Integrable semi-discretizations of integrable PDEs

Dmitry Pelinovsky

Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

is posed for real u on $(t, x) \in \mathbb{R} \times \mathbb{R}$.

Question: Can we numerically approximate this equation on a equally spaced grid $x_n = nh$, $n \in \mathbb{Z}$ with step size h?

Answer: Yes, in many different ways, for example, with accuracy $\mathcal{O}(h^2)$:

$$\frac{du_n}{dt} + \frac{u_{n+1} + u_{n-1}}{2}\frac{u_{n+1} - u_{n-1}}{2h} + \frac{u_{n+2} - 2u_{n+1} + 2u_{n-1} - u_{n-2}}{2h^3} = 0$$

However,

- Such discretizations have many problems with stability of iterations.
- Such discretizations do not preserve integrability properties of KdV.

Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

is posed for real u on $(t, x) \in \mathbb{R} \times \mathbb{R}$.

Question: Can we numerically approximate this equation on a equally spaced grid $x_n = nh$, $n \in \mathbb{Z}$ with step size h?

Answer: Yes, in many different ways, for example, with accuracy $\mathcal{O}(h^2)$:

$$\frac{du_n}{dt} + \frac{u_{n+1} + u_{n-1}}{2}\frac{u_{n+1} - u_{n-1}}{2h} + \frac{u_{n+2} - 2u_{n+1} + 2u_{n-1} - u_{n-2}}{2h^3} = 0$$

However,

- Such discretizations have many problems with stability of iterations.
- Such discretizations do not preserve integrability properties of KdV.

The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

is a compatibility condition of the spectral problem

$$\left[\frac{\partial^2}{\partial x^2} + u\right]\psi = \lambda\psi$$

and the linear time-evolution problem

$$\frac{\partial\psi}{\partial t} = \left[4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3\frac{\partial u}{\partial x}\right]\psi.$$

This gives

- infinitely many conserved quantities,
- infinitely many exact solutions,
- Bäcklund–Darboux transformations between solutions,
- inverse scattering for the Cauchy problem,

and many many more.

Recent surprising discovery: Bäcklund–Darboux transformations also define integrable discretizations of integrable PDEs.

The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

is a compatibility condition of the spectral problem

$$\left[\frac{\partial^2}{\partial x^2} + u\right]\psi = \lambda\psi$$

and the linear time-evolution problem

$$\frac{\partial \psi}{\partial t} = \left[4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3\frac{\partial u}{\partial x}\right]\psi.$$

This gives

- infinitely many conserved quantities,
- infinitely many exact solutions,
- Bäcklund–Darboux transformations between solutions,
- inverse scattering for the Cauchy problem,

and many many more.

Recent surprising discovery: Bäcklund–Darboux transformations also define integrable discretizations of integrable PDEs.

What is a Bäcklund–Darboux transformation?

Consider

$$\left[\frac{\partial^2}{\partial x^2} + u\right]\psi = \lambda\psi$$

and define any nonzero solution ψ_0 for any fixed λ_0. Then,

$$\tilde{\psi} = \frac{\partial \psi}{\partial x} - \frac{1}{\psi_0}\frac{\partial \psi_0}{\partial x}\psi$$

satisfies

$$\left[\frac{\partial^2}{\partial x^2} + \tilde{u}\right]\tilde{\psi} = \lambda\tilde{\psi}$$

with new

$$\tilde{u} = u - 2\frac{\partial^2}{\partial x^2}\log\psi_0.$$

If u is a solution to the KdV, then $\tilde{u}$ is a new solution to the KdV.

Example

$u = 0$ is a trivial solution to the KdV.

Fix $\lambda_0 > 0$ and solve:

$$\frac{\partial^2}{\partial x^2}\psi_0 = \lambda_0 \psi_0 \quad \Rightarrow \quad \psi_0 = c_1 e^{\sqrt{\lambda_0} x} + c_2 e^{-\sqrt{\lambda_0} x},$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary.

