Orbital stability of Dirac solitons

(the massive Thirring model)

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Université de Cergy–Pontoise, France, November 18, 2013

The problem

The nonlinear Dirac equations in one spatial dimension,

$$\begin{cases} i(u_t + u_x) + v = \partial_{\bar{u}} W(u,v), \\ i(v_t - v_x) + u = \partial_{\bar{v}} W(u,v), \end{cases}$$

where $W(u,v) : \mathbb{C}^2 \to \mathbb{R}$ satisfies the following three conditions:

- symmetry $W(u,v) = W(v,u)$;
- gauge invariance $W(e^{i\theta}u, e^{i\theta}v) = W(u,v)$ for any $\theta \in \mathbb{R}$;
- polynomial in (u,v) and $(\bar{u},\bar{v})$.

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Examples of nonlinear potentials:

- Bragg resonance: $W = |u|^4 + 4|u|^2|v|^2 + |v|^4$.
- Gross–Neveu model: $W = (\bar{u}v + u\bar{v})^2$.
- Massive Thirring model: $W = |u|^2|v|^2$

Massive Thirring Model (MTM)

The MTM in laboratory coordinates

$$\begin{cases} i(u_t + u_x) + v = 2|v|^2 u, \\ i(v_t - v_x) + u = 2|u|^2 v, \end{cases}$$

First three conserved quantities are

$$Q = \int_{\mathbb{R}} \left(|u|^2 + |v|^2\right) dx,$$

$$P = \frac{i}{2} \int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u} + v\bar{v}_x - v_x\bar{v})\, dx,$$

$$H = \frac{i}{2} \int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u} - v\bar{v}_x + v_x\bar{v})\, dx + \int_{\mathbb{R}} \left(-v\bar{u} - u\bar{v} + 2|u|^2|v|^2\right) dx.$$

An infinite set of conserved quantities is available thanks to the integrability of the MTM.

Local and global existence

Theorem

Assume $\mathbf{u}_0 \in H^s(\mathbb{R})$ *for any fixed* $s > \frac{1}{2}$. *There exists* $T > 0$ *such that the nonlinear Dirac equations admit a unique solution*

$$\mathbf{u}(t) \in C([0,T], H^s(\mathbb{R})) \cap C^1([0,T], H^{s-1}(\mathbb{R})) : \quad \mathbf{u}(0) = \mathbf{u}_0,$$

which depends continuously on the initial data.

Theorem

Assume that W *is a polynomial in variables* $|u|^2$ *and* $|v|^2$. *A local solution in* $H^{[s]}$ *is extended globally as* $\mathbf{u}(t) \in C(\mathbb{R}_+, H^{[s]}(\mathbb{R}))$.

References: Delgado (1978); Goodman-Weinstein-Holmes (2001); Selberg-Tesfahun (2010); Huh (2011); Zhang (2013).

Quick proof of global well-posedness in $H^1(\mathbb{R})$

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$$\partial_t \left(|u|^{2p+2} + |v|^{2p+2}\right) + \partial_x \left(|u|^{2p+2} - |v|^{2p+2}\right)$$
$$= i(p+1)(v\bar{u} - \bar{v}u)(|u|^{2p} - |v|^{2p}).$$

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- By Gronwall's inequality, we have

$$\|\mathbf{u}(t)\|_{L^{2p+2}} \leq e^{2|t|}\|\mathbf{u}(0)\|_{L^{2p+2}}, \quad t \in [0,T],$$

which holds for any $p \geq 0$ including $p \to \infty$.

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- This allows to control

$$\frac{d}{dt}\|\partial_x \mathbf{u}(t)\|^2_{L^2} \leq C_W e^{4(N-1)|t|}\|\partial_x \mathbf{u}(t)\|^2_{L^2},$$

where N is the degree of W in variables $|u|^2$ and $|v|^2$.

