Traveling waves and breathers in the nonlocal NLS models

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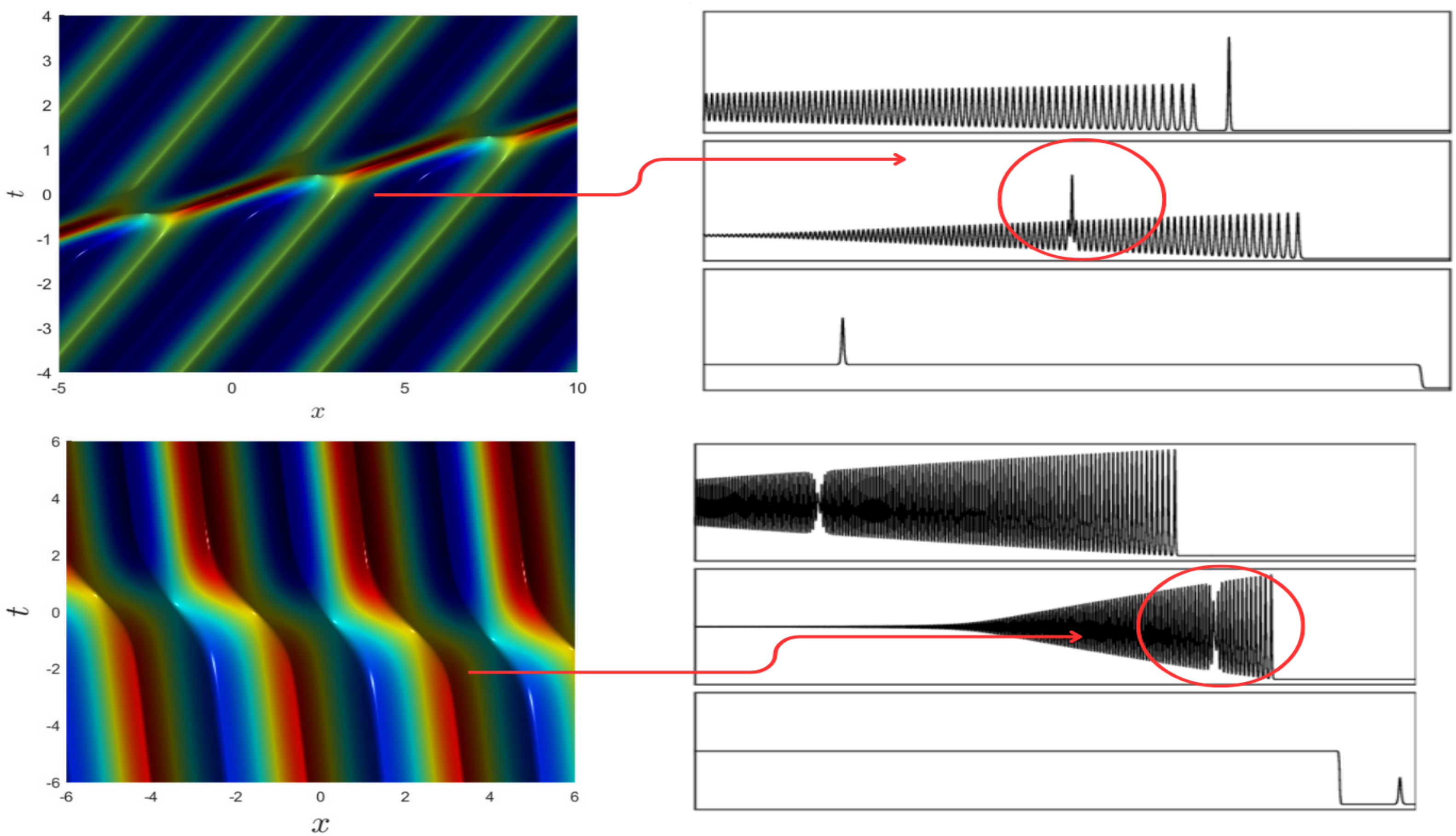
Section 1. Motivations

The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \qquad \text{(KdV)}$$

with the step-like data $\lim_{x\to-\infty} u(t,x) = u_-$, $\lim_{x\to+\infty} u(t,x) = u_+$, $u_- \neq u_+$, has been used for both analysis and applications.



M. Ablowitz, J. Cole, G. El, M. Hoefer, X. Luo, Stud. Appl. Math. **151** (2023) 795–856

Y. Mao, S. Chandramouli, W. Xu, M. Hoefer, Phys. Rev. Lett. **131** (2023) 147201

Breathers = solitary waves on the traveling periodic wave

Traveling waves $u(x,t) = \phi(x - ct)$ satisfy

$$u_t + 6uu_x + u_{xxx} = 0 \quad \Rightarrow \quad \phi''' + 6\phi\phi' - c\phi' = 0.$$

After integration(s), it yields

$$\phi'' + 3\phi^2 - c\phi = b \quad \Rightarrow \quad (\phi')^2 + 2\phi^3 - c\phi^2 = 2b\phi + d$$

with three parameters (b, c, d). Due to scaling transformation

$$u(x,t) \mapsto \alpha u(\alpha x, \alpha^3 t)$$

and Galilean transformation

$$u(x,t) \mapsto \beta + u(x - 6\beta t, t),$$

only one parameter is independent and the normalized TW solution is

$$\phi(x) = 2k^2 \mathrm{cn}^2(x;k), \;\; b_0 = 4k^2(1-k^2), \;\; c_0 = 4(2k^2 - 1), \;\; d_0 = 0.$$

Stability of the traveling periodic wave

The KdV equation with the TW solution $u(x,t) = \phi(x-ct)$ has the Lax pair

$$\mathcal{L}v = \lambda v, \quad \frac{\partial v}{\partial t} = \mathcal{M}v,$$

where $\mathcal{L} = -\partial_x^2 - \phi(x-ct)$ and $\mathcal{M} = -4\partial_x^3 - 6\phi(x-ct)\partial_x - 3\phi'(x-ct)$.

Separation of the variables as $v(x,t) = w(x-ct)e^{\omega(\lambda)t}$ yields the characteristic polynomial

$$\omega^2 + 16P(\lambda) = 0,$$

where

$$P(\lambda) = \lambda^3 + \frac{c}{2}\lambda^2 + \frac{c^2-4b}{16}\lambda - \frac{d+bc}{16} = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3).$$

The eigenfunction $w \in L^\infty(\mathbb{R})$ is defined by the Schrödinger equation $\mathcal{L}w = \lambda w$ with the elliptic potential. If $\phi(x+L) = \phi(x)$ is spatially periodic, then $w(x+L) = w(x)e^{i\kappa x}$ is quasi-periodic (Floquet's theorem) if and only if λ belongs to the Lax spectrum:

$$\lambda \in \sigma_L := [\lambda_1, \lambda_2] \cup [\lambda_3, \infty).$$

G. Lamé, 1837.

Traveling periodic waves are spectrally stable

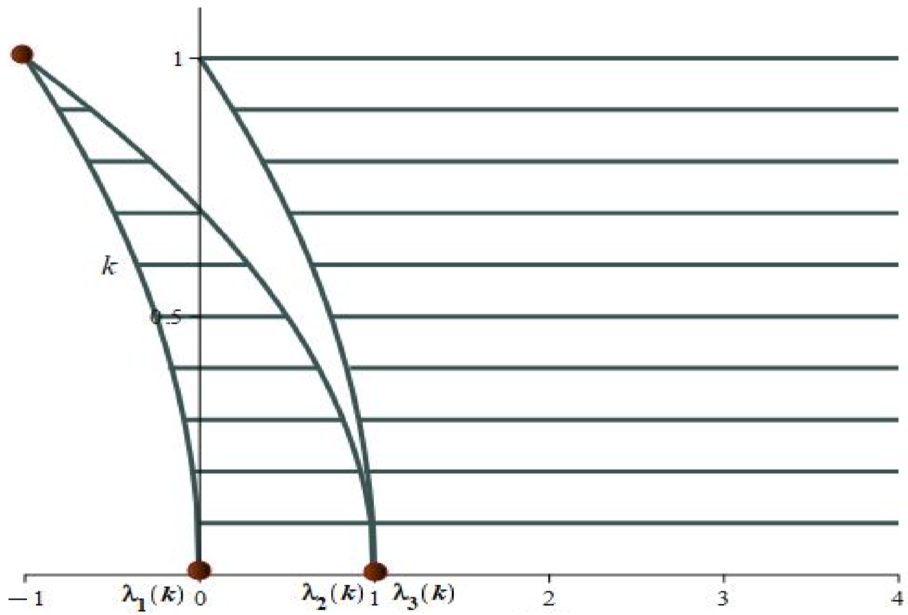
Each bounded solution $w \in L^\infty(\mathbb{R})$ of the Lax pair generates bounded solutions of the linearized KdV equation with the perturbation

$$u(x,t) = \phi(x - ct) + \tilde{u}(x,t),$$

by means of the squared eigenfunction relation

$$\tilde{u}(x,t) = w(x - ct) w'(x - ct) e^{2\omega(\lambda) t}.$$

M. Ablowitz, D. Kaup, A. Newell, and H. Segur, 1974.

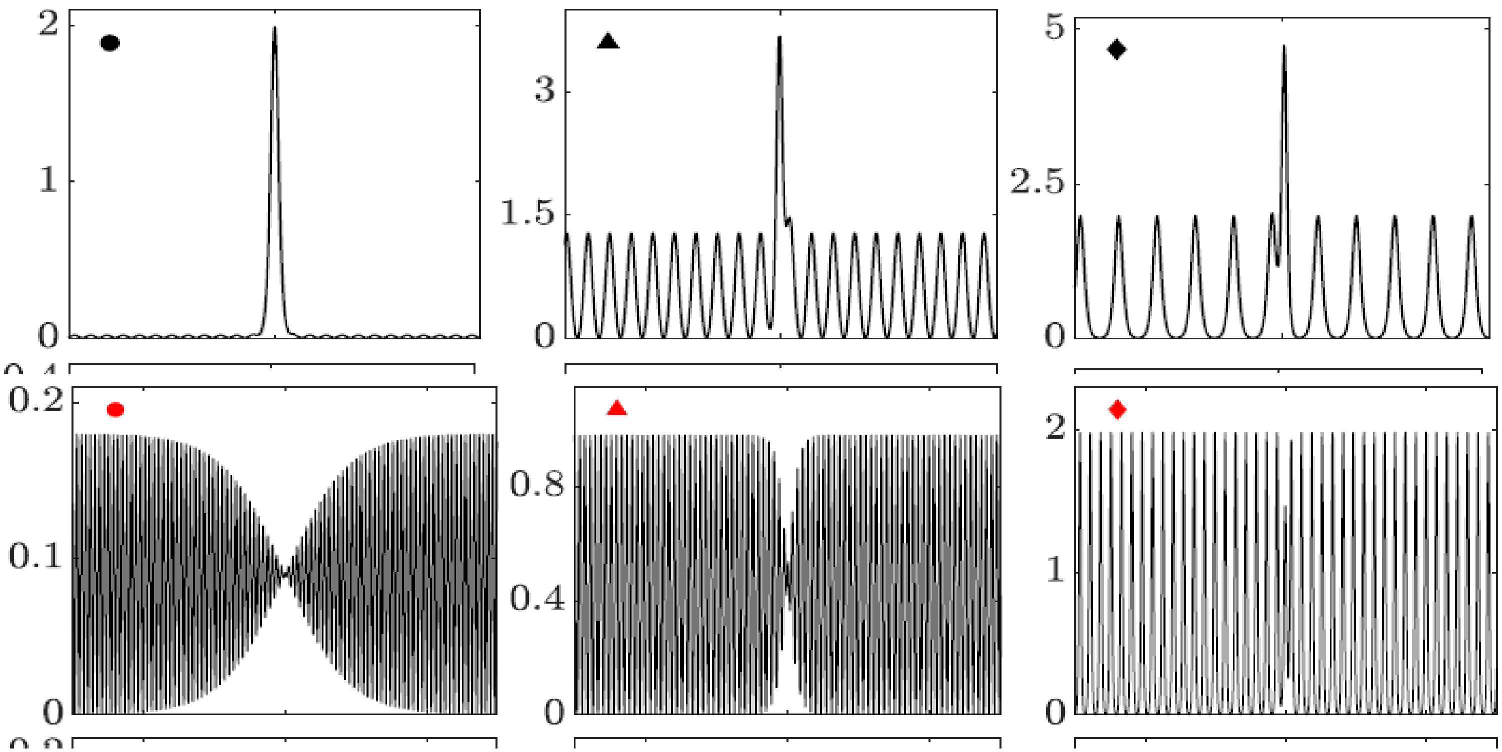


Since $P(\lambda) > 0$ for $\lambda \in \sigma_L$, then $\omega = \pm 4i\sqrt{P(\lambda)} \in i\mathbb{R}$, and the TW is spectrally stable.

Breather solutions are generated by the Darboux transformation:
$\tilde{u} = u + 2\partial_x^2 \log(v)$ with a general eigenfunction v for a selected value of $\lambda \in \mathbb{R}$.

Breathers on the stable traveling periodic wave

- Elevation (bright) breathers correspond to adding a point $\lambda_0 \in (-\infty, \lambda_1)$ to σ_L.
- Depression (dark) breathers correspond to adding a point $\lambda \in (\lambda_2, \lambda_3)$ to σ_L.



E. Kuznetsov, A. Mikhailov, JETP **40** (1974) 855

F. Gesztesy, R. Svirsky, Memoirs AMS **118** (1995) 1–88

X.R. Hu, S.Y. Lou, Y. Chen, Phys. Rev. E **85** (2012) 056607

A. Nakayashiki, Lett. Math. Phys. **111** (2021) 85

M. Hoefer, A. Mucalica and D.E. Pelinovsky, J. Phys. A: Mathem. Theor. 56 (2023) 185701

M. Bertola, R. Jenkins, A. Tovbis, Nonlinearity 36 (2023) 3622

Section 2. The BO and NLS-BO models

The Benjamin–Ono equation is a model for long internal waves in deep fluid:

$$u_t + uu_x + H(u_{xx}) = 0, \quad H(u) := \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{u(y,t)dy}{y-x}. \tag{BO}$$

B.T. Benjamin, J. Fluid Mech. **29** (1967) 559

H. Ono, J. Phys. Soc. Japan **39** (1975) 1082–1091.

Question: Given a slow modulation of the linear waves,

$$u(x,t) = \varepsilon A(\varepsilon(x-2|k|t), \varepsilon t)e^{ikx-ik|k|t} + \varepsilon \bar{A}(\varepsilon(x-2|k|t), \varepsilon t)e^{-ikx+ik|k|t} + \mathcal{O}(\varepsilon^2),$$

what is the normal form for the complex amplitude $A = A(\xi, \tau)$?

For the KdV equation $u_t + 6uu_x + u_{xxx} = 0$, the answer to this question is the defocusing cubic NLS equation

$$iA_\tau + A_{\xi\xi} - |A|^2 A = 0, \tag{NLS}$$

where the constant-amplitude background is linearly and nonlinearly stable and stable dark solitons propagate at the stable periodic traveling background.

