Dynamics of shocks in the modular Burgers equation

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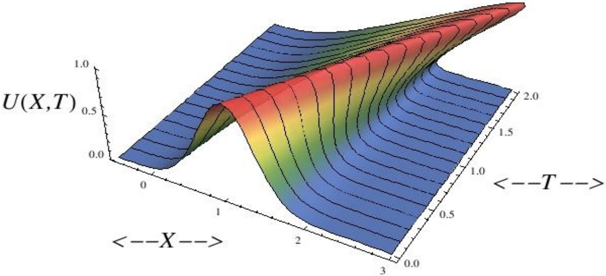
Inviscid Shocks

- Dynamics of a Conservation Law

$$\partial_t v + \partial_x f(v) = 0$$

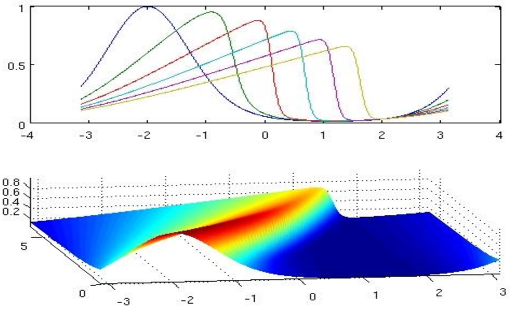
generate shock singularities in finite time from a large class of smooth data and for smooth $f(v)$.



Viscous Shocks

- Diffusive regularization leads to a viscous Burgers equation

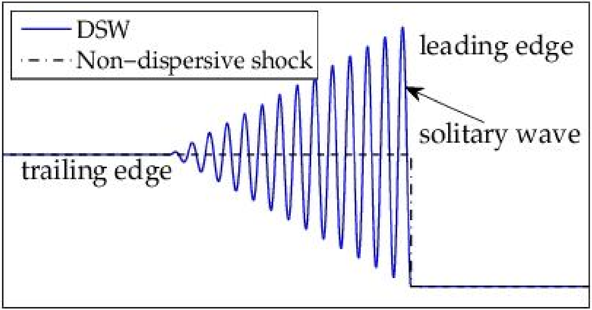
$$\partial_t v + \partial_x f(v) = \varepsilon^2 \partial_x^2 v.$$



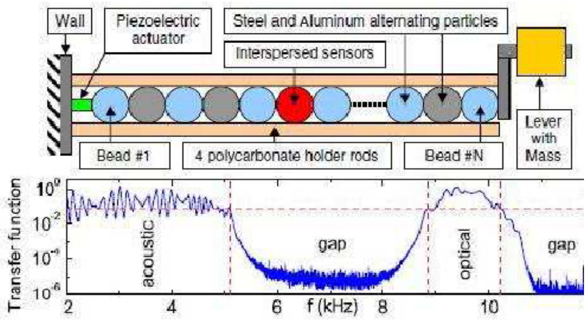
Dispersive Shocks

- Dispersive regularization leads to the Korteweg–de Vries equation

$$\partial_t v + \partial_x f(v) + \varepsilon^3 \partial_x^3 v = 0.$$



Granular chains



- Granular chains contain densely packed, elastically interacting particles with Hertzian contact forces.
- N. Boechler, G. Theocharis, P.G. Kevrekidis, M.A. Porter, C. Daraio.

Logarithmic models

Granular chains are modeled with Newton's equations of motion:

$$x_n''(t) = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z},$$

where x_n is the displacement of the nth particle and V is the interaction potential for spherical beads (H. Hertz, 1882):

$$V(x) = |x|^{1+\alpha} H(-x), \quad \alpha = \frac{3}{2},$$

where H is the step (Heaviside) function. For hollow materials, $\alpha \to 1$.

- The conservative model yields the logarithmic KdV equation

$$\partial_t v + \partial_x (v \log |v|) + \partial_x^3 v = 0$$

- The dissipative model yields the logarthmic Burgers equation

$$\partial_t v + \partial_x (v \log |v|) = \partial_x^2 v$$

G. James & D. P., 2014; G. James, 2021

Modular nonlinearity

In a similar context of dynamics of particles with piecewise interaction potentials, models with modular nonlinearities have been derived:

- The modular KdV equation

$$\partial_t v + \partial_x |v| + \partial_x^3 v = 0$$

- The modular Burgers equation

$$\partial_t v = \partial_x |v| + \partial_x^2 v$$

C. M. Hedberg, O. V. Rudenko, 2016–2018

The models are linear for sign-definite solutions. Nonlinear waves correspond to the sign-changing solutions, for which the modeling problem becomes a moving interface problem between solutions of linear equations.

