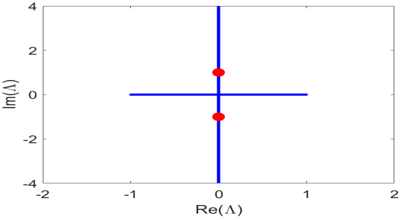


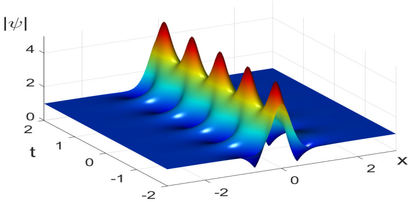
Other rogue waves - Kuznetsov-Ma breathers

Temporally periodic soliton was constructed by E. A. Kuznetsov (1977) and Y.-C. Ma (1979):

$$\psi(x,t) = \left[1 - \frac{2(\lambda^2-1)\cos(\beta\lambda t) + i\beta\lambda\sin(\beta\lambda t)}{\lambda\cosh(\beta x) - \cos(\beta\lambda t)}\right] e^{it},$$

where $\beta = 2\sqrt{\lambda^2-1}$ and $\lambda \in (1,\infty)$ is the only free parameter. There is a unique solution for each temporal period $T = \frac{2\pi}{\beta\lambda} = \frac{\pi}{\lambda\sqrt{\lambda^2-1}} > 0$ with $k = i\beta$.





Traveling periodic waves (elliptic background)

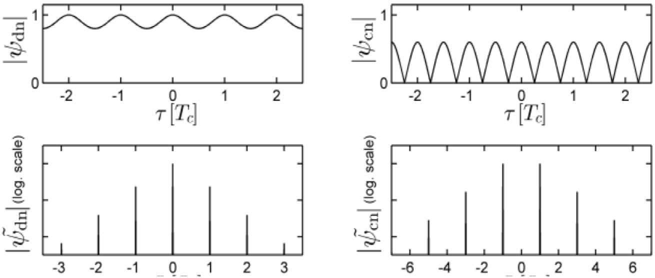
The focusing nonlinear Schrödinger (NLS) equation

$$i\partial_t \psi + \partial_x^2 \psi + |\psi|^2 \psi = 0$$

also admits the periodic solutions, e.g. the dnoidal and cnoidal waves:

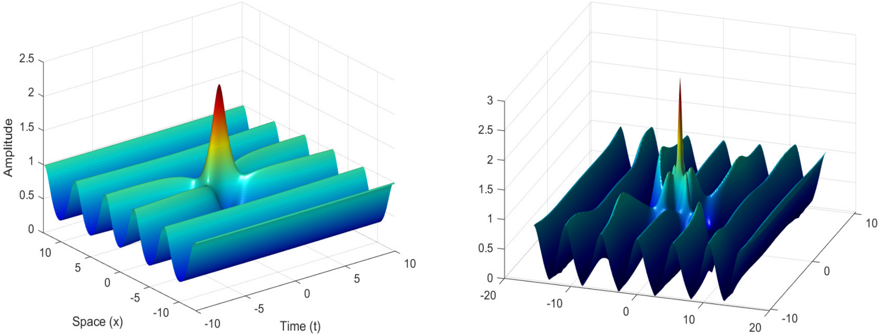
$$\psi_{\rm dn}(x,t) = {\rm dn}(x;k) e^{i(1-k^2/2)t}, \quad \psi_{\rm cn}(x,t) = k{\rm cn}(x;k) e^{i(k^2-1/2)t},$$

where $k \in (0,1)$ is elliptic modulus.