Existence of solitary waves

Time-periodic space-localized solutions

$$u(x,t) = U_\omega(x) e^{-i\omega t}, \quad v(x,t) = V_\omega(x) e^{-i\omega t}$$

satisfy a system of stationary Dirac equations. They are known in the closed analytic form

$$\left\{ \begin{array}{l} u(x,t) = i \sin(\gamma)\, \text{sech} \left[x \sin\gamma - i \frac{\gamma}{2} \right] \, e^{-it\cos\gamma}, \\ v(x,t) = -i \sin(\gamma)\, \text{sech} \left[x \sin\gamma + i \frac{\gamma}{2} \right] \, e^{-it\cos\gamma}. \end{array} \right.$$

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- Translations in x and t can be added as free parameters.
- Constraint $\omega = \cos\gamma \in (-1, 1)$ exists because spectrum of linear waves is located for $(-\infty, -1] \cup [1, \infty)$.
- Moving solitons can be obtained from the stationary solitons with the Lorentz transformation.

Orbital stability of solitary waves

Definition

We say that the solitary wave $e^{-i\omega t}\mathbf{U}_\omega(x)$ is orbitally stable if for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$, such that if

$$\|\mathbf{u}(\cdot, 0) - \mathbf{U}_\omega(\cdot)\|_{H^1} \leq \delta(\epsilon)$$

then

$$\inf_{\theta, a \in \mathbb{R}} \|\mathbf{u}(\cdot, t) - e^{-i\theta}\mathbf{U}_\omega(\cdot + a)\|_{H^1} \leq \epsilon,$$

for all $t > 0$.

- Spectral stability of Dirac solitons was mainly studied numerically, with the exception of recent results by A. Comech and his coauthors (N. Boussaid, S. Gustafson).
- Asymptotic stability of Dirac solitons was proved for quintic nonlinearities in 1D by Pelinovsky–Stefanov (2012) and in 3D by Boussaid–Cuccagna (2012).

Orbital stability of MTM solitons in H^1

Theorem

There is $\omega_0 \in (0,1]$ such that for any fixed $\omega = \cos\gamma \in (-\omega_0, \omega_0)$, the MTM soliton is a local non-degenerate minimizer of R in $H^1(\mathbb{R}, \mathbb{C}^2)$ under the constraints of fixed values of Q and P.

The higher-order Hamiltonian R is

$$R = \int_{\mathbb{R}} \Big[|u_x|^2 + |v_x|^2 - \frac{i}{2}(u_x \overline{u} - \overline{u}_x u)(|u|^2 + 2|v|^2) + \frac{i}{2}(v_x \overline{v} - \overline{v}_x v)(2|u|^2 + |v|^2)$$

$$-(u\overline{v} + \overline{u}v)(|u|^2 + |v|^2) + 2|u|^2|v|^2(|u|^2 + |v|^2) \Big] \, dx.$$

R is a conserved quantity of the MTM in addition to the standard Hamiltonian H, the charge Q, and the momentum P.

Similar works

- Sachs and Maddocks (1993) used higher-order conserved quantities of the KdV equation to prove orbital stability of n-solitons in $H^n(\mathbb{R})$.

- Kapitula (2006) used higher-order conserved quantities of the NLS equation to prove spectral and orbital stability of n-solitons in $H^n(\mathbb{R})$.

- Deconinck and Kapitula (2010) proved orbital stability of periodic waves in the KdV equation by adding lower-order Hamiltonians to the higher-order Hamiltonian, which has no minimum property at the periodic waves.

- Alejo and Munoz (2013) proved orbital stability of breathers in the modified KdV equation in $H^2(\mathbb{R})$ by using an additional conserved quantity.

The energy functionals

- Critical points of $H + \omega Q$ for a fixed $\omega \in (-1, 1)$ satisfy the stationary MTM equations. After the reduction $(u, v) = (U, \overline{U})$, we obtain the first-order equation

$$i\frac{dU}{dx} - \omega U + \overline{U} = 2|U|^2 U,$$

which is satisfied by the MTM soliton $U = U_\omega$.

