Existence and stability of singular nonlinear waves in mathematical models of nonlinear optics

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Section 1

Introduction: NLS models

Solitary waves in NLS models

Bright soliton $\psi(t,x) = e^{it}\mathrm{sech}(x)$ of the focusing NLS equation

$$i\partial_t\psi + \partial_x^2\psi + 2|\psi|^2\psi = 0$$

with $|\psi(t,x)| \to 0$ as $|x| \to \infty$

Dark soliton $\psi(t,x) = e^{-2it}\tanh(x)$ of the defocusing NLS equation

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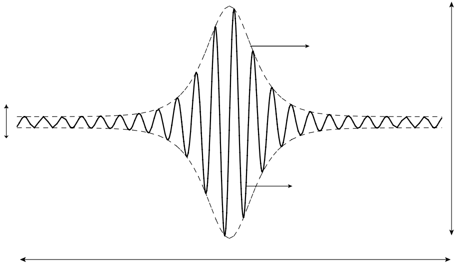

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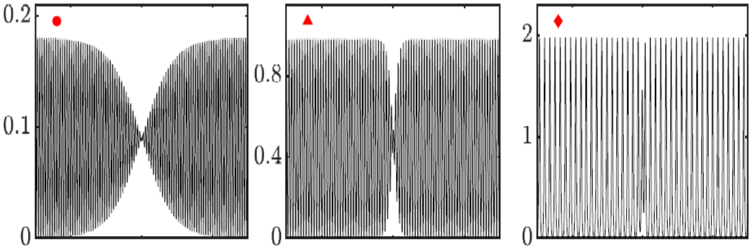

Generalized NLS models

The classical NLS equation realizes a balance between nonlinearity and dispersion for propagation of nonlinear dispersive waves.

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$$i\psi_t + \alpha\psi_{xx} + \beta|\psi|^2\psi = 0. \tag{NLS}$$

Taking into account higher-order nonlinearity and dispersion gives an extended version of the NLS equation:

$$\begin{aligned} i\psi_t + \alpha\psi_{xx} + \beta|\psi|^2\psi + i\alpha_1\psi_{xxx} + \alpha_2\psi_{xxxx} \\ + i\beta_1|\psi|^2\psi_x + i\beta_2(|\psi|^2\psi)_x + \gamma|\psi|^4\psi = 0. \end{aligned}$$

Well-posedness of initial-value problem, stability of nonlinear waves, global dynamics (scattering versus blowup in a finite time), ...

NLS models with intensity-dependent dispersion

A new NLS model where the dispersion depends on the wave intensity and vanishes at a selected intensity:

$$i\psi_t + \alpha(1 - |\psi|^2)\psi_{xx} + \beta|\psi|^2\psi = 0. \qquad \text{(NLS-IDD)}$$

C.Y. Lin, J.H. Chang, G. Kurizki, and R.K. Lee, Optics Letters **45** (2020) 1471–1474

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A.D. Greentree, D. Richards, J.A. Vaccaro, *et al.*, Phys. Rev. A **67** (2003), 023818

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Another (regularized) NLS model where the dispersion is bounded:

$$i(1 - \epsilon^2\partial_x^2)\psi_t + \alpha\psi_{xx} + \beta|\psi|^2\psi = 0.$$

M. Colin, D. Lannes SIMA **41** (2009) 708–732

D. Lannes, Proc. R. Soc. Edinburgh Ser A **141** (2011) 253–286

Section 2

Bright solitons in NLS-IDD

NLS-IDD model

For NLS-IDD,

$$i\psi_t + (1 - |\psi|^2)\psi_{xx} = 0, \qquad \text{(NLS-IDD)}$$

two conserved quantities exist:

$$Q(\psi) = -\int_{\mathbb{R}} \log|1 - |\psi|^2| dx$$

and

$$E(\psi) = \int_{\mathbb{R}} |\psi_x|^2 dx.$$

Local solutions exist in $H^\infty(\mathbb{R})$

M. Poppenberg, Nonlinear Anal. **45** (2001) 723

$\|\psi(t,\cdot)\|_{H^1}$ is controlled for if $\|\psi(t,\cdot)\|_{L^\infty} \leq C < 1$.