A. Mucalica, D. Pelinovsky, Lett. Math. Phys. **114** (2024) 100

Modulation equation for the BO equation

For the BO equation, the dispersion relation $\omega(k) = k|k|$ yields $\omega''(k) = 2$ for $k > 0$ so that the linear Schrödinger equation $iA_\tau + A_{\xi\xi} = 0$ holds for linear perturbations. However, the coefficient of the cubic term $|A|^2 A$ is zero and the next nonlinear term is the cubic derivative term $iA(|A|^2)_\xi$.

M. Tanaka, J. Phys. Soc. Japan **51** (1982) 2686

However, the mean field was computed incorrectly in the derivation of the local cubic derivative term. With the account of the mean field and the asymptotic multi-scale expansions, one can obtain the correct modulation equation as

$$iA_\tau + A_{\xi\xi} + A(i + H)(|A|^2)_\xi = 0. \qquad \text{(NLS-BO+)}$$

D. Pelinovsky, Phys. Lett. A **197** (1995) 401-406

R. Grimshaw, D. Pelinovsky, J. Math. Phys. **36** (1995) 4203–4219

Similarly to the defocusing cubic NLS equation, the constant-amplitude solutions of NLS-BO+ are linearly stable and stable dark solitons propagate on the constant background.

Y. Matsuno, Phys. Lett. A **278** (2000) 53-58
Y. Matsuno, J. Phys. Soc. Japan **73** (2004) 3285–3293

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Another version of the NLS-BO equation

In the continuous approximation of dynamics of particles of the Calogero–Moser–Sutherland system, a continuum limit was derived as

$$iA_\tau + A_{\xi\xi} - A(i+H)(|A|^2)_\xi = 0. \qquad \text{(NLS-BO-)}$$

A. Abanov, E. Betterlheim, and P. Wiegmann, J. Phys. A: Math. Theor. **42** (2009) 135201

This equation has recently been studied in many details:

- Well-posedness, blow-up in a finite time, and stable bright solitons

  P. Gérard, E. Lenzmann, Comm. Pure Appl. Math. **77** (2024) 4008–4062
  Y. Matsuno, Stud. Appl. Math. **151** (2023) 883-922

  J. Hogan, M. Kowalski (2024); R. Killip, T. Laurens, M. Visan (2024); K. Kim, T. Kim, S. Kwon (2024)

- Periodic traveling waves, Lax spectrum, and well-posedness

  R. Badreddine, Pure Appl. Anal. **6** (2023) 379-414

  R. Badreddine, Ann. Inst. H. Poincaré C (2024)

- Coupled integrable systems of the NLS-BO-type

  B. K. Berntson, E. Langmann, J. Lenells, Lett. Math. Phys. **112** (2022) 50
  B. K. Berntson, A. Fagerlund, Physica D **451** (2024) 133762

  R. Sun, Lett. Math. Phys. **114** (2024) 74

Our contributions

For the BO equation $u_t + uu_x + H(u_{xx}) = 0$, we study

- how (bright) and (dark) breathers are obtained from the (well-known) multi-periodic solutions and how their existence is related to the Lax spectrum of the periodic traveling waves.

J. Chen, D. Pelinovsky, Wave Motion **126** (2024) 147–173

For the NLS–BO equation $iu_t = u_{xx} \pm u(i + H)(|u|^2)_x$, we study

- if the constant-amplitude background is stable/unstable in the defocusing/focusing cases

- how the Lax spectrum combines embedded and isolated bands and eigenvalues

- how many families of (bright) and (dark) breathers exist in each case

- if any rogue waves exist in the focusing case as in $iu_t + u_{xx} + |u|^2 u = 0$

J. Chen, D. Pelinovsky, Nonlinearity (2025), submitted

Section 3. Breathers in the BO equation

Consider the exact solutions of the BO equation:

$$u_t + uu_x + H(u_{xx}) = 0, \quad u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$$

Traveling periodic waves are expressed in elementary functions:

$$u(x,t) = \frac{k \sinh \phi}{\cos k(x-ct) + \cosh \phi}, \quad \tanh \phi = \frac{k}{c},$$

where $k > 0$ ($k = 1$ due to scaling transformation) and $c \in (k, \infty)$ is arbitrary.

Traveling waves are spectrally stable. This is based on the direct analysis of eigenfunctions, no relations to the squared eigenfunctions.

M. Spector, T. Miloh, SIAM J. Appl. Math. **54** (1994) 688–707

Lax spectrum for the traveling periodic waves

The BO equation is a compatibility condition of the linear system:

$$\left\{ \begin{array}{l} i\varphi_x^+ + \lambda(\varphi^+ - \varphi^-) + u\varphi^+ = 0, \\ i\varphi_t^\pm - 2i\lambda\varphi_x^\pm + \varphi_{xx}^\pm \mp 2i\varphi^\pm \mathcal{P}^\pm(u_x) = 0, \end{array} \right.$$

where $\varphi^\pm$ are analytic functions in $\mathbb{C}^\pm$ such that $\mathcal{P}^\pm \varphi^\pm = \varphi^\pm$ and $\mathcal{P}^\pm \varphi^\mp = 0$.

- Lax spectrum in $L^2(\mathbb{R})$ is obtained from the spectrum of the self-adjoint operator $\mathcal{L} : H^1(\mathbb{R}) \cap L^2_+ \mapsto L^2_+ = \{f \in L^2(\mathbb{R}) : \ \mathcal{P}^+ f = f\}$ given by

$$\mathcal{L} = -i\partial_x - \mathcal{P}^+(u\cdot).$$

  Lax spectrum in $L^2(\mathbb{R})$ is $\cup_{j=0}^\infty [\lambda_j, \lambda_j + k]$,
  where $\{\lambda_j\}_{j=0}^\infty$ are eigenvalues in $L^2_{\rm per}(0, \frac{2\pi}{k})$.

  P. Gérard, T. Kappeler, Comm. Pure Appl. Math. **74** (2021) 1685–1747

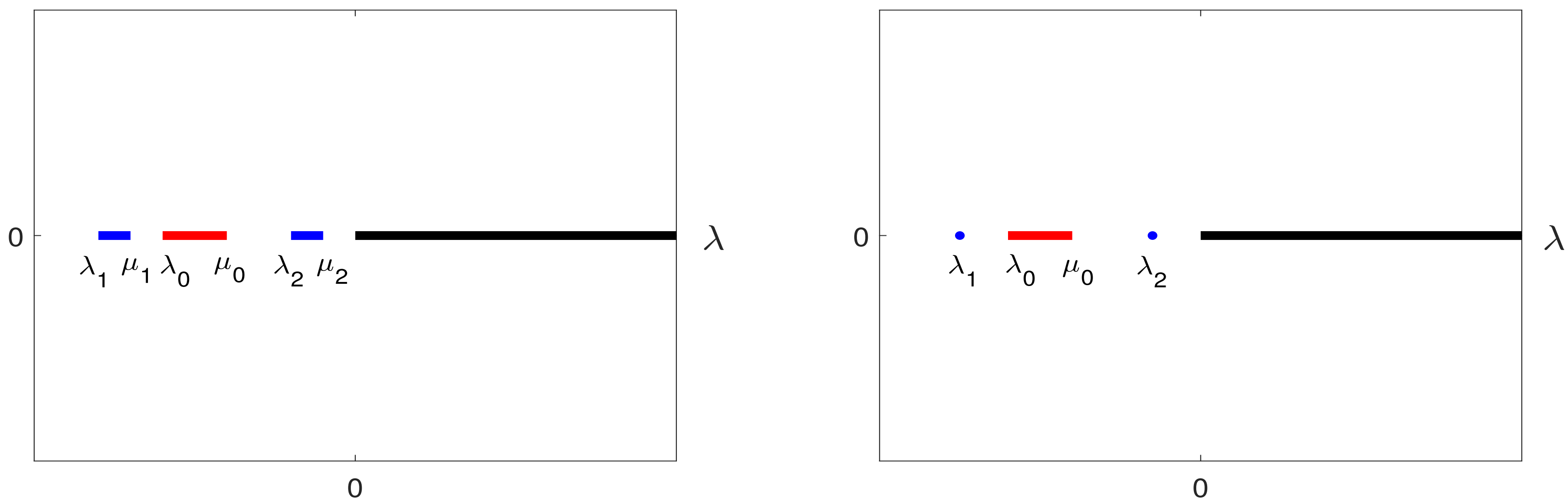
- Lax spectrum of the traveling periodic waves is located at

$$[\lambda_0, \lambda_0 + k] \cup [0, \infty), \qquad \lambda_0 := -\frac{c+k}{2}.$$

  S. Dobrokhotov, I. Krichever, Math. Notices **49** (1991) 583–594

Degeneration of multi-periodic solutions

Breather solutions are obtained by degeneration of multi-periodic solutions in the long-wave limit, e.g. for two eigenvalues below.