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  which is satisfied by the MTM soliton $U = U_\omega$.

- Critical points of $R + \Omega Q$ for some fixed $\Omega \in \mathbb{R}$ satisfy another system of equations. After the reduction $(u, v) = (U, \overline{U})$, we obtain the second-order equation

$$\frac{d^2U}{dx^2} + 6i|U|^2\frac{dU}{dx} - 6|U|^4 U + 3|U|^2\bar{U} + U^3 = \Omega U.$$

  Nice surprise is that $U = U_\omega$ satisfies this second-order equation if $\Omega = 1 - \omega^2$.

The Lyapunov functional for MTM solitons

We define the energy functional in $H^1(\mathbb{R}, \mathbb{C}^2)$

$$\Lambda_\omega := R + (1-\omega^2)Q, \quad \omega \in (-1,1),$$

where $Q = \|u\|_{L^2}^2 + \|v\|_{L^2}^2$.

- U_ω is a critical point of Λ_ω.
- The second variation of Λ_ω is determined by the 4×4 matrix differential operator, which can be block-diagonalized (Chugunova and Pelinovsky, 2006):

$$S^T L S = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix},$$

where L_+ and L_- are 2×2 matrix Schrödinger operators.

The Linearized Operators

We want strict positivity of L in

$$S^T L S = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix}.$$

Unfortunately, operators L_+ and L_- have negative and zero eigenvalues. At least, the continuous spectrum of $L_\pm$ is strictly positive if $\omega^2 < 1$: $\sigma_c(L_\pm) = [1-\omega^2, \infty)$.

$$L_+ = \begin{bmatrix} \mathcal{L}_+ & -6\omega U_\omega^2 \\ -6\omega \overline{U}_\omega^2 & \overline{\mathcal{L}}_+ \end{bmatrix}, \quad L_- = \begin{bmatrix} \mathcal{L}_- & 2\omega U_\omega^2 \\ 2\omega \overline{U}_\omega^2 & \overline{\mathcal{L}}_- \end{bmatrix},$$

where

$$\mathcal{L}_+ = -\frac{d^2}{dx^2} - 6i|U_\omega|^2 \frac{d}{dx} U_\omega + 6|U_\omega|^4 - 3U_\omega^2 + 3\overline{U}_\omega^2 - 6\omega |U_\omega|^2 + 1 - \omega^2,$$

$$\mathcal{L}_- = -\frac{d^2}{dx^2} - 2i|U_\omega|^2 \frac{d}{dx} U_\omega - 2|U_\omega|^4 - U_\omega^2 + \overline{U}_\omega^2 - 2\omega |U_\omega|^2 + 1 - \omega^2.$$

The spectral problem of the operator L_-

Lemma

For any $\omega \in (-1, 1)$, L_- has exactly two eigenvalues below the continuous spectrum. One eigenvalue is zero for any ω. The other eigenvalue is positive for $\omega \in (0, 1)$, negative for $\omega \in (-1, 0)$, and zero for $\omega = 0$.

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By setting $u(x) = \varphi(x) e^{-i \int_0^x |U_\omega(x')|^2 dx'}$ in the spectral problem $L_- \mathbf{u} = \mu \mathbf{u}$, we obtain an equivalent spectral problem $\widetilde{L}\vec{\phi} = \mu \vec{\phi}$ with

$$\widetilde{L} = \begin{bmatrix} -\partial_x^2 + 1 - \omega^2 - 2\omega|U_\omega|^2 - 3|U_\omega|^4 & 2\omega|U_\omega|^2 \\ 2\omega|U_\omega|^2 & -\partial_x^2 + 1 - \omega^2 - 2\omega|U_\omega|^2 - 3|U_\omega|^4 \end{bmatrix}.$$