Solitary waves (bright solitons)

Standing waves have the form $\psi(x,t) = e^{i\omega t}u(x)$ with (ω, u) satisfying

$$\omega u(x) = (1 - u^2)u''(x).$$

Solitary waves with $u(x) \to 0$ as $|x| \to \infty$ exist only if $\omega > 0$, in which case ω can be scaled out by $u(x) \mapsto u(\sqrt{\omega}x)$.

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Equation $(1 - u^2) u'' = u$ is integrable with the first invariant:

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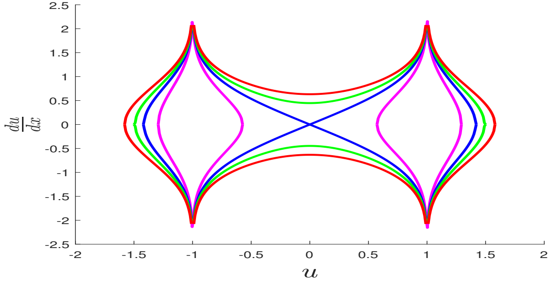
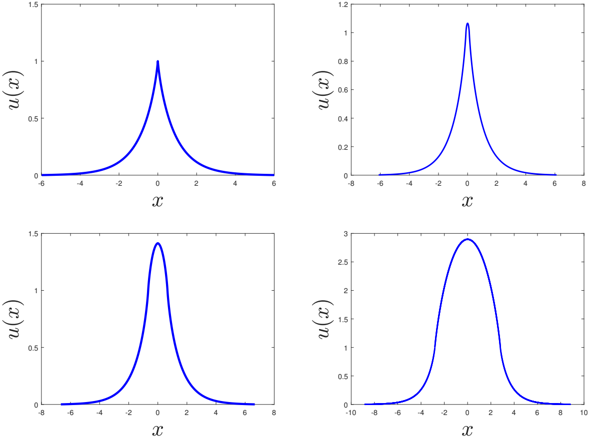
From the phase portrait, bright solitons are singular at $u = \pm 1$.



Possible solitary waves

Gluing the stable and unstable curves with another integral curve gives a one-parameter family of single-humped solitary waves:



Top left: “cusped soliton". Others: “bell-shaped solitons".

Questions on existence and stability of these solitary waves

- In what space (in what sense) do they exist?
- What is the nature of singularity at $u = \pm 1$?
- Can these solutions be characterized variationally?

Existence result

Definition

We say that $u \in H^1(\mathbb{R})$ is a weak solution of the differential equation $u = (1-u^2)u''$ if it satisfies the following equation

$$\langle u, \varphi \rangle + \langle (1-u^2)u', \varphi' \rangle - 2\langle u(u')^2, \varphi \rangle = 0, \quad \text{for every } \varphi \in H^1(\mathbb{R}),$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R})$.

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where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R})$.

Theorem (Ross–Kevrekidis–P, Q.Appl.Math. 79 (2021) 641)

There exists a one-parameter continuous family of weak, positive, and single-humped solutions of $u = (1 - u^2)u''$ parametrized by C.

After orbits have been glued, we need to show that

- ▷ $u \in H^1(\mathbb{R})$;
- ▷ $\lim_{x \to x_0} (1 - u^2(x))u'(x) = 0$ for each x_0 where $u(x_0) = 1$.

Nature of singularity at $u = 1$

It follows from the first invariant

$$\frac{1}{2}(u')^2 + \frac{1}{2}\log|1-u^2| = C,$$

that the cusped soliton is defined by the implicit function

$$|x| = \int_u^1 \frac{d\xi}{\sqrt{-\log(1-\xi^2)}}, \quad u \in (0,1).$$

Asymptotic analysis gives as $|x| \to 0$:

$$u(x) = 1 - |x|\sqrt{\log(1/|x|)}\left[1 + \mathcal{O}\left(\frac{\log\log(1/|x|)}{\log(1/|x|)}\right)\right].$$

[Alfimov–Korobeinikov–Lustri–P, Nonlinearity 32 (2019) 3445]

Hence, $u'(x) \sim \sqrt{\log(1/|x|)}$ and $(1-u^2)u'(x) \sim |x|\log(1/|x|)$.