Traveling waves in the modular Burgers equation

Starting with

$$\partial_t v = \partial_x |v| + \partial_x^2 v,$$

we can think of the traveling wave solutions $v(t,x) = W(x - ct)$, where

$$W''(x) + \mathrm{sign}(W) W'(x) + cW'(x) = 0, \quad x \in \mathbb{R}.$$

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Q What is the function space for solutions?

A Space of piecewise C^2 functions satisfying the interface conditiion

$$[W'']_-^+(x_0) = -2|W'(x_0)|$$

at each interface located at x_0, where $[f]_-^+(x_0) = f(x_0^+) - f(x_0^-)$ is the jump of a piecewise continuous function f across x_0.

Traveling waves in the modular Burgers equation

Integrating once yields

$$W'(x) + |W(x)| + cW(x) = d, \quad x \in \mathbb{R},$$

where the constant of integration is identical for all pieces of piecewise C^2 function $W(x) : \mathbb{R} \to \mathbb{R}$.

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If $W_\pm = \lim_{x \to \pm\infty} W(x)$, then bounded solutions only exist if $W_- < 0 < W_+$ with uniquely selected speed

$$c = \frac{W_+ + W_-}{W_+ - W_-}$$

and uniquely defined profile W up to spatial translations:

$$W(x) = \begin{cases} W_+(1 - e^{-(1+c)x}), & x > 0, \\ W_-(1 - e^{(1-c)x}), & x < 0. \end{cases}$$

If $W_+ = -W_-$, then $c = 0$ and $W(-x) = -W(x)$ is odd.

Motivational questions

1. Is the viscous shock W stable in the time evolution of the modular Burgers equation?
2. How does the interface moves in the time evolution depending on the initial conditions?
3. Is there the finite-time extinction of the area between two consequent interfaces?
4. How can we model the moving interface problems numerically?

Interface equation

It is natural to look for solutions of the modular Burgers equation

$$\partial_t v = \partial_x |v| + \partial_x^2 v$$

in class of piecewise C^2 functions.

If $v(t,\xi(t)) = 0$ defines the interface at $x = \xi(t)$, then

$$[v_t]_-^+(\xi(t)) = 0 \quad \text{and} \quad [v_x]_-^+(\xi(t)) = 0,$$

whereas

$$[v_{xx}]_-^+(\xi(t)) = -2|v_x(t,\xi(t))|$$

determines the interface equation for $\xi(t)$, $t > 0$.

Simple case: odd data

It follows from

$$\partial_t v = \partial_x |v| + \partial_x^2 v$$

that if $v(0,-x) = -v(0,x)$ is odd at $t = 0$, then $v(t,-x) = -v(t,x)$ remains odd for all $t > 0$. The interface is located at $\xi(t) = 0$, $t > 0$.

Adding an odd perturbation $w(t,x)$ to the odd viscous shock $W(x) = (1 - e^{-|x|})\mathrm{sgn}(x)$ with $c = 0$ as $v(t,x) = W(x) + w(t,x)$, we get the linear initial-boundary-value problem

$$\begin{cases} w_t = w_x + w_{xx}, & x > 0, & t > 0, \\ w(t,0) = 0, & & t > 0, \\ w(t,x) \to 0 & \text{as } x \to +\infty, & t > 0, \\ w(0,x) = w_0(x), & x > 0, & \end{cases}$$

Main result: odd data

Theorem (Le, Pelinovsky, Poullet, 2021)

For every $\epsilon > 0$ there is $\delta > 0$ such that for every odd v_0 satisfying

$$\|v_0 - W\|_{H^2} < \delta,$$

there exists a unique odd solution $v(t,x)$ with $v(0,x) = v_0(x)$ satisfying

$$\|v(t,\cdot) - W\|_{H^2} < \epsilon, \quad t > 0$$

and

$$\|v(t,\cdot) - W\|_{W^{2,\infty}} \to 0 \quad \text{as} \quad t \to +\infty.$$

- Since $W(0) = 0$, $W'(0) = 1$, and H^2 is embedded into C^1, we have $v(t,x) = W(x) + w(t,x) > 0$ for every $x > 0$ and $t > 0$.
- The result is extended to $W(x - ct)$ under suitably scaled data.