Furthermore, if we set $\psi_\pm := \varphi(x) \pm \bar{\varphi}(x)$, $z := \sqrt{1-\omega^2}x$, and $\mu := (1-\omega^2)\lambda$, we obtain two uncoupled spectral problems

$$-\frac{d^2\psi_+}{dz^2} + \left[1 - \frac{3(1-\omega^2)}{(\omega + \cosh(2z))^2}\right]\psi_+ = \lambda\psi_+ \qquad (1)$$

and

$$-\frac{d^2\psi_-}{dz^2} + \left[1 - \frac{3(1-\omega^2)}{(\omega + \cosh(2z))^2} - \frac{4\omega}{\omega + \cosh(2z)}\right]\psi_- = \lambda\psi_-. \qquad (2)$$

- The eigenfunction of Eq (2) for $\lambda = 0$ for any $\omega \in (-1, 1)$ is

$$\psi_0(z) = \frac{1}{(\omega + \cosh(2z))^{1/2}} > 0.$$

By Sturm's theory, there is no negative eigenvalue.

- For the problem with a deeper potential well

$$-\frac{d^2\psi_-}{dz^2} + \left[1 - \frac{8(1-\omega^2)}{(\omega + \cosh(2z))^2} - \frac{4\omega}{\omega + \cosh(2z)}\right]\psi_- = \lambda\psi_-,$$

there is the end-point resonance at $\lambda = 1$:

$$\psi_c(z) = \frac{\sinh(2z)}{\omega + \cosh(2z)}$$

By Sturm's theory, $\lambda = 0$ is the only isolated eigenvalue.

- The difference of potentials between Eq (1) and Eq (2) is

$$\Delta V(z) := \frac{4\omega}{\omega + \cosh(2z)}.$$

The zero eigenvalue for $\omega = 0$ is a positive eigenvalue for $\omega > 0$ and a negative eigenvalue for $\omega < 0$.

- For the problem with a deeper potential well

$$-\frac{d^2\psi}{dz^2} + \left[1 - \frac{3(1-\omega^2)}{(\omega + 1 + 2z^2)^2}\right]\psi = \lambda\psi,$$

there is the end-point resonance at $\lambda = 1$:

$$\tilde{\psi}_c(y) = \frac{z}{\sqrt{\omega + 1 + 2z^2}}.$$

By Sturm's theory, the eigenvalue above is the only isolated eigenvalue.

The spectral problem of the operator L_+

Lemma

There is $\omega_0 \in (0,1]$ such that for any fixed $\omega \in (-\omega_0, \omega_0)$, operator L_+ has exactly two eigenvalues below the continuous spectrum. One eigenvalue is zero for any ω. The other eigenvalue is positive for $\omega \in (-\omega_0, 0)$, negative for $\omega \in (0, \omega_0)$, and zero for $\omega = 0$.

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By setting $u(x) = \varphi(x) e^{-3i \int_0^x |U_\omega(x')|^2 dx'}$ in the spectral problem $L_+ \mathbf{u} = \mu \mathbf{u}$, where $\mathbf{u} = (u, \overline{u})^t$ and setting $z := \sqrt{1-\omega^2} x$ and $\mu := (1-\omega^2)\lambda$, we obtain an equivalent spectral problem

$$\begin{bmatrix} -\partial_z^2 + 1 + V_1(z) & V_2(z) \\ \overline{V}_2(z) & -\partial_z^2 + 1 + V_1(z) \end{bmatrix} \begin{bmatrix} \varphi \\ \bar{\varphi} \end{bmatrix} = \lambda \begin{bmatrix} \varphi \\ \bar{\varphi} \end{bmatrix},$$

where

$$V_1(z) := -\frac{3(1-\omega^2)}{(\omega + \cosh(2z))^2} - \frac{6\omega}{\omega + \cosh(2z)}$$

and

$$V_2(z) := -6\omega \frac{\left(1 + \omega \cosh(2z) + i\sqrt{1-\omega^2} \sinh(2z)\right)^2}{(\omega + \cosh(2z))^3}.$$