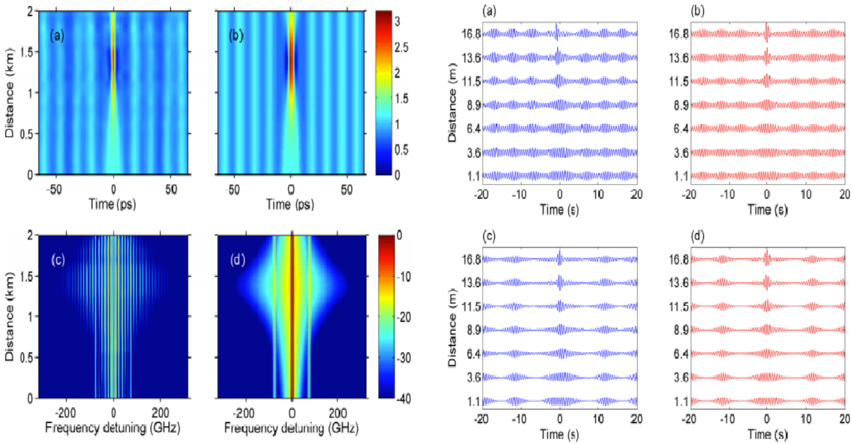
Rogue wave on background of periodic waves

J. Chen, D. P., Proceedings A (2018)
J. Chen, D. P., R. White, Physica D (2020)



Experimental observations of rogue waves

The same rogue waves were observed in optics and hydrodynamics:
G. Xu, A. Chabchoub, D.P., B. Kibler, Physical Review Research (2020)



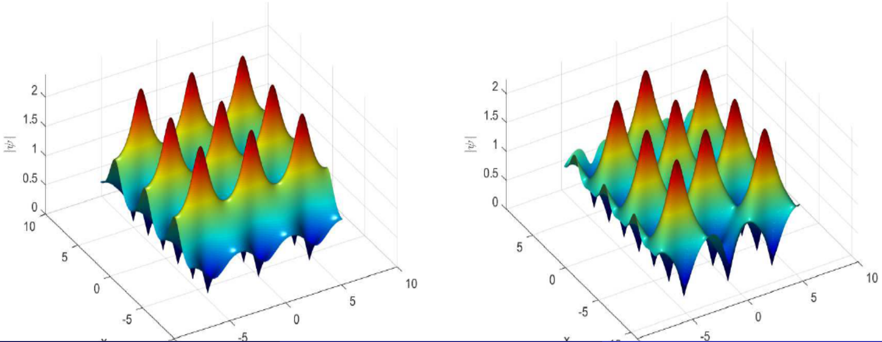
Double-periodic waves (elliptic background)

Double-periodic solutions (Akhmediev, Eleonskii, Kulagin, 1987):

$$\psi(x,t) = k\frac{\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa) + i\sqrt{1+k}\mathrm{sn}(t;k)\mathrm{dn}(\sqrt{1+k}x;\kappa)}{\sqrt{1+k}\mathrm{dn}(\sqrt{1+k}x;\kappa) - \mathrm{dn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa)} e^{it},$$

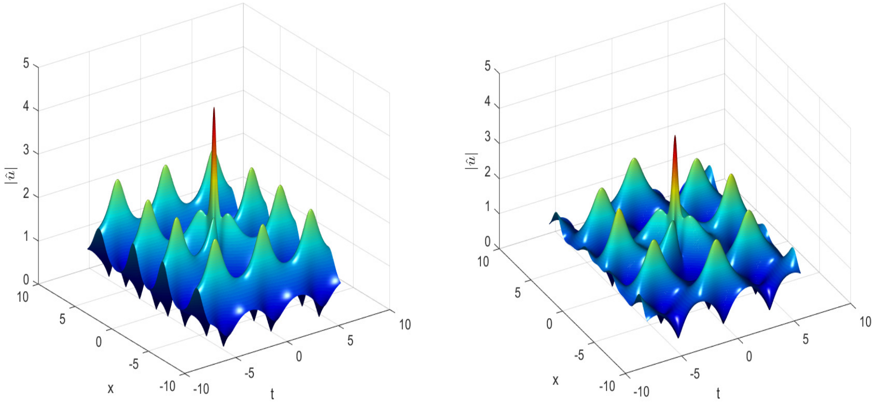
$$\psi(x,t) = \frac{\mathrm{dn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa) + i\sqrt{k(1+k)}\mathrm{sn}(t;k)}{\sqrt{1+k} - \sqrt{k}\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa)} e^{ikt},$$

where $k \in (0,1)$ is elliptic modulus and $\kappa \in (0,1)$ is determined by k.