General case: single interface

Consider the viscous shock $W(x) = (1 - e^{-|x|})\mathrm{sgn}(x)$ with $c = 0$ but make no assumption on the symmetry of the perturbations. With the decomposition

$$v(t,x) = W(x - \xi(t)) + w(t, x - \xi(t)), \quad y = x - \xi(t),$$

we have now the linear initial-boundary-value problem

$$\left\{ \begin{array}{lll} w_t = (\xi'(t) \pm 1) w_y + w_{yy} + \xi'(t) W'(y), & \pm y > 0, & t > 0, \\ w(t,0) = 0, & & t > 0, \\ w(t,x) \to 0 & \text{as} \;\; y \to \pm\infty, & t > 0, \\ w(0,y) = w_0(y), & y \in \mathbb{R}, & \end{array} \right.$$

The two equations on half-lines are coupled by the interface conditions

$$(\xi'(t) \pm 1) w_y(t, 0^{\pm}) + w_{yy}(t, 0^{\pm}) + \xi'(t) = 0,$$

which are consistent due to the conditions $[u_{xx}]_-^+(\xi(t)) = -2|u_x(t,\xi(t))|$.

Main result: general data

Theorem (Le, Pelinovsky, Poullet, 2021)

Fix $\alpha \in \left(0, \frac{1}{2}\right)$. *For every* $\epsilon > 0$ *there is* $\delta > 0$ *s.t. for every* v_0 *s.t.*

$$\|v_0 - W\|_{H^2 \cap W^{2,\infty}} + \|e^{\alpha|\cdot|}(v_0 - W)\|_{W^{2,\infty}} < \delta$$

there exists a unique solution $v(t,x)$ *with* $v(0,x) = v_0(x)$ *satisfying*

$$\|v(t, \cdot + \xi(t)) - W\|_{H^2 \cap W^{2,\infty}} < \epsilon, \quad t > 0$$

and

$$\|v(t, \cdot + \xi(t)) - W\|_{W^{2,\infty}} \to 0 \quad \text{as} \quad t \to +\infty,$$

with $\xi' \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ *and* $\xi_\infty := \lim_{t \to +\infty} \xi(t)$.

Reformulation for numerical approximations

The original problem for general perturbation $w(t, y)$ with $y = x - \xi(t)$:

$$\begin{cases} w_t = (\xi'(t) \pm 1) w_y + w_{yy} + \xi'(t) e^{-y}, & \pm y > 0, & t > 0, \\ w(t,0) = 0, & & t > 0, \\ w(t,x) \to 0 & \text{as } \; y \to \pm\infty, & t > 0, \\ w(0,y) = w_0(y), & y \in \mathbb{R}, & \end{cases}$$

By using variables $v^{\pm}(t, y) := w(t, y) \mp w(t, -y)$ with $y > 0$ we obtain the coupled system

$$\begin{cases} v_t^+ = v_y^+ + v_{yy}^+ + \xi'(t) v_y^-, & y > 0, \\ v_t^- = v_y^- + v_{yy}^- + \xi'(t) v_y^+ + 2\xi'(t) e^{-y}, & y > 0, \end{cases}$$

subject to $v^{\pm}(t, 0) = 0$, $v_y^-(t, 0) = 0$, and $\xi'(t) = -\frac{v_{yy}^-(t,0)}{2 + v_y^+(t,0)}$.

Remarks on the numerical method

- Central-difference approximation of spatial derivatives.
- Neumann condition for $v_y^-(t,0) = 0$ is modelled with an extra grid point $v_{-1}^-(t) = v_1^-(t)$.
- The smoothness condition for $v_y^+(t,0) + v_{yy}^+(t,0) = 0$ is modelled with an extra grid point

$$v_{-1}^+(t) = -\frac{2+h}{2-h} v_1^+(t).$$

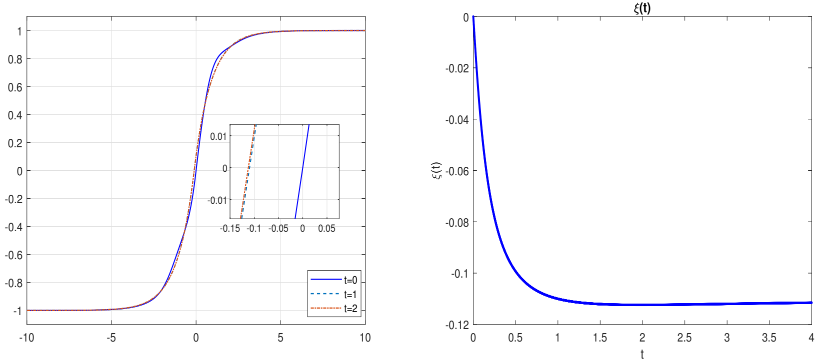
- The interface condition $\xi'(t) = -\frac{v_{yy}^-(t,0)}{2+v_y^+(t,0)}$ is resolved as