- $\lambda = 0$ is an eigenvalue for all $\omega \in (-1,1)$ with the eigenvector $(\varphi_0, \bar{\varphi}_0)$,

$$\varphi_0(z) = \frac{\omega \sinh(2z) + i\sqrt{1-\omega^2}\cosh(2z)}{(\omega + \cosh(2z))^{3/2}}.$$

- For $\omega = 0$, the zero eigenvalue is double, the end-points have no resonances, and no other eigenvalues exist.

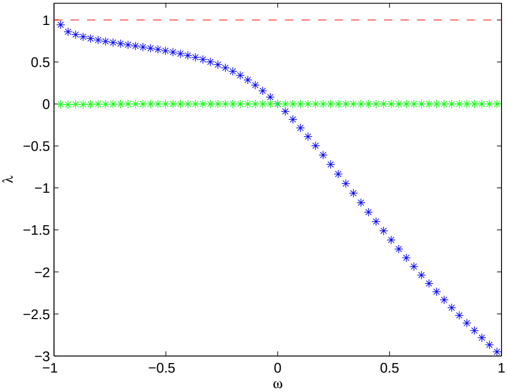
- The assertion is proved by the perturbation theory:

$$\begin{aligned} \left\langle \begin{bmatrix} \varphi_0 \\ -\bar{\varphi}_0 \end{bmatrix}, L_+ \begin{bmatrix} \varphi_0 \\ -\bar{\varphi}_0 \end{bmatrix} \right\rangle &= -12\omega \int_{\mathbb{R}} \frac{3 - \cosh(4z)}{\cosh(2z)^4} dz \\ &= -16\omega + \mathcal{O}(\omega^2). \end{aligned}$$

Conjecture on eigenvalues of the operator L_+

Conjecture

Operator L_+ has exactly two isolated eigenvalues and no end-point resonances for all $\omega \in (-1, 1)$. The non-zero eigenvalue is positive for all $\omega \in (-1, 0)$ and negative for all $\omega \in (0, 1)$.



Convexity of the energy functional

Consider again the energy functional in $H^1(\mathbb{R}, \mathbb{C}^2)$

$$\Lambda_\omega := R + (1 - \omega^2)Q, \quad \omega \in (-1, 1),$$

where $Q = \|u\|_{L^2}^2 + \|v\|_{L^2}^2$.

- U_ω is a critical point of Λ_ω.
- The second variation of Λ_ω at U_ω is associated with the matrix operator

$$S^T L S = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix},$$

  which has exactly one negative eigenvalue for $\omega < 0$ and $\omega > 0$ and a quadripole zero eigenvalue for $\omega = 0$.

Constrained Hilbert spaces

Let us assume that $(u,v) \in L^2(\mathbb{R};\mathbb{C}^2)$ satisfies the complex-valued constraints:

$$\int_{\mathbb{R}} \left(\bar{U}_\omega u + U_\omega v\right) dx = 0, \qquad (1)$$

$$\int_{\mathbb{R}} \left(\bar{U}'_\omega u + U'_\omega v\right) dx = 0, \qquad (2)$$

- Real part of Eq (1) corresponds to fixed Q (charge).
- Imaginary part of Eq. (2) corresponds to fixed P (momentum).
- Imaginary part of Eq. (1) corresponds to orthogonality to the gauge translation mode $u \mapsto ue^{i\alpha}$, $v \mapsto ve^{i\alpha}$.
- Real part of Eq. (2) corresponds to orthogonality to the space translation mode $u(x) \mapsto u(x+x_0)$, $v(x) \mapsto v(x+x_0)$.