Rogue wave on background of double-periodic waves

J. Chen, D. P., R. White, Phys. Rev. E (2019)



NLS hierarchy

The focusing nonlinear Schrödinger (NLS) equation

$$i\partial_t\psi + \partial_x^2\psi + |\psi|^2\psi = 0$$

is a member of the NLS hierarchy

$$\frac{d}{dt_k}\left[\begin{array}{c} u \\ \bar{u} \end{array}\right] = J\nabla H_k(u), \quad \nabla H_{k+1}(u) = R\nabla H_k(u),$$

where

$$J = i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad R = i\begin{bmatrix} \partial_x + 2\bar{u}\partial_x^{-1}u & -2\bar{u}\partial_x^{-1}\bar{u} \\ 2u\partial_x^{-1}u & -\partial_x - 2u\partial_x^{-1}\bar{u} \end{bmatrix},$$

Thus, we obtain

$$H_0 = \int_{\mathbb{R}} |u|^2 dx, \quad H_1 = \frac{i}{2}\int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u})\, dx,$$

$$H_2 = \int_{\mathbb{R}} \left(|u_x|^2 - |u|^4\right) dx, \quad H_3 = \frac{i}{2}\int_{\mathbb{R}} \left[u_x\bar{u}_{xx} - u_{xx}\bar{u}_x - 3|u|^2(u\bar{u}_x - u_x\bar{u})\right] dx.$$

Stationary Lax-Novikov equations

The stationary (Lax–Novikov) equations are given by

$$\begin{aligned} &\nabla H_1(u) + 2c\nabla H_0(u) = 0, \\ &\nabla H_2(u) + 2c\nabla H_1(u) + 4b\nabla H_0(u) = 0, \\ &\nabla H_3(u) + 2c\nabla H_2(u) + 4b\nabla H_1(u) + 8a\nabla H_0(u) = 0, \end{aligned}$$

or explicitly,

$$\begin{aligned} &u'(x) + 2icu = 0, \\ &u''(x) + 2|u|^2 u + 2icu' + 4bu = 0, \\ &u'''(x) + 6|u|^2 u' + 2ic(u'' + 2|u|^2 u) + 4bu' + 8iau = 0, \end{aligned}$$

where c, b, a are constants.

Solutions of stationary Lax-Novikov equations

In terms of the NLS equation

$$i\partial_t\psi + \partial_x^2\psi + |\psi|^2\psi = 0$$

the stationary Lax–Novikov equations

$$u' + 2icu = 0,$$
$$u'' + 2|u|^2u + 2icu' + 4bu = 0,$$
$$u''' + 6|u|^2u' + 2ic(u'' + 2|u|^2u) + 4bu' + 8iau = 0,$$

generate the following solutions:

1. Constant-amplitude wave $\psi(x,t) = Ae^{-2ic(x+ct)+iA^2t}$,
2. Traveling standing wave $\psi(x,t) = u(x+ct)e^{-2ibt}$
3. Double-periodic wave $\psi(x,t) = [q(x,t) + i\beta(t)]e^{it+i\alpha(t)}$, where $q(x+L,t) = q(x,t+T) = q(x,t)$, $\beta(t+T) = \beta(t)$, $\alpha(t+T) = \alpha(t)$.

Characterization of $u'' + 2|u|^2 u + 2icu' + 4bu = 0$

Consider the Lax system of linear equations

$$\varphi_x = U(\lambda, u)\varphi, \qquad U(\lambda, u) = \left(\begin{array}{cc} \lambda & u \\ -\bar{u} & -\lambda \end{array} \right)$$

and

$$\varphi_t = V(\lambda, u)\varphi, \qquad V(\lambda, u) = i \left(\begin{array}{cc} \lambda^2 + \frac{1}{2}|u|^2 & \frac{1}{2}u_x + \lambda u \\ \frac{1}{2}\bar{u}_x - \lambda \bar{u} & -\lambda^2 - \frac{1}{2}|u|^2 \end{array} \right).$$

Fix $\lambda = \lambda_1 \in \mathbb{C}$ with $\varphi = (p_1, q_1) \in \mathbb{C}^2$ and set $u = p_1^2 + \bar{q}_1^2$. The spectral problem $\varphi_x = U(\lambda, u)\varphi$ becomes the Hamiltonian system generated by

$$H = \lambda_1 p_1 q_1 + \bar{\lambda}_1 \bar{p}_1 \bar{q}_1 + \frac{1}{2}(p_1^2 + \bar{q}_1^2)(\bar{p}_1^2 + q_1^2).$$

with additional constant $F = i(p_1 q_1 - \bar{p}_1 \bar{q}_1)$.