$$\xi'(t) = -\frac{(2-h)v_1^-(t)}{hv_1^+(t) + h^2(2-h)}.$$

- Time steps are performed with the implicit Crank-Nicholson method

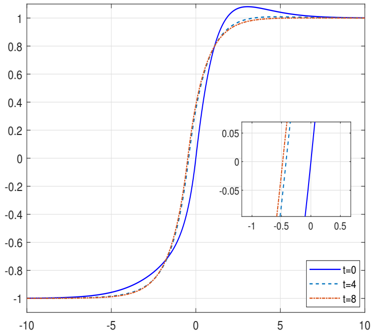
Initial data with Gaussian decay

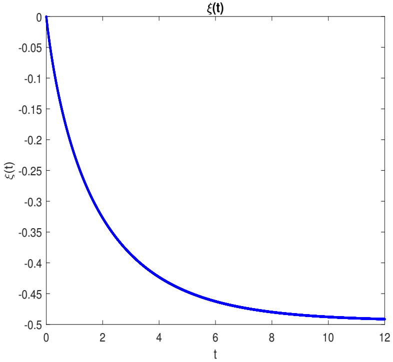
$$v^{+}(0,y) = 0.1(y - 0.5y^2)e^{-y^2}, \quad v^{-}(0,y) = 0.5y^2 e^{-y^2}.$$



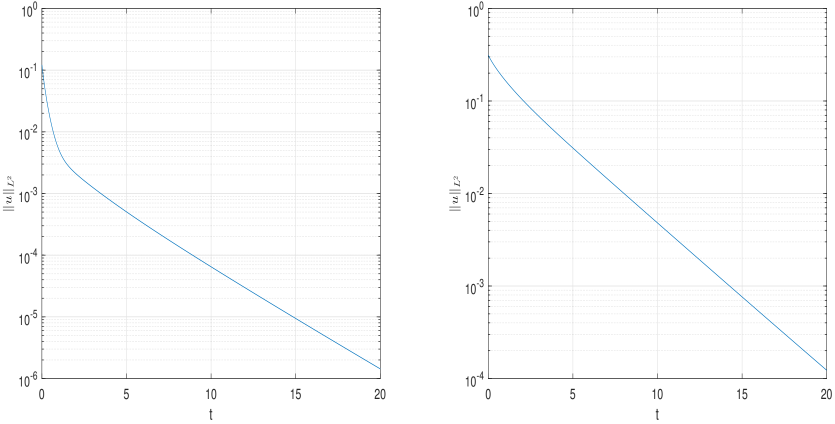
Initial data with exponential decay

$$v^+(0,y) = 0.1(y + 0.5y^2)e^{-y}, \quad v^-(0,y) = 0.5y^2 e^{-y},$$



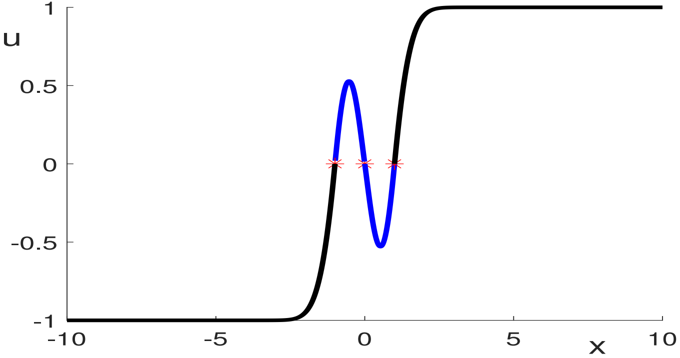


Convergence in time for L^2-norm of perturbation



Initial data with multiple interfaces

Main question: Is there the finite-time extinction of the area between two consequent interfaces?



Interface at $x = 0$ persists for odd data. Interfaces at $x = \pm\xi(t)$ move.