Convexity of the energy functional

Theorem

There is $\omega_0 \in (0,1]$ such that for any fixed $\omega \in (-\omega_0, \omega_0)$, the Lyapunov functional Λ_ω is strictly convex at $(u,v) = (U_\omega, \bar{U}_\omega)$ in the orthogonal complement of the complex-valued constraints (1) and (2).

The second variation of Λ_ω at U_ω is associated with the matrix operator

$$S^T L S = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix},$$

The constraints remove the negative eigenvalue of L_+ and L_- for $\omega > 0$ and $\omega < 0$ and the zero eigenvalue for all ω.

Orbital stability result

- R, Q, and P are conserved in time t.
- Strict positivity of L implies

$$\langle L\mathbf{u}, \mathbf{u}\rangle_{L^2} \geq C\|\mathbf{u}\|_{H^1}$$

  for all $\mathbf{u} \in H^1(\mathbb{R}; \mathbb{C}^2)$ in the constrained space.
- Then, we obtain the lower bound via standard arguments:

$$\Lambda_\omega(\mathbf{u}) - \Lambda_\omega(\mathbf{U}_\omega) \geq \inf_{\theta, x_0} \|\mathbf{u}(\cdot, t) - e^{i\theta}\mathbf{U}_\omega(\cdot + x_0)\|_{H^1}$$

- This yields orbital stability of $\mathbf{U}_\omega$ for $\omega \in (-\omega_0, \omega_0)$.

Orbital stability of MTM solitons in L^2

Well-posedness (Candy, 2011): For any $(u_0, v_0) \in L^2(\mathbb{R})$, there exists a unique solution of the MTM $(u, v) \in C(\mathbb{R}, L^2(\mathbb{R}))$:

$$\|u(\cdot,t)\|_{L^2}^2 + \|v(\cdot,t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2.$$

Theorem

Let $(u, v) \in C(\mathbb{R}; L^2(\mathbb{R}))$ be a solution of the MTM system and λ_0 be a complex non-zero number. There exist a real positive constant ϵ such that if the initial value $(u_0, v_0) \in L^2(\mathbb{R})$ satisfies

$$\|u_0 - u_{\lambda_0}(\cdot,0)\|_{L^2} + \|v_0 - v_{\lambda_0}(\cdot,0)\|_{L^2} \leq \epsilon,$$

then for every $t \in \mathbb{R}$, there exists $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_0| \leq C\epsilon$,

$$\inf_{a,\theta\in\mathbb{R}} \left(\|u(\cdot+a,t) - e^{-i\theta} u_\lambda(\cdot,t)\|_{L^2} + \|v(\cdot+a,t) - e^{-i\theta} v_\lambda(\cdot,t)\|_{L^2}\right) \leq C\epsilon,$$

where the constant C is independent of ϵ and t.

Lax operators for the MTM

The MTM is obtained from the compatibility condition of the linear system

$$\vec{\phi}_x = L\vec{\phi} \quad \text{and} \quad \vec{\phi}_t = A\vec{\phi},$$

where

$$L = \frac{i}{2}(|v|^2 - |u|^2)\sigma_3 - \frac{i\lambda}{\sqrt{2}}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{\sqrt{2}\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\frac{1}{\lambda^2} - \lambda^2\right)\sigma_3$$

and

$$A = -\frac{i}{4}(|u|^2 + |v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\sigma_3$$

References:
Kaup–Newell (1977); Kuznetsov–Mikhailov (1977).

Bäcklund transformation for the MTM

- Let (u, v) be a C^1 solution of the MTM system.
- Let $\vec{\phi} = (\phi_1, \phi_2)^t$ be a C^2 nonzero solution of the linear system associated with (u, v) and $\lambda = \delta e^{i\gamma/2}$.