Multi-periodic solutions are available in elementary (exponential) functions. When all periods go to ∞, they become multi-soliton solutions expressed by the rational functions.

J. Satsuma, Y. Ishimori, J. Phys. Soc. Japan **46** (1979) 681–687

Y. Matsuno, J. Phys. Soc. Japan **73** (2004) 3285–3293

Here we keep one period fixed and all others go to ∞ to get breathers.

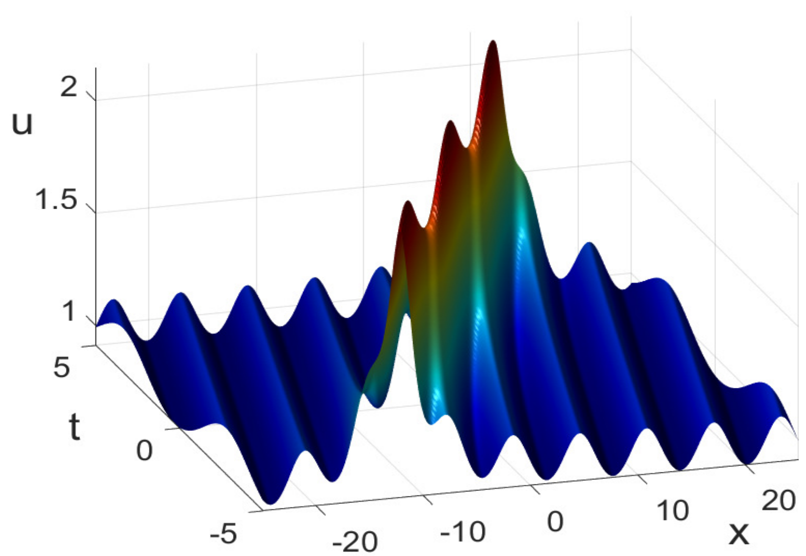
Single breather solutions

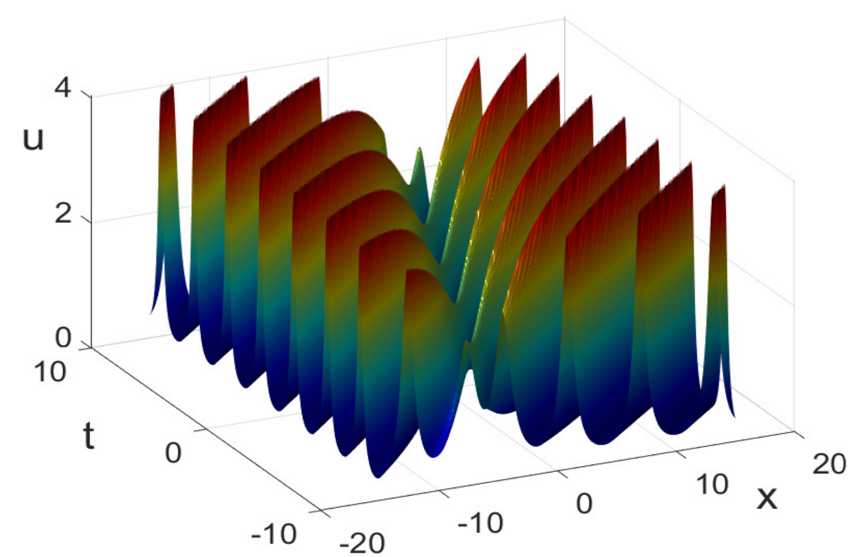
Breather solution contains a mixture of exponential and power functions:

$$u = \frac{2(c_b + k\beta)\cosh\phi + [k(1+\beta^2 + c_b^2\eta^2) + 2\beta c_b]\sinh\phi + 2c_b\cos(k\xi)}{(1+\beta^2+c_b^2\eta^2)\cosh\phi + 2\beta\sinh\phi + (1-\beta^2+c_b^2\eta^2)\cos(k\xi) + 2\beta c_b\eta\sin(k\xi)},$$

where $\xi = x - ct$, $\eta = x - c_b t - \eta_0$, $\beta = \frac{2c_b k}{(c_b - c)^2 - k^2}$, and

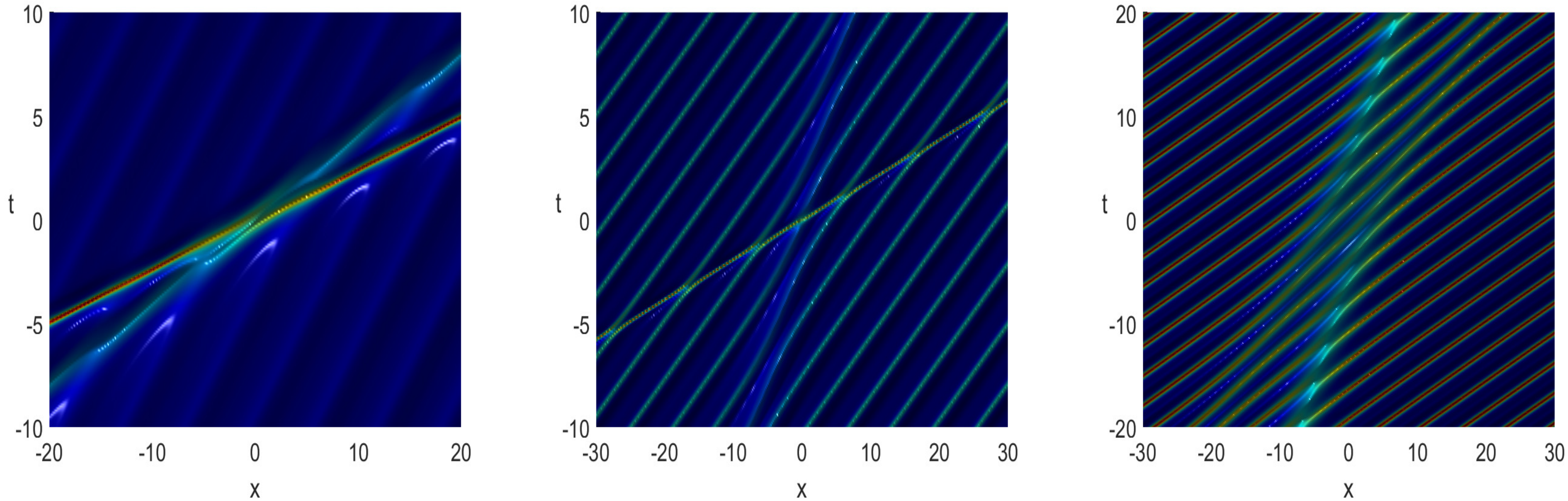
- either $c_b > c + k$ (bright breathers) with $\{-\frac{c_b}{2}\} \cup [\lambda_0, \lambda_0 + k] \cup [0, \infty)$
- or $c_b < c - k$ (dark breathers) with $[\lambda_0, \lambda_0 + k] \cup \{-\frac{c_b}{2}\} \cup [0, \infty)$





Two-soliton breathers

- Bright-bright breathers with $\{\lambda_1, \lambda_2\} \cup [\lambda_0, \lambda_0 + k] \cup [0, \infty)$
- Bright-dark breathers with $\{\lambda_1\} \cup [\lambda_0, \lambda_0 + k] \cup \{\lambda_2\} \cup [0, \infty)$
- Dark-dark breathers with $[\lambda_0, \lambda_0 + k] \cup \{\lambda_1, \lambda_2\} \cup [0, \infty)$



- Solitons impair no phase shift upon interaction with the traveling wave.
- Solitons have the same speed as at the zero background.