Stability result ?

Recall the conserved quantities:

$$Q(\psi) = -\int_{\mathbb{R}} \log|1 - |\psi|^2| dx, \quad E(\psi) = \int_{\mathbb{R}} |\psi_x|^2 dx.$$

Solitary wave $\psi(x,t) = u(x)e^{i\omega t}$ is a critical point of the action

$$\Lambda_\omega(u) = E(u) + \omega Q(u).$$

Expanding near the solitary wave yields

$$\Lambda_\omega(u+\varphi) - \Lambda_\omega(u) = 2\langle u', \varphi' \rangle + 2\omega \langle (1-u^2)^{-1} u, \varphi \rangle + \mathcal{O}(\|\varphi'\|_{L^2}^2 + \|(1-u^2)^{-1}\varphi\|_{L^2 \cap L^\infty}^2),$$

which is not compatible with the definition of weak solutions:

$$u \in H^1(\mathbb{R}) : \quad \omega\langle u, \varphi\rangle + \langle (1-u^2)u', \varphi' \rangle - 2\langle u(u')^2, \varphi \rangle = 0,$$

for every $\varphi \in H^1(\mathbb{R})$.

New definition of weak solutions

Definition

Fix $L > 0$ and define

$$X_L := \left\{ u \in H^1(\mathbb{R}) : \; u(x) > 1, \; x \in (-L, L) \text{ and } u(x) \leq 1, \; |x| \geq L \right\}.$$

Pick $u_L \in X_L$ satisfying

$$\lim_{|x| \to L} \frac{u_L(x) - 1}{(L - |x|)\sqrt{|\log|L - |x|||}} = 1.$$

We say that $u \in X_L \subset H^1(\mathbb{R})$ is a weak solution if it satisfies the following equation

$$\langle u', \varphi' \rangle + \omega \langle (1 - u^2)^{-1} u, \varphi \rangle = 0, \quad \text{for every } \varphi \in H^1_L,$$

where $H^1_L := \left\{ \varphi \in H^1(\mathbb{R}) : \; (1 - u_L^2)^{-1} \varphi \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \right\}$.

Stability result

Theorem (P–Ross–Kevrekidis, J. Phys. A 54 (2021) 445701)

For every $\mu > 0$ and $L > 0$, there exists a unique minimizer of the constrained variational problem

$$\mathcal{Q}_{\mu,L} := \inf_{u \in X_L} \{Q(u) : \quad E(u) = \mu\}.$$

For the proof, we need to complete three steps:

- Monotonicity of mappings $C \mapsto E(u_C)$ and $C \mapsto \ell_C$, where $2\ell_C$ is the length of the bell head;
- Scaling transformation from ℓ_C to L
- Convexity of action $\Lambda_{\omega=1}$ at u_C.

Monotonicity of mappings $C \mapsto E(u_C)$ and $C \mapsto \ell_C$

It follows from $(u')^2 + \log|1-u^2| = 2C$ that

$$E(u_C) = E(u_{\text{cusp}}) + 2\int_1^{\sqrt{1+e^{2C}}} \sqrt{2C - \log(u^2-1)}du$$

and

$$\ell_C = \int_1^{\sqrt{1+e^{2C}}} \frac{du}{\sqrt{2C - \log(u^2-1)}}$$

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$\frac{dE(u_C)}{dC} > 0$ follows from

$$\frac{dE(u_C)}{dC} = 2\int_1^{\sqrt{1+e^{2C}}} \frac{du}{\sqrt{2C-\log(u^2-1)}} = 2\ell_C.$$

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$\frac{d\ell_C}{dC} > 0$ follows from a longer computation, where we use **the period function** for periodic orbits on the phase plane.