A simple argument suggesting finite-time coalescence

Let $z(t,x) := 1 - u(t,x)$. It satisfies $z_t = -|1-z|_x + z_{xx}$.
If $z(0,\cdot) : (0,\infty) \to \mathbb{R}$ is positive and integrable, then $z(t,\cdot) : (0,\infty) \to \mathbb{R}$ is positive and integrable for $t > 0$ by comparison principle.

We have for some time $t \in [0, \tau_0)$

$$0 < \xi(t) \leq \int_0^{\xi(t)} z(t,x)dx \leq \int_0^{\infty} z(t,x)dx =: M(t),$$

because $z(t,x) \geq 1$ for $x \in [0, \xi(t)]$ and $z(t,x) \geq 0$ for $x \in [\xi(t), \infty)$.

On the other hand,

$$\frac{dM}{dt} = -1 - z_x(t,0) \leq -1.$$

Hence, $M(t) \leq M(0) - t$ and we have finite-time coalescence: $\xi(\tau_0) = 0$.

Reformulation for numerical approximations

The original problem is

$$\begin{cases} u_t = -u_x + u_{xx}, & u(t,x) < 0, & 0 < x < \xi(t), \\ u_t = u_x + u_{xx}, & u(t,x) > 0, & \xi(t) < x < \infty, \\ u(t,0) = 0, & u(t,\xi(t)) = 0, & \lim_{x\to+\infty} u(t,x) = 1, \end{cases}$$

By using $y := x/\xi(t)$, the boundary-value problem is mapped to the time-independent regions:

$$\begin{cases} u_t = \xi^{-1}(\xi' y - 1)u_y + \xi^{-2} u_{yy}, & u(t,y) < 0, & 0 < y < 1, \\ u_t = \xi^{-1}(\xi' y + 1)u_y + \xi^{-2} u_{yy}, & u(t,y) > 0, & 1 < y < \infty, \\ u(t,0) = 0, & u(t,1) = 0, & \lim_{y\to+\infty} u(t,y) = 1, \end{cases}$$

closed with the interface condition:

$$\xi'(t) = -1 - \frac{u_{yy}(t,1^+)}{\xi(t) u_y(t,1)} = +1 - \frac{u_{yy}(t,1^-)}{\xi(t) u_y(t,1)}.$$

Remarks on the numerical method

- Central-difference approximation of spatial derivatives.
- The grid on $[0, 1]$ is complemented with the extra grid point $y_{N+1} = 1 + h$ and the approximation u^*_{N+1}. The grid on $[1, L]$ with $L = 10$ is complemented with the extra grid point $y_{N-1} = 1 - h$ and the approximation u^*_{N-1}. Note that $u^*_{N\pm1} \neq u_{N\pm1}$.
- The additional variables u^*_{N+1} and u^*_{N-1} are found from the interface conditions: $[u_y]^+_-(1) = 0$ and $[u_{yy}]^+_-(1) = -2\xi(t)|u_y(t, 1)|$. This yields the relation between linear advection-diffusion equation and

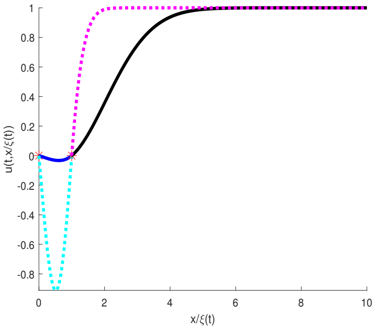
$$\xi'(t) = -\frac{(2 - h\xi)(u_{N+1} + u_{N-1})}{h\xi(u_{N+1} - u_{N-1})}.$$

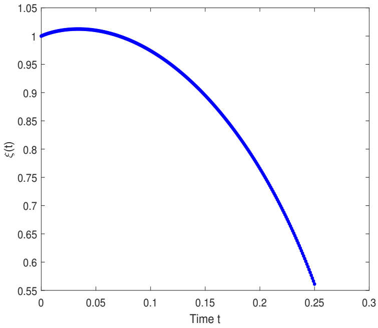
- Time steps are performed with the implicit Crank-Nicholson method

Initial data and evolution: $\alpha = 1.5$

$$u_0(x) = \begin{cases} x(1-x)(ax^2+bx+c), & 0 < x < 1, \\ 1 - e^{-\alpha(x^2-1)}, & 1 < x < \infty, \end{cases}$$

with $\xi'(0) = 2(\alpha - 1)$, where a, b, c are uniquely defined by α.