General breathers

General breathers can be expressed explicitly in the determinant form:

$$u(x,t) = -k + i\partial_x \log \frac{\det(\bar{F})}{\det(F)},$$

with

$$F = \begin{bmatrix} 1 + e^{ik\xi+\phi} & \frac{2k}{k+c-c_1} & \cdots & \frac{2k}{k+c-c_N} \\ \frac{2c_1}{k+c_1-c} & -ic_1\eta_1 - 1 - \frac{2kc_1}{(c-c_1)^2-k^2} & \cdots & \frac{2c_1}{c_1-c_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2c_N}{k+c_N-c} & \frac{2c_N}{c_N-c_1} & \cdots & -ic_N\eta_N - 1 - \frac{2kc_N}{(c-c_N)^2-k^2} \end{bmatrix},$$

where for $1 \leq j \leq N$, we have defined $\eta_j = x - c_j t - x_j$ with arbitrary $x_j \in \mathbb{R}$ and arbitrary distinct $c_j > 0$ satisfing $|c_j - c| > k$.

Theorem 1 (J. Chen & D.P., 2024)**.** *The general breather solution is bounded for every* $(x,t) \in \mathbb{R} \times \mathbb{R}$ *and every* $N \in \mathbb{N}$.

S. Dobrokhotov, I. Krichever, Math. Notices **49** (1991) 583–594

Section 4. Stability in the NLS–BO equation

For the NLS–BO equation,

$$iu_t = u_{xx} + \sigma u(i + H)(|u|^2)_x, \quad u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \quad \sigma = \pm 1,$$

the main question is whether the focusing $(\sigma = -1)$ and defocusing $(\sigma = +1)$ cases have different conclusions in the stability of the constant solution as it happens for the local NLS equation

$$iu_t = -u_{xx} + \sigma |u|^2 u, \quad u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \quad \sigma = \pm 1.$$

Due to symmetries:

$$u(x,t) \mapsto u(x + x_0, t + t_0)e^{i\theta_0}, \quad u(x,t) \mapsto \alpha u(\alpha^2 x, \alpha^4 t), \quad x_0, t_0, \theta_0, \alpha \in \mathbb{R},$$

it is sufficient to normalize the background to unity: $u = 1$.

Linear stability theorem

Theorem 2 (J. Chen & D.P., 2025)**.** *Let $u = 1 + v$ and consider the linearized equations of motion*

$$iv_t = v_{xx} + \sigma(i + H)(v_x + \bar{v}_x).$$

In the defocusing case $\sigma = +1$, for every initial data $v_0 \in H^s(\mathbb{R})$, $s \geq 0$, the unique solution $v \in C^0(\mathbb{R}, H^s(\mathbb{R}))$ with $v|_{t=0} = v_0$ satisfies

$$\|v(\cdot,t)\|_{H^s} \leq C\|v_0\|_{H^s} \quad \textit{for every} \quad t \in \mathbb{R},$$

for some constant $C > 0$.

Remark 3. *In the focusing case $\sigma = -1$, there is a resonance of Fourier modes which suggests the linear instability of the constant solution $u = 1$ in the space of 2π-periodic functions. This linear instability is missed in $L^2_{\rm per}(0,T)$ if the spatial period T is not divisible by 2π. The linear instability is also missed in $L^2(\mathbb{R})$ if the Fourier transform of $v|_{t=0} = v_0$ is zero at the resonant modes.*

Proof of the linear stability (defocusing case)

Separating the real and imaginary parts as $v = A + iB$ and using Fourier transform in x with Fourier parameter $k \in \mathbb{R}$ yields the system

$$\left\{ \begin{array}{l} \hat{A}_t = -k^2 \hat{B} + 2\sigma i k \hat{A}, \\ \hat{B}_t = k^2 \hat{A} + 2\sigma |k| \hat{A}, \end{array} \right.$$

from which we obtain the characteristic equation,

$$\lambda^2 - 2i\sigma k \lambda + k^2 (k^2 + 2\sigma |k|) = 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} \lambda_1(k) = -ik|k|, \\ \lambda_2(k) = ik(2\sigma + |k|). \end{array} \right.$$

For $\sigma = 1$, there is no resonance $\lambda_1(k) = \lambda_2(k)$ with $k \neq 0$ so that there exists $C > 0$ such that

$$|\hat{A}(k,t)| + |\hat{B}(k,t)| \leq C \left(|\hat{A}(k,0)| + |\hat{B}(k,0)| \right), \qquad t \in \mathbb{R},$$

which implies

$$\|v(\cdot,t)\|_{H^s} \leq C \|v_0\|_{H^s} \quad \text{for every} \quad t \in \mathbb{R}.$$

Proof of the linear stability (focusing case)

For $\sigma = -1$, there is resonance $\lambda_1(k) = \lambda_2(k)$ at $k = \pm 1$ with linear growth of $\hat{A}(\pm 1, t)$ and $\hat{B}(\pm 1, t)$:

$$\begin{cases} \hat{A}(\pm 1, t) = (\hat{c}_1 + \hat{c}_2 t)\, e^{\mp it}, \\ \hat{B}(\pm 1, t) = (\mp i\hat{c}_1 + (\mp it - 1)\hat{c}_2)\, e^{\mp it}, \end{cases}$$

For every $k \in \mathbb{R} \backslash \{+1, -1\}$, we get the solution

$$\begin{cases} \hat{A}(k, t) = k\hat{C}_1(k) e^{-ik|k|t} + \hat{C}_2(k) e^{ik(|k|-2)t}, \\ \hat{B}(k, t) = i(|k| - 2)\hat{C}_1(k) e^{-ik|k|t} - i\text{sgn}(k)\hat{C}_2(k) e^{ik(|k|-2)t}, \end{cases}$$

where

$$\hat{C}_1(k) = \frac{i\text{sgn}(k)\hat{A}(k,0) + \hat{B}(k,0)}{2i(|k|-1)}, \qquad \hat{C}_1(k) = \frac{i(|k|-2)\hat{A}(k,0) + \text{sgn}(k)\hat{B}(k,0)}{2i(|k|-1)},$$

which implies

$$\|v(\cdot, t)\|_{H^s} \leq C\|v_0\|_{H^s \cap L^{2,2}}, \quad \text{for every} \quad t \in \mathbb{R}$$

only if $\hat{A}(k, 0)$, $\hat{B}(k, 0)$ are C^1 at $k = \pm 1$ with $\hat{A}(\pm 1, 0) = \hat{B}(\pm 1, 0) = 0$.

Nonlinear stability theorem (defocusing case)

Theorem 4 (J. Chen & D.P., 2025). *For every fixed* $T > 0$*, there exists* $\delta > 0$ *such that for every* $v_0 \in H^1_{\rm per}((0,T),\mathbb{C})$ *with* $\|v_0\|_{H^1_{\rm per}} \leq \delta$*, the unique solution* $u \in C^0(\mathbb{R}, H^1_{\rm per}((0,T),\mathbb{C}))$ *with* $u|_{t=0} = 1 + v_0$ *satisfies*

$$\|e^{-i\theta(t)} u(\cdot,t) - 1\|_{H^1_{\rm per}} \leq C \|v_0\|_{H^1_{\rm per}} \quad \text{for every} \quad t \in \mathbb{R},$$

for some constant $C > 0$ *and some function* $\theta \in C^0(\mathbb{R})$.