(Cao–Geng, 1990) (Cao–Wu–Geng, 1999) (R.Zhou, 2009) (Chen–P, 2018)

Second-order Lax–Novikov equation

By differentiating of the constraints between u and (p_1, q_1), we obtain

$$\begin{aligned} u &= p_1^2 + \bar{q}_1^2, \\ u' + 2iFu &= 2(\lambda_1 p_1^2 - \bar{\lambda}_1 \bar{q}_1^2), \\ u'' + 2|u|^2 u + 2iFu' - 4Hu &= 4(\lambda_1^2 p_1^2 + \bar{\lambda}_1^2 \bar{q}_1^2), \end{aligned}$$

which yields the second-order Lax–Novikov equation:

$$u'' + 2|u|^2 u + 2icu' + 4bu = 0,$$

where $c := F + i(\lambda_1 - \bar{\lambda}_1)$ and $b := -H - iF(\lambda_1 - \bar{\lambda}_1) - |\lambda_1|^2$.

The second-order equation admits two conserved quantities:

$$\begin{aligned} i(u'\bar{u} - u\bar{u}') - 2c|u|^2 &= 4a, \\ |u'|^2 + |u|^4 + 4b|u|^2 &= 8d. \end{aligned}$$

Algebraic polynomial for $u'' + 2|u|^2 u + 2icu' + 4bu = 0$

Further analysis show that admissible values of λ_1 for the reduction $u = p_1^2 + \bar{q}_1^2$ appears to be roots of the characteristic polynomial $P(\lambda)$ given by

$$P(\lambda) = \lambda^4 + 2ic\lambda^3 + (2b - c^2)\lambda^2 + 2i(a + bc)\lambda + b^2 - 2ac + 2d,$$

where constants (a, b, c, d) are the same as in the second-order Lax-Novikov equation and its two conserved quantities:

$$\begin{aligned} u'' + 2|u|^2 u + 2icu' + 4bu &= 0, \\ i(u'\bar{u} - u\bar{u}') - 2c|u|^2 &= 4a, \\ |u'|^2 + |u|^4 + 4b|u|^2 &= 8d. \end{aligned}$$

Four roots exist due to properties of $P(\lambda)$:

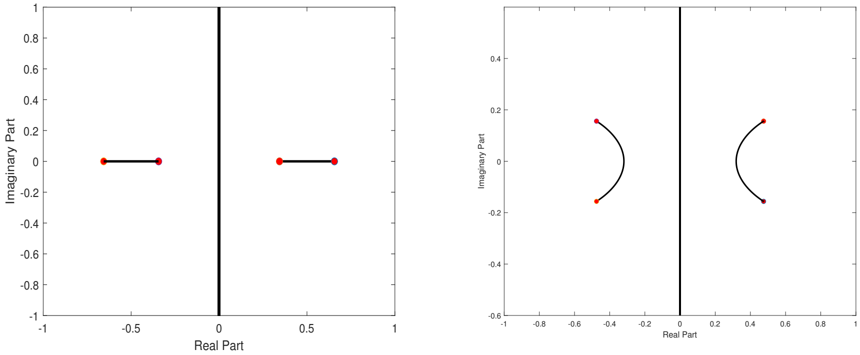
$$P(\lambda) = (\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)(\lambda - \lambda_2)(\lambda + \bar{\lambda}_2).$$

Lax spectrum for the standing periodic waves

Two possible solutions for the standing periodic waves ($a = c = 0$):

$$u(x) = \mathrm{dn}(x;k), \quad u(x) = k\mathrm{cn}(x;k).$$

Solutions are periodic with some period and the Lax spectrum of $\varphi_x = U(\lambda, u)\varphi$ coincides with the Floquet spectrum.



Red dots show roots of $P(\lambda)$, e.g., eigenvalues of the nonlinearization method.