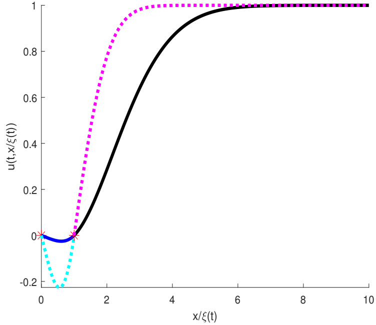


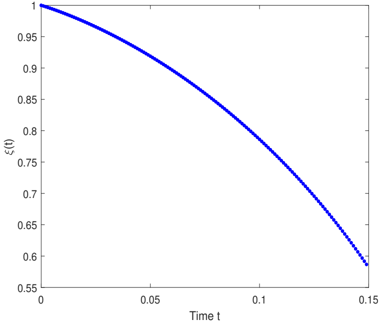


Initial data and evolution: $\alpha = 0.5$

$$u_0(x) = \begin{cases} x(1-x)(ax^2+bx+c), & 0 < x < 1, \\ 1 - e^{-\alpha(x^2-1)}, & 1 < x < \infty, \end{cases}$$

with $\xi'(0) = 2(\alpha - 1)$, where a, b, c are uniquely defined by α.





Conjecture based on numerical data [P., de Rijk, 2023]

There exists $t_0 \in (0, \infty)$ such that

$$\xi(t) \sim \sqrt{t_0 - t}, \quad u_x(t, \xi(t)) \sim (t_0 - t), \quad u_{xx}(t, \xi(t)^-) \sim \sqrt{t_0 - t}.$$

This is in agreement with

$$\xi'(t) = +1 - \frac{u_{xx}(t, \xi(t)^-)}{u_x(t, 1)}.$$

Furthermore, we conjecture

$$\left| \int_0^{\xi(t)} u(t, x) dx \right| \sim (t_0 - t)^2, \quad \int_0^{\xi(t)} u^2(t, x) dx \sim \sqrt{(t_0 - t)^7},$$

in agreement with the balance laws

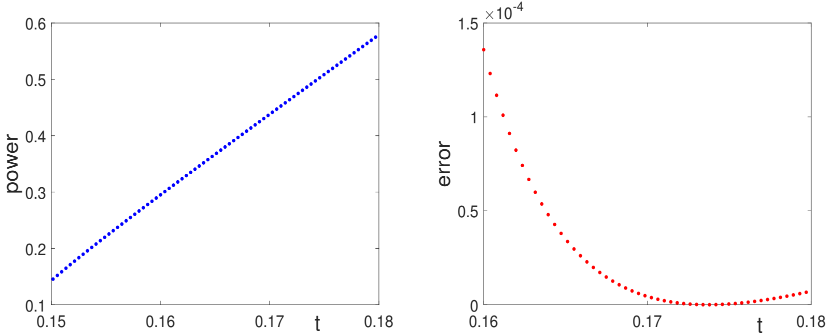
$$\frac{d}{dt} \int_0^{\xi(t)} u dx = u_x(t, \xi(t)) - u_x(t, 0), \quad \frac{d}{dt} \int_0^{\xi(t)} u^2 dx = -2 \int_0^{\xi(t)} u_x^2 dx.$$

The method of data extraction, e.g. for $\xi(t) \sim \sqrt{t_0 - t}$

For a fixed value of t_0 (past the termination time of our computations) , we compute c_1 (left) and c_2 in the linear regression

$$\log(\xi(t)) \quad \text{versus} \quad c_1 \log(t_0 - t) + c_2$$

as well as the approximation error (right). The minimal error of 10^{-9} is attained at $t_0 = 0.17$ with $c_1 = 0.492$.



Summary

- Evolution of the modular Burgers equation is considered.
- Asymptotic stability of a traveling viscous shock is proven and illustrated numerically.
- It is shown that shock waves with multiple interfaces extinct in a finite time due to finite-time coalesence of interfaces
- A precise scaling law of the finite-time coalescence is suggested based on the numerical data.

Open question

1. Numerical approximations of shock waves with multiple interfaces as a problem with moving boundaries.

2. Numerical approximation of solitary waves with multiple interfaces in the modular KdV equation.

3. Analytical proof of well-posedness of the linear evolution with multiple interfaces.

4. Analytical proof of the precise scaling law of the finite-time coalescence.

5. Expanding methods to the Burgers and KdV models with logartihmic nonlinearities...

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