- Local well-posedness in $H^1(\mathbb{R})$ was proven in

  R. Moura, D. Pilod, Adv. Diff. Eqs. **15** (2010) 925–952

- There exist infinitely many conserved quantities:

$$I_1(u) = \oint (|u|^2 - 1) dx,$$

$$I_2(u) = i \oint (u \bar{u}_x - \bar{u} u_x) dx + \sigma \oint (|u|^4 - 1) dx,$$

$$I_3(u) = \oint \left(|u_x|^2 - \frac{i}{2} \sigma |u|^2 (\bar{u} u_x - \bar{u}_x u) - \frac{1}{2} \sigma |u|^2 H(|u|^2)_x + \frac{1}{3} (|u|^6 - 1) \right) dx.$$

R. Grimshaw, D. Pelinovsky, J. Math. Phys. **36** (1995) 4203–4219

Lyapunov functional

Combining three conserved quantities together yields the Lyapunov functional

$$\Lambda(v) := I_3(1+v) - \sigma I_2(1+v) + I_1(1+v)$$
$$= \oint \left(|v_x|^2 + \frac{1}{2}\sigma(v+\bar{v})K(v+\bar{v}) + N(v) \right) dx,$$

where $K := -H\partial_x \geq 0$ is a self-adjoint operator in L^2 with $\mathrm{Dom}(K) = H^1_{\rm per}$.

If $\sigma = +1$, then

$$\oint \left[|v_x|^2 + \frac{1}{2}(v+\bar{v})K(v+\bar{v}) \right] dx = \sum_{n\in\mathbb{Z}} \frac{4\pi^2 n^2}{T} |\hat{v}_n|^2 + \pi|n||\hat{v}_n + \bar{\hat{v}}_{-n}|^2$$
$$\geq \frac{1}{2}\|v_x\|^2_{L^2} + \frac{2\pi^2}{T^2}\|v - \hat{v}_0\|^2_{L^2}.$$

If $\hat{v}_0(t)$ is controlled, then coercivity and Banach algebra of $H^1_{\rm per}$ for $N(v)$ yields

$$\|v(\cdot,t)\|_{H^1_{\rm per}} \leq \|\hat{v}_0(t)\|_{L^2} + \|v(\cdot,t) - \hat{v}_0(t)\|_{H^1_{\rm per}}$$
$$\leq \sqrt{T}|\hat{v}_0(t)| + C\sqrt{\Lambda(v)} \leq C\|v_0\|_{H^1_{\rm per}}.$$

Control of the mean-value term

$\mathrm{Re}(\hat{v}_0)(t)$ can be controlled from conservation of

$$I_1(1+v) = \oint (v+\bar{v}+|v|^2)dx \quad \Rightarrow \quad |\mathrm{Re}(\hat{v}_0)(t)| \leq \frac{\sqrt{T+I_1(1+v)}}{\sqrt{T}} - 1 \leq C\|v_0\|_{L^2},$$

provided that $\mathrm{Im}(\hat{v}_0)(t) = 0$ from the orthogonal decomposition

$$u(x,t) = e^{i\theta(t)}\left[1+v(x,t)\right], \qquad \oint \mathrm{Im}(v)(x,t)dx = 0.$$

The latter is achieved with the implicit function theorem for $f(\theta) : \mathbb{R} \to \mathbb{R}$,

$$f(\theta) = \oint \mathrm{Im}(e^{-i\theta}u - 1)dx$$

in a ball with small $\inf\limits_{\theta \in \mathbb{R}} \|e^{-i\theta}u - 1\|_{H^1_{\mathrm{per}}} \leq C\|v_0\|_{H^1_{\mathrm{per}}}$.

Nonlinear stability (focusing case)

The quadratic part if coercive if $T < \pi$:

$$\oint \left[|v_x|^2 - \frac{1}{2}(v+\bar{v})K(v+\bar{v}) \right] dx = \sum_{n\in\mathbb{Z}} \frac{4\pi^2 n^2}{T} (|\widehat{\mathrm{Re}(v)}_n|^2 + |\widehat{\mathrm{Im}(v)}_n|^2) - 4\pi|n||\widehat{\mathrm{Re}(v)}|^2.$$

Recall the resonance and the linear instability if $T = 2\pi$ or if T is multiple to 2π.

Nonlinear stability of the constant background in the focusing case is an open problem in $H^1_{\mathrm{per}}(0,T)$ with $T \geq \pi$.

Section 5. Breathers in the NLS–BO equation

Let us look at the exact solutions of the NLS–BO equation,

$$iu_t = u_{xx} + 2i\sigma u \mathcal{P}^+(|u|^2)_x, \quad u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \quad \sigma = \pm 1.$$

Traveling periodic waves are expressed in elementary functions:

$$u = \frac{g}{f^+}, \quad \bar{u} = \frac{h}{f^-}, \quad |u(x,t)|^2 = 1 - i\sigma \frac{\partial}{\partial x} \ln \frac{f^+}{f^-} = 1 - \frac{\sigma k \sinh \phi}{\cos k(x-ct) + \cosh \phi},$$

where $k > 0$ and $c \in \mathbb{R}$ are arbitrary parameters such that

$$e^{2\phi} = \frac{(c-k)(c+k+2\sigma)}{(c+k)(c-k+2\sigma)} > 1, \qquad \frac{c+k}{c-k} > 0.$$

- If $\sigma = 1$, then $k \in (0,1)$ and $c \in (-2+k, -k)$.
- If $\sigma = -1$, then $k \in (0,\infty)$ and either $c \in (k+2, \infty)$ or $c \in (-\infty, -k)$.

Traveling wave solutions are defined in $H^1 \cap L^2_+$, $L^2_+ = \{u \in L^2 : \; \mathcal{P}^+ u = u\}$.

Lax spectrum for the traveling periodic waves

Solutions of the NLS-BO is the compatibility condition for the linear system:

$$\left\{ \begin{array}{l} ip_x + \lambda p + uq^+ = 0, \\ q^+ - \mu q^- + \sigma \bar{u} p = 0, \\ ip_t + \lambda^2 p + \lambda u q^+ + i(uq^+_x - u_x q^+) = 0, \\ iq^\pm_t - 2i\lambda q^\pm_x + q^\pm_{xx} \pm 2i\sigma q^\pm \mathcal{P}^\pm(|u|^2)_x = 0, \end{array} \right.$$

where λ is the spectral parameter, and $q^\pm \in L^2_\pm$.