Algebraic polynomial for the third-order equation

Solutions of the third-order Lax–Novikov equation

$$u''' + 6|u|^2 u' + 2ic(u'' + 2|u|^2 u) + 4bu' + 8iau = 0$$

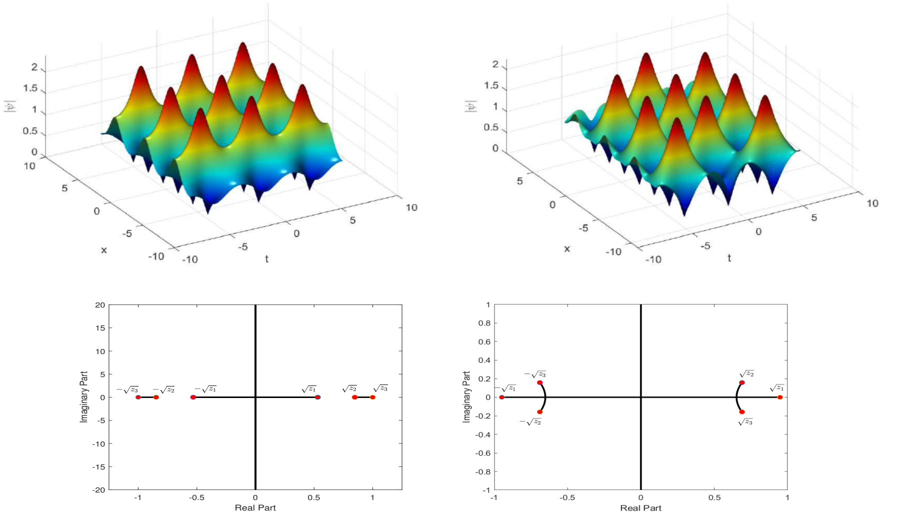
can be characterized similar (Chen-P-White, 2019) with the polynomial:

$$P(\lambda) = \lambda^6 + 2ic\lambda^5 + (2b - c^2)\lambda^4 + 2i(a + bc)\lambda^3 + (b^2 - 2ac + 2d)\lambda^2 \\ + 2i(e + ab + cd)\lambda + f + 2bd - 2ce - a^2.$$

where constants (a, b, c, d, e, f) are incorporated from the third-order Lax-Novikov equation and its three conserved quantities.

Double-periodic solutons are obtained from solutions of the third-order equation. Akhmediev and Kuznetsov–Ma breathers are degenerate cases of such double-periodic solutions.

Lax spectrum for the double-periodic solutions



Red dots show roots of $P(\lambda)$, eigenvalues of the nonlinearization method.

Linearized NLS equation

Let ψ be a standing periodic wave solution of the NLS equation

$$i\partial_t\psi + \partial_x^2\psi + |\psi|^2\psi = 0.$$

Using $\psi + \chi$ in NLS with perturbation χ and neglecting χ^2, χ^3 yields the linearized NLS equation

$$i\partial_t\chi + \partial_x^2\chi + 2|\psi|^2\chi + \psi^2\bar{\chi} = 0,$$

For the standing periodic waves, the variables can be separated:

$$\psi(x,t) = u(x+ct)e^{-2ibt}, \quad \chi(x,t) = v(x+ct)e^{-2ibt+\Lambda t}.$$

The spectral parameter Λ is found from the condition that $v(x)$ is bounded. Since $u(x+L) = u(x)$ is periodic, then by Floquet theory, $v(x) = w(x)e^{i\theta x}$, where $\theta \in [-\pi/L, \pi/L]$ and $w(x+L) = w(x)$.

If there exists Λ with $\mathrm{Re}(\Lambda) > 0$ for some $\theta \in [-\pi/L, \pi/L]$, then the standing periodic wave is unstable in the time evolution of the NLS equation. It is modulationally unstable if the band with $\mathrm{Re}(\Lambda) > 0$ intersects $\Lambda = 0$ as $\theta \to 0$.