Then

$$\tilde{u} = u - 2\frac{\partial^2}{\partial x^2}\log\psi_0 = \lambda_0 \mathrm{sech}^2(\sqrt{\lambda_0}(x - x_0))$$

is the KdV soliton at $t = 0$ with x_0 expressed by (c_1, c_2).

Hence BT maps 0-solution to 1-soliton: $BT_{\lambda_0}(0) = u_{\lambda_0}$.

Bianchi's permutability theorem

$$\tilde{u} = BT_\lambda(u), \quad \hat{u} = BT_\mu(u) \quad \Rightarrow \quad BT_\mu(\tilde{u}) = BT_\lambda(u) =: \hat{\tilde{u}}.$$

Moreover,

$$(\hat{\tilde{w}} - w)(\tilde{w} - \hat{w}) = 4(\lambda - \mu),$$

where w is the potential for u: $u = \frac{\partial w}{\partial x}$.

Interpret this as the lattice equation with

$$w := w_{n,m}, \quad \tilde{w} = w_{n+1,m}, \quad \hat{w} = w_{n,m+1}, \quad \hat{\tilde{w}} = w_{n+1,m+1}$$

and denote $4\lambda = p^2$, $4\mu = q^2$. Then, the permutability theorem gives the fully discrete KdV equation (in the potential form):

$$(w_{n+1,m+1} - w_{n,m})(w_{n+1,m} - w_{n,m+1}) = p^2 - q^2.$$

The fully discrete equation is completely integrable!
J. Hietarinta, N. Joshi, and F. Nijhoff, *Discrete systems and Integrability* (Cambridge University Press, 2016)

Bianchi's permutability theorem

$$\tilde{u} = BT_\lambda(u), \quad \hat{u} = BT_\mu(u) \quad \Rightarrow \quad BT_\mu(\tilde{u}) = BT_\lambda(u) =: \hat{\tilde{u}}.$$

Moreover,

$$(\hat{\tilde{w}} - w)(\tilde{w} - \hat{w}) = 4(\lambda - \mu),$$

where w is the potential for u: $u = \frac{\partial w}{\partial x}$.

Interpret this as the lattice equation with

$$w := w_{n,m}, \quad \tilde{w} = w_{n+1,m}, \quad \hat{w} = w_{n,m+1}, \quad \hat{\tilde{w}} = w_{n+1,m+1}$$

and denote $4\lambda = p^2$, $4\mu = q^2$. Then, the permutability theorem gives the fully discrete KdV equation (in the potential form):

$$(w_{n+1,m+1} - w_{n,m})(w_{n+1,m} - w_{n,m+1}) = p^2 - q^2.$$

The fully discrete equation is completely integrable!

J. Hietarinta, N. Joshi, and F. Nijhoff, *Discrete systems and Integrability* (Cambridge University Press, 2016)

How does discrete KdV represent continuous KdV?

$$(w_{n+1,m+1} - w_{n,m})(w_{n+1,m} - w_{n,m+1}) = p^2 - q^2.$$

Set $w_{n,m} = np + mq + v_{n,m}$ to have $v = 0$ as a trivial solution. Then, the semi-continuous limit $v_{n,m} = V_n(m/q)$ as $q \to \infty$ yields

$$v_{n,m+1} = V_n(\tau) + q^{-1}\partial_\tau V_n(\tau) + \mathcal{O}(q^{-2}), \quad \tau := mq^{-1}$$

leading to the integrable semi-discretization in the formal limit $q \to \infty$:

$$\partial_\tau (V_{n+1} + V_n) = 2p(V_{n+1} - V_n) - (V_{n+1} - V_n)^2.$$

By taking another continuous limit $V_n(\tau) = V(\tau, n/p)$ as $p \to \infty$, we can recover the continuous KdV equation (in the potential form):

$$\partial_\tau V = \partial_\xi V + p^{-2}\left[\frac{1}{6}\partial_\xi^3 V + (\partial_\xi V)^2\right] + \mathcal{O}(p^{-4}), \quad \xi := np^{-1}.$$