Hence $q^+ = -\sigma \mathcal{P}^+(\bar{u}p)$ and the Lax spectrum is defined by the self-adjoint operator in $\mathcal{L} : H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ given by

$$\mathcal{L} := -i\partial_x + \sigma u \mathcal{P}^+(\bar{u}\, \cdot)$$

If $u \in H^1 \cap L^2_+$, then $p \in H^1 \cap L^2_+$, that is, the Lax spectrum is the set of admissible values of $\mathcal{L}|_{L^2_+}$ given by

$$\mathcal{L}_u|_{L^2_+} = -i\partial_x + \sigma \mathcal{P}^+ u \mathcal{P}^+(\bar{u}\, \cdot)$$

P. Gérard, E. Lenzmann, Comm. Pure Appl. Math. **77** (2024) 4008–4062

R. Badreddine, Ann. Inst. H. Poincaré C (2024)

Exact eigenfunctions for the Lax spectrum

By using the bilinear form, we computed eigenfunctions for the Lax spectrum:

$$\Sigma = [\lambda_0, \lambda_0 + k] \cup [\sigma, \infty), \qquad \lambda_0 := -\frac{c+k}{2}.$$

For $[\sigma, \infty)$, $q^- = 0$ and q^+ is analytic in $\mathbb{C}_+$ and bounded as $\mathrm{Im}(x) \to +\infty$:

$$q^+ = e^{i(\lambda-\sigma)x + i(\lambda^2-1)t} \frac{1 + e^{ik(x-ct)+\phi}}{1 + e^{ik(x-ct)-\phi}}.$$

For $[\lambda_0, \lambda_0 + k]$, both q^+ and q^- are bounded as $\mathrm{Im}(x) \to \pm\infty$ and co-periodic:

$$q^+ = \frac{1}{1 + e^{ik(x-ct)-\phi}} \left[1 + \frac{c + 2\lambda + k}{c + 2\lambda - k} e^{ik(x-ct)-\phi} \right],$$

$$q^- = \frac{1}{1 + e^{-ik(x-ct)-\phi}} \left[e^{-ik(x-ct)-\phi} + \frac{c + 2\lambda + k}{c + 2\lambda - k} \right].$$

We have $\oint q^- dx = 0$ at $\lambda = \lambda_0$, for which $c + 2\lambda + k = 0$.

P. Gérard, T. Kappeler, Comm. Pure Appl. Math. **74** (2021) 1685–1747

No difference between isolated and embedded bands?

- If $\sigma = +1$, then $[\lambda_0, \lambda_0 + k] \subset (0,1)$ is isolated from $[1,\infty)$.

- If $\sigma = -1$, then either $c \in (k+2, \infty)$ for which $[\lambda_0, \lambda_0 + k]$ is isolated from $[-1, \infty)$ or $c \in (-\infty, -k)$ for which $[\lambda_0, \lambda_0 + k]$ is embedded into $[-1, \infty)$.

However, all families of traveling periodic waves are symmetric about the midpoint:

$$\sigma = +1: \qquad c + 1 \in (-1 + k, 1 - k), \qquad k \in (0,1)$$

$$\sigma = -1: \qquad c - 1 \in (-\infty, -1 - k) \cup (1 + k, \infty), \qquad k \in (0, \infty).$$

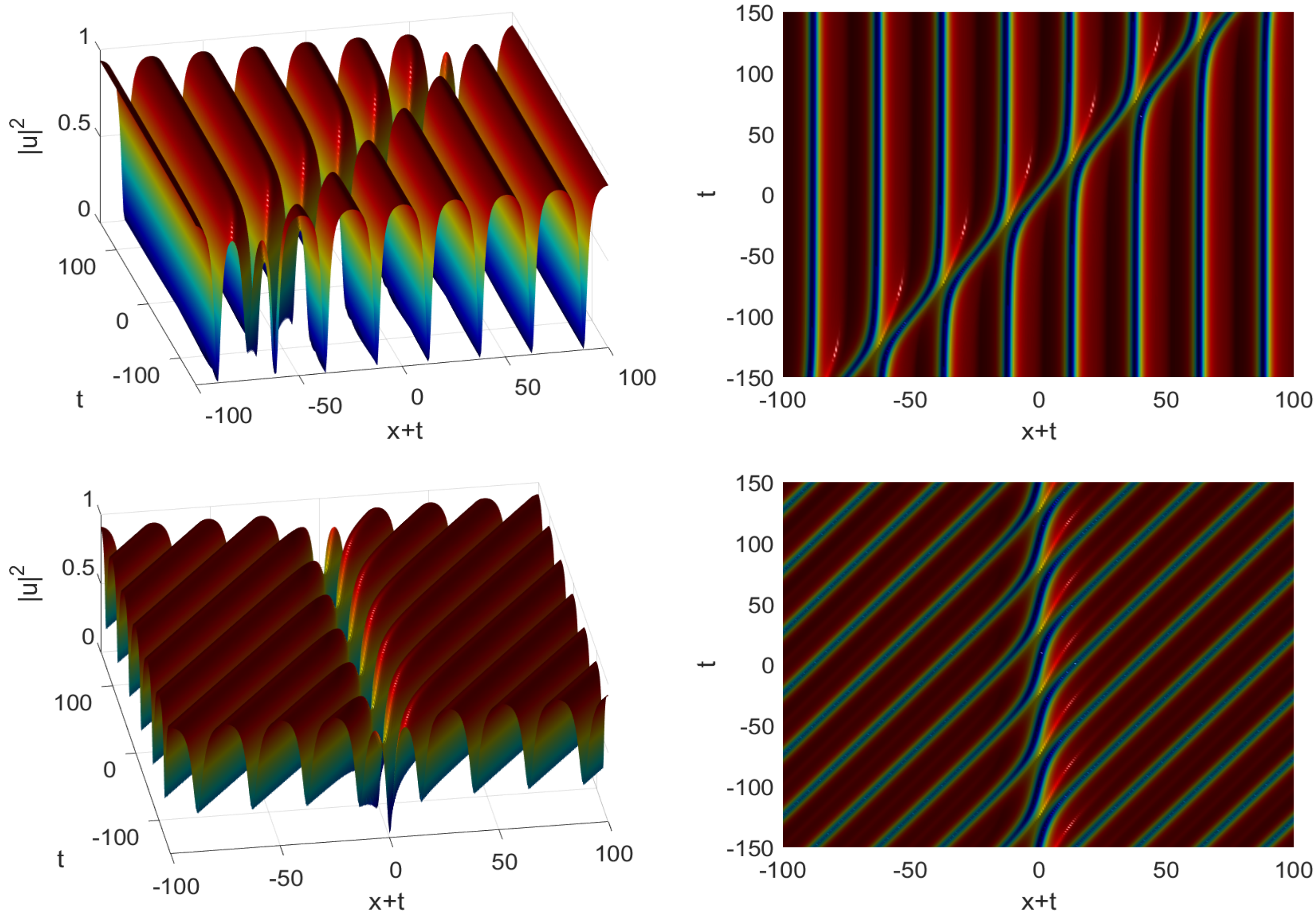
with no difference in dynamics.

In both defocusing and focusing case, we have only obtained breathers steadily propagating on the traveling periodic background. We observed no rogue waves or instability of the traveling periodic wave.

Breathers in the defocusing case $\sigma = +1$

Top: $k = \frac{1}{4}$, $c = -1$, $c_b = -\frac{1}{2}$, Lax spectrum $\{-\frac{c_b}{2}\} \cup [\lambda_0, \lambda_0 + k] \cup [1, \infty)$