Relation to squared eigenfunctions

Recall the linear Lax system:

$$\varphi_x = U(\lambda,\psi)\varphi, \qquad U(\lambda,\psi) = \begin{pmatrix} \lambda & \psi \\ -\bar{\psi} & -\lambda \end{pmatrix}$$

and

$$\varphi_t = V(\lambda,\psi)\varphi, \qquad V(\lambda,\psi) = i\begin{pmatrix} \lambda^2 + \frac{1}{2}|\psi|^2 & \frac{1}{2}\psi_x + \lambda\psi \\ \frac{1}{2}\bar{\psi}_x - \lambda\bar{\psi} & -\lambda^2 - \frac{1}{2}|\psi|^2 \end{pmatrix},$$

where ψ is a solution of the NLS equation.

If φ and ϕ are two linearly independent solutions of the Lax system, then

Pair I	Pair II	Pair III
$\chi = \varphi_1^2 - \bar{\varphi}_2^2$	$\chi = \varphi_1\phi_1 - \bar{\varphi}_2\bar{\phi}_2$	$\chi = \phi_1^2 - \bar{\phi}_2^2$
$\chi = i\varphi_1^2 + i\bar{\varphi}_2^2$	$\chi = i\varphi_1\phi_1 + i\bar{\varphi}_2\bar{\phi}_2$	$\chi = i\phi_1^2 + i\bar{\phi}_2^2$

are solutions of the linearized NLS equation.

Relation to squared eigenfunctions

Theorem

Let λ belongs to the Lax spectrum so that

$$\varphi(x,t) = \xi(x+ct)e^{-2ib\sigma_3 t + \Omega t}$$

with $\xi \in L^\infty(\mathbb{R})$. Then, $\Omega = \pm i\sqrt{P(\lambda)}$, where $P(\lambda)$ is the polynomial for the second-order Lax–Novikov equation:

$$P(\lambda) = \lambda^4 + 2ic\lambda^3 + (2b - c^2)\lambda^2 + 2i(a+bc)\lambda + b^2 - 2ac + 2d$$

Consequently, $\Lambda = 2\Omega = \pm 2i\sqrt{P(\lambda)}$.

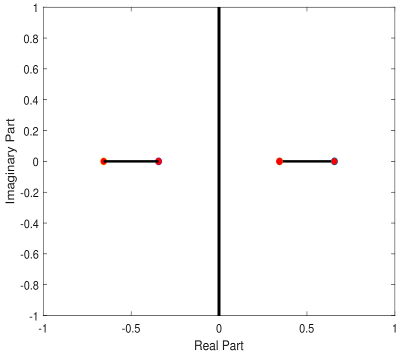
The proof follows from separation of variables for

$$\xi_x = U(\lambda,u)\xi, \qquad U(\lambda,u) = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix}$$

$$\Omega\xi + c\xi_x - 2ib\sigma_3\xi = V(\lambda,u)\xi, \quad V(\lambda,u) = i\begin{pmatrix} \lambda^2 + \frac{1}{2}|u|^2 & \frac{1}{2}u_x + \lambda u \\ \frac{1}{2}\bar{u}_x - \lambda\bar{u} & -\lambda^2 - \frac{1}{2}|u|^2 \end{pmatrix},$$

Instability of the dnoidal periodic waves

$$u(x) = \mathrm{dn}(x;k), \quad L = 2K(k).$$



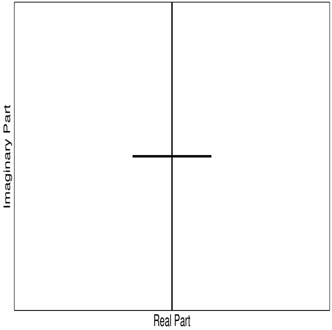


Figure: Left: Lax spectrum. Right: stability spectrum related by $\Lambda = \pm 2i\sqrt{P(\lambda)}$.