A new C^1 solution of the MTM system is given by

$$\mathbf{u} = -u\frac{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2} + \frac{2i\delta^{-1}\sin\gamma\overline{\phi}_1\phi_2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}$$

$$\mathbf{v} = -v\frac{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2} - \frac{2i\delta\sin\gamma\overline{\phi}_1\phi_2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2},$$

A new C^2 nonzero solution $\vec{\psi} = (\psi_1, \psi_2)^t$ of the linear system associated with $(\mathbf{u}, \mathbf{v})$ and same λ is given by

$$\psi_1 = \frac{\overline{\phi}_2}{|e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2|}, \quad \psi_2 = \frac{\overline{\phi}_1}{|e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2|}.$$

Bäcklund transformation $0 \leftrightarrow 1$ soliton

Let $(u, v) = (0, 0)$ and define

$$\begin{cases} \phi_1 = e^{\frac{i}{4}(\lambda^2-\lambda^{-2})x+\frac{i}{4}(\lambda^2+\lambda^{-2})t}, \\ \phi_2 = e^{-\frac{i}{4}(\lambda^2-\lambda^{-2})x-\frac{i}{4}(\lambda^2+\lambda^{-2})t}. \end{cases}$$

Then, $(\mathbf{u}, \mathbf{v}) = (u_\lambda, v_\lambda)$.

If $\lambda = e^{i\gamma/2}$ (stationary case), the vector $\vec{\psi}$ is given by

$$\begin{cases} \psi_1 = e^{\frac{1}{2}x\sin\gamma+\frac{i}{2}t\cos\gamma} \left|\operatorname{sech}\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right|, \\ \psi_2 = e^{-\frac{1}{2}x\sin\gamma-\frac{i}{2}t\cos\gamma} \left|\operatorname{sech}\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right|. \end{cases}$$

It decays exponentially as $|x| \to \infty$.

Note that if $(u, v) = (u_\lambda, v_\lambda)$ and $\vec{\phi} = \vec{\psi}$, then $(\mathbf{u}, \mathbf{v}) = (0, 0)$.

Similar works

- Merle and Vega (2003) used the Miura transformation to prove asymptotic stability of KdV solitons in L^2.
- Mizumachi and Tzvetkov (2011) applied the same transformation to prove L^2-stability of line solitons in the KP-II equation under periodic transverse perturbations.
- Mizumachi and Pego (2008); Hoffman and Wayne (2009) used Bäcklund transformation to prove asymptotic stability of Toda lattice one-soliton and multi-solitons.
- Mizumachi and Pelinovsky (2012); Contreras and Pelinovsky (2013) used Bäcklund transformation to prove orbital stability of NLS one-soliton and multi-solitons in L^2.

Steps in the proof of the main result

- Step 1: From a perturbed one-soliton to a small solution at the initial time $t = 0$.
- Step 2: Time evolution of the small solution for $t \in \mathbb{R}$.
- Step 3: From the small solution to the perturbed one-soliton for every $t \in \mathbb{R}$.
- Step 4: Approximation arguments in $H^2(\mathbb{R})$ to control the compatibility condition of the linear system for every $t \in \mathbb{R}$.

Asymptotic stability of MTM solitons ?

To prove asymptotic stability of MTM solitons, one needs first to establish the space where small initial data (u_0, v_0) produce no eigenvalues in the spectral problem

$$\vec{\phi}_x = L(u_0, v_0, \lambda)\vec{\phi},$$

where

$$L = \frac{i}{2}(|v|^2 - |u|^2)\sigma_3 - \frac{i\lambda}{\sqrt{2}}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{\sqrt{2}\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\frac{1}{\lambda^2} - \lambda^2\right)\sigma_3$$

For NLS-type problems, it is well known that $\|u_0\|_{L^1}$ has to be small, e.g. if $\|\sqrt{1+x^2}u_0\|_{L^2}$ is small. Asymptotic stability of NLS solitons follows from an application of the auto–Backlund transformation (Deift–Park, 2011; Cuccagna–Pelinovsky, 2013).

For MTM systems, the precise conditions when the spectral problem has no eigenvalues are unknown...