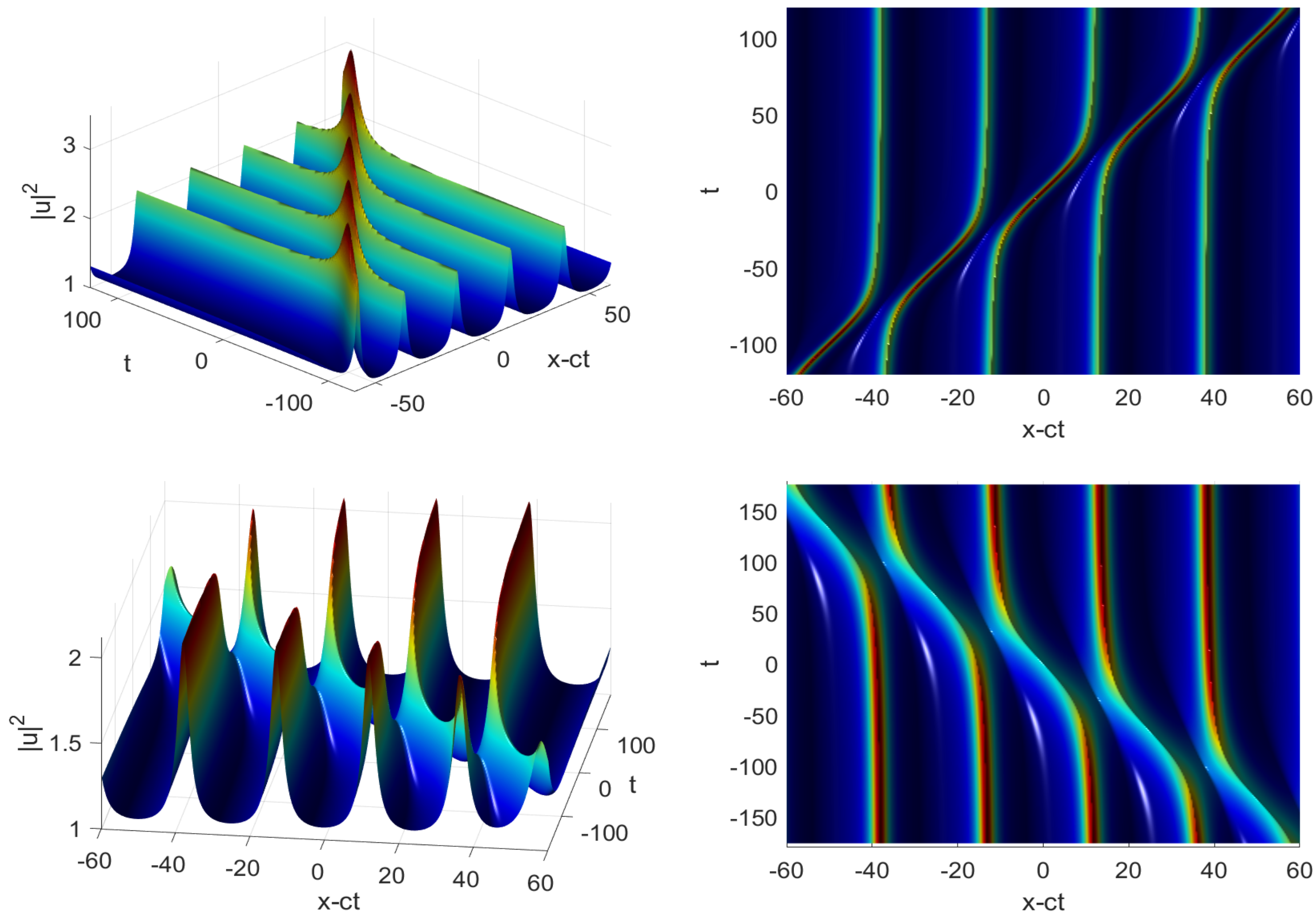
Bottom: $k = \frac{1}{4}$, $c = -\frac{1}{2}$, $c_b = -1$, Lax spectrum $[\lambda_0, \lambda_0 + k] \cup \{-\frac{c_b}{2}\} \cup [1, \infty)$



Breathers in the focusing case $\sigma = -1$

Top: $k = \frac{1}{4}$, $c = 2 + 2k$, $c_b = c + 2k$, Lax spectrum $\{-\frac{c_b}{2}\} \cup [\lambda_0, \lambda_0 + k] \cup [-1, \infty)$

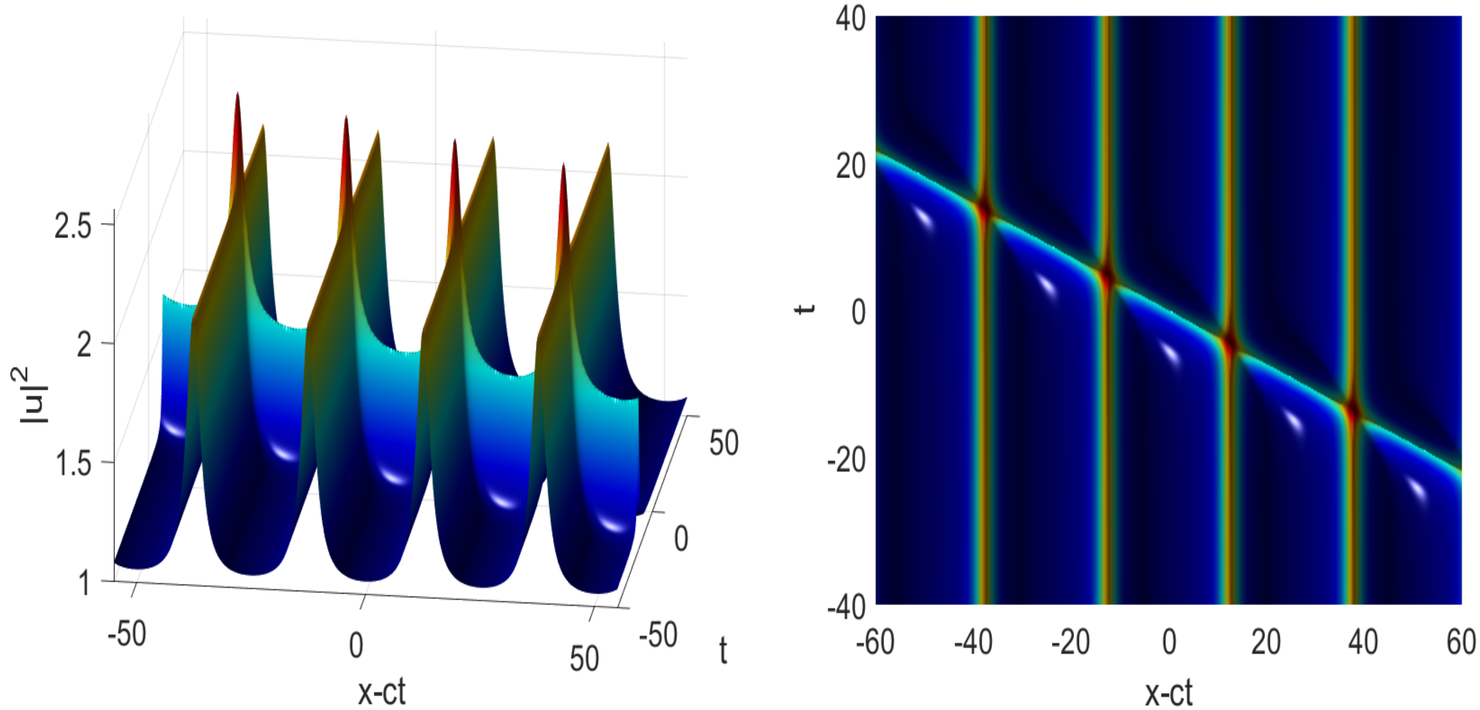
Bottom: $k = \frac{1}{4}$, $c = 2 + 2k$, $c_b = c - \frac{3}{2}k$, Lax sp. $[\lambda_0, \lambda_0 + k] \cup \{-\frac{c_b}{2}\} \cup [-1, \infty)$



Breathers in the focusing case $\sigma = -1$

$k = \frac{1}{4}$, $c = 2 + 2k$, $c_b = -k$

Lax spectrum $[\lambda_0, \lambda_0 + k] \cup [-1, \infty)$ with $-\frac{c_b}{2} \in [-1, \infty)$ embedded.



If $c \in (-\infty, -k)$ so that $[\lambda_0, \lambda_0 + k]$ is embedded into $[-1, \infty)$, then the same breather solutions hold after symmetrical reflection.

Section 6. Conclusion

We have considered stability and breathers on the background of the traveling periodic waves in the NLS–BO equation.

- Defocusing case: Constant background is linearly and nonlinearly stable, stable dark breathers propagate on the TW background.

- Focusing case: Both bright and dark breathers propagate on the TW background, no rogue waves or instabilities are detected, no difference in dynamics between isolated or embedded eigenvalues.

- Open Problem 1: Stability of the constant wave for long periodic perturbations in the focusing case.

- Open Problem 2: Stability of the traveling periodic wave in both cases.