Instability of the cnoidal periodic waves

$$u(x) = k\mathrm{cn}(x;k), \quad L = 4K(k).$$

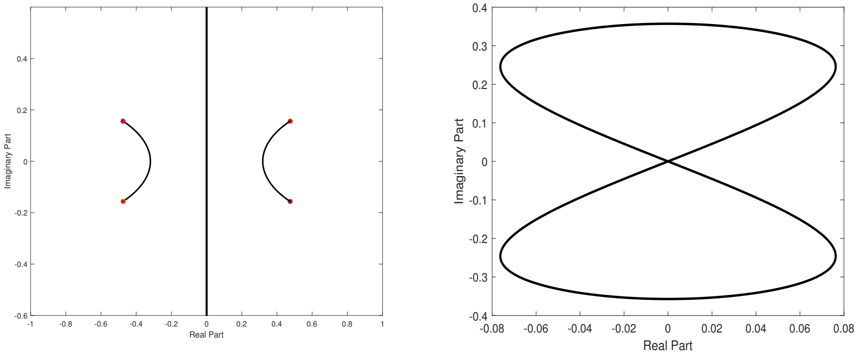


Figure: Left: Lax spectrum. Right: stability spectrum related by $\Lambda = \pm 2i\sqrt{P(\lambda)}$.

Spectral stability of double-periodic waves

For the double-periodic waves, the variables can not be separated:

$$\psi(x,t) = [q(x,t) + i\beta(t)]e^{it+i\alpha(t)},$$

where $q(x+L,t) = q(x,t+T) = q(x,t)$, $\beta(t+T) = \beta(t)$, $\alpha(t+T) = \alpha(t)$. Perturbation $\chi(x,t)$ to $\psi(x,t)$ satisfies the linearized NLS equation

$$i\partial_t \chi + \partial_x^2 \chi + 2|\psi|^2 \chi + \psi^2 \bar{\chi} = 0,$$

Due to periodicity both in x and t, Floquet theory yields solutions in the form

$$\chi(x,t) = v(x,t)e^{it+i\theta x+\Lambda t},$$

where $v(x+L,t) = v(x,t+T) = v(x,t)$, $\theta \in [-\pi/L, \pi/L]$, and where Λ defines stability (unique if $\mathrm{Im}(\Lambda) \in [-\pi/T, \pi/T]$).

Spectral stability of double-periodic waves

Recall the linear Lax system

$$\varphi_x = U(\lambda,\psi)\varphi, \qquad U(\lambda,\psi) = \begin{pmatrix} \lambda & \psi \\ -\bar{\psi} & -\lambda \end{pmatrix}$$

and

$$\varphi_t = V(\lambda,\psi)\varphi, \qquad V(\lambda,\psi) = i \begin{pmatrix} \lambda^2 + \frac{1}{2}|\psi|^2 & \frac{1}{2}\psi_x + \lambda\psi \\ \frac{1}{2}\bar{\psi}_x - \lambda\bar{\psi} & -\lambda^2 - \frac{1}{2}|\psi|^2 \end{pmatrix},$$

where ψ is a solution of the NLS equation.

By the Floquet theory both with respect to x and t, we write

$$\varphi(x,t) = \xi(x,t)e^{i\theta x + t\Omega},$$

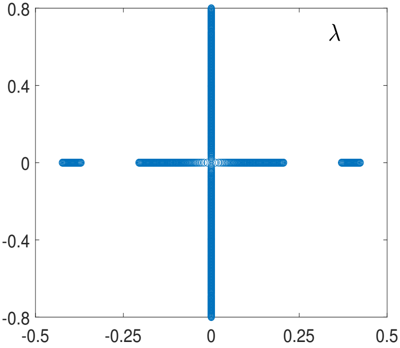
$\xi(x+L,t) = \xi(x,t+T) = \xi(x,t)$, $\theta \in [-\pi/L, \pi/L]$, $\mathrm{Im}(\Omega) \in [-\pi/T, \pi/T]$.

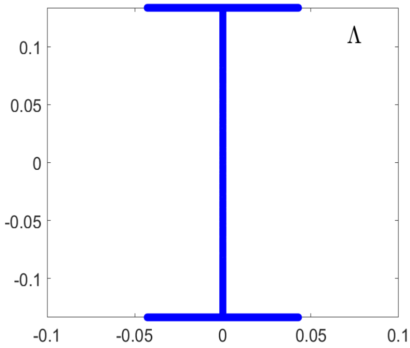
- λ is found from the Lax spectrum for $\varphi_x = U(\lambda,\psi)$.
- Ω is found from $\varphi_t = V(\lambda,\psi)\varphi$.