Massive Thirring Model

$$\begin{cases} i(u_t + u_x) + v = 2|v|^2 u, \\ i(v_t - v_x) + u = 2|u|^2 v, \end{cases} \quad \text{or} \quad \begin{cases} i\psi_t - \varphi_x - \psi = (\psi^2 + \varphi^2)\bar{\psi}, \\ i\varphi_t + \psi_x + \varphi = (\psi^2 + \varphi^2)\bar{\varphi}. \end{cases}$$

- One of the two examples of relativistically invariant nonlinear Dirac equations in (1+1) dimensions.
- Derived in relativistic field theory by W. Thirring (1958).
- Integrable by inverse scattering since the works of A. Mikhailov (1976).
- Admits stable solitary waves [Y. Shimabukuro (2016)].
- No integrable semi-discretizations are known [T. Tsuchida (2015)]

Integrable semi-discretization of the MTM system

$$\left\{\begin{array}{l} 4i\frac{dU_n}{dt} + Q_{n+1} + Q_n + \frac{2i}{h}(R_{n+1} - R_n) + U_n^2(\bar{R}_n + \bar{R}_{n+1}) \\ -U_n(|Q_{n+1}|^2 + |Q_n|^2 + |R_{n+1}|^2 + |R_n|^2) - \frac{ih}{2}U_n^2(\bar{Q}_{n+1} - \bar{Q}_n) = 0, \\ -\frac{2i}{h}(Q_{n+1} - Q_n) + 2U_n - |U_n|^2(Q_{n+1} + Q_n) = 0, \\ R_{n+1} + R_n - 2U_n + \frac{ih}{2}|U_n|^2(R_{n+1} - R_n) = 0, \end{array}\right.$$

In the continuum limit

$$U_n(t) = U(x = hn, t), \quad R_n(t) = R(x = hn, t), \quad Q_n(t) = Q(x = nh, t),$$

we obtain $U = R$ and

$$\left\{\begin{array}{l} 2i\frac{\partial R}{\partial t} + i\frac{\partial R}{\partial x} + Q - R|Q|^2 = 0, \\ -i\frac{\partial Q}{\partial x} + R - |R|^2 Q = 0, \end{array}\right.$$

which yields the MTM for $R(t,x) = u(t-x,x)$ and $Q(t,x) = v(t-x,x)$.

Integrable semi-discretization of the MTM system

$$\left\{\begin{array}{l} 4i\frac{dU_n}{dt} + Q_{n+1} + Q_n + \frac{2i}{h}(R_{n+1} - R_n) + U_n^2(\bar{R}_n + \bar{R}_{n+1}) \\ -U_n(|Q_{n+1}|^2 + |Q_n|^2 + |R_{n+1}|^2 + |R_n|^2) - \frac{ih}{2}U_n^2(\bar{Q}_{n+1} - \bar{Q}_n) = 0, \\ -\frac{2i}{h}(Q_{n+1} - Q_n) + 2U_n - |U_n|^2(Q_{n+1} + Q_n) = 0, \\ R_{n+1} + R_n - 2U_n + \frac{ih}{2}|U_n|^2(R_{n+1} - R_n) = 0, \end{array}\right.$$

In the continuum limit

$$U_n(t) = U(x = hn, t), \quad R_n(t) = R(x = hn, t), \quad Q_n(t) = Q(x = nh, t),$$

we obtain $U = R$ and

$$\left\{\begin{array}{l} 2i\frac{\partial R}{\partial t} + i\frac{\partial R}{\partial x} + Q - R|Q|^2 = 0, \\ -i\frac{\partial Q}{\partial x} + R - |R|^2 Q = 0, \end{array}\right.$$

which yields the MTM for $R(t,x) = u(t-x,x)$ and $Q(t,x) = v(t-x,x)$.

The integrable semi-discretization is a starting point for

- Derivation of discrete Dirac solitons and analysis of their stability.
- Comparison of numerical simulations between different discretizations of the MTM system.
- Derivation of an integrable semi-discretization of another fundamental model in the field theory, the sine–Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = 0.$$

- Derivation of fully discrete version of the MTM system.