Open question: a relation between Ω and $P(\lambda)$.

Instabilities of the first solution

$k = 0.85$ (Pelinovsky, 2021):

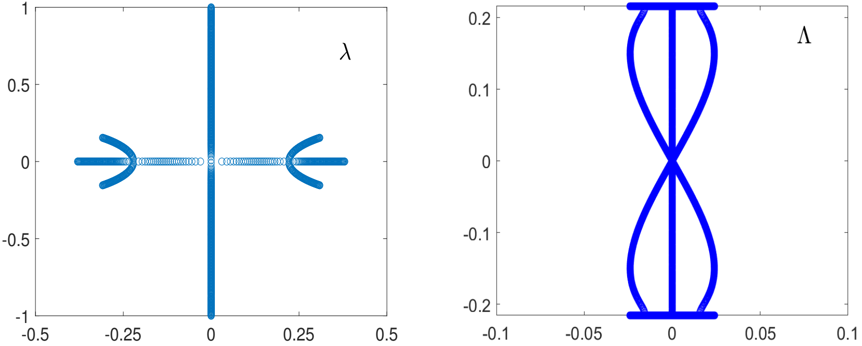




Left: Lax spectrum. Right: stability spectrum.

Instabilities of the second solution

$k = 0.6$ (Pelinovsky, 2021):



Left: Lax spectrum. Right: stability spectrum.

Akhmediev breathers under periodic perturbation

A family of Akhmediev breathers with parameter $\lambda \in (0,1)$:

$$\psi(x,t) = e^{it}\left[1 - \frac{2(1-\lambda^2)\cosh(k\lambda t) + ik\lambda\sinh(k\lambda t)}{\cosh(k\lambda t) - \lambda\cos(kx)}\right],$$

If the perturbation is periodic, the Lax and stability spectra are purely discrete.
There was an open question if the Akhmediev breather is linearly unstable.
P. Grinevich & P. Santini, Nonlinearity **34** (2021) 8331–8358
M. Haragus & D. Pelinovsky, J. Nonlinear Science **32** (2022) 66

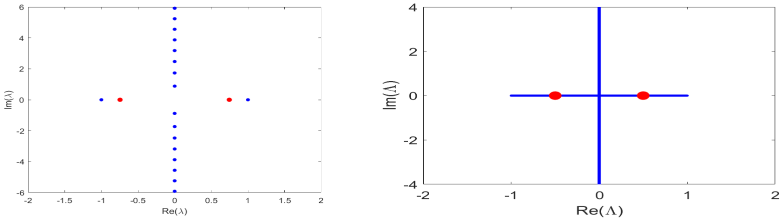


Figure: Lax spectrum (left) and stability spectrum (right) of Akhmediev breather.

Other examples of integrable Hamiltonian systems

- Modified Korteweg–de Vries equation

$$\partial_t u + 6u^2 \partial_x u + \partial_x^3 u = 0$$

Dnoidal periodic waves are modulationally stable.
Cnoidal periodic waves are modulationally unstable.
J. Chen & D. Pelinovsky, Nonlinearity **31** (2018) 1955–1980

- Sine–Gordon equation

$$\partial_t^2 u - \partial_x^2 u + \sin(u) = 0$$

Same conclusion.
D. Pelinovsky & R. White, Proceedings A **476** (2020) 20200490

- Derivative NLS equation

$$i\partial_t \psi + \partial_x^2 \psi + i\partial_x(|\psi|^2 \psi) = 0.$$

There exist modulationally stable periodic waves.
J. Chen, D. Pelinovsky, & J. Upsal, J. Nonlinear Science **31** (2021) 58

Summary

- Standing periodic waves are solutions of the second-order Lax–Novikov equation. Double-periodic waves are solutions of the third-order Lax–Novikov equation. Akhmediev and Kuznetsov–Ma breathers are particular cases of double-periodic solutions.
- Standing periodic waves are spectrally (modulationally) unstable, their instability is computed from separation of variables and Floquet theory.
- Double-periodic waves are also linearly unstable, their instability is computed from double Floquet theory (both in x and t).
- Akhmediev and Kuznetsov–Ma breathers are also linearly unstable.

Many thanks for your attention!