Monotonicity of mappings $C \mapsto E(u_C)$ and $C \mapsto \ell_C$

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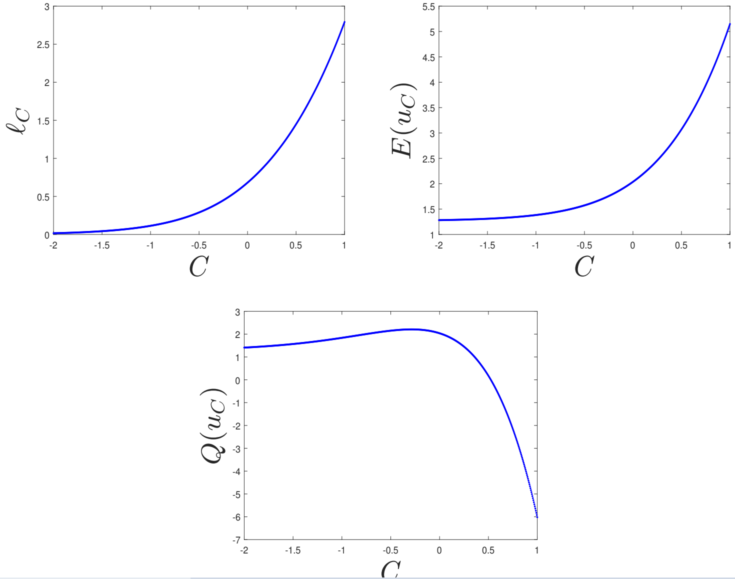
$$\ell_C = \int_1^{\sqrt{1+e^{2C}}} \frac{du}{\sqrt{2C - \log(u^2-1)}}$$

The mapping $C \mapsto Q(u_C)$ is non-monotone.

Numerical illustrations of mappings $C \mapsto \ell_C, E(u_C), Q(u_C)$



Scaling transformation

The variational problem for $\mu > 0$ and $L > 0$:

$$\mathcal{Q}_{\mu,L} := \inf_{u \in X_L} \{Q(u) : \quad E(u) = \mu\},$$

is associated with the Euler–Lagrange equation $\omega u = (1 - u^2)u''$.

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Let u_C be a solution of $u = (1 - u^2)u''$. Then, $u_\omega(x) = u_C(\sqrt{\omega}x)$ is a solution of the Euler–Lagrange equation so that

$$Q(u_\omega) = \frac{1}{\sqrt{\omega}} Q(u_C), \quad E(u_\omega) = \sqrt{\omega} E(u_C)$$

and

$$L = \frac{1}{\sqrt{\omega}} \ell_C, \quad \mu = \sqrt{\omega} E(u_C).$$

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is associated with the Euler–Lagrange equation $\omega u = (1 - u^2)u''$.

Transformation $(\omega, C) \mapsto (\mu, L)$ is invertible because the Jacobian is

$$\begin{vmatrix} \frac{\partial \mu}{\partial \omega} & \frac{\partial \mu}{\partial C} \\ \frac{\partial L}{\partial \omega} & \frac{\partial L}{\partial C} \end{vmatrix} = \frac{1}{2\omega} \left[E(u_C) \frac{d\ell_C}{dC} + \ell_C \frac{dE(u_C)}{dC} \right] > 0.$$

Hence the mapping $(\omega, C) \mapsto (\mu, L)$ is invertible and there exists a unique $C = C_{\mu,L}$ for every $\mu > 0$ and $L > 0$. In fact, $\ell_C E(u_C) = L\mu$.

Convexity of action Λ_ω

Let $v + iw$ with real $v, w \in H^1_{\ell_C} \subset H^1(\mathbb{R})$ be a perturbation to u_C. Then, the action is expanded as

$$\Lambda_{\omega=1}(u_C + v + iw) = \Lambda_{\omega=1}(u_C) + Q_+(v) + Q_-(w) + R(v, w),$$

where $R(v, w)$ is the remainder term

$$R(v, w) = \int_{\mathbb{R}} \left[\log\left(1 - \frac{2u_C v + v^2 + w^2}{1 - u_C^2}\right) + \frac{2u_C v}{1 - u_C^2} + \frac{(1 + u_C^2)v^2}{(1 - u_C^2)^2} + \frac{w^2}{1 - u_C^2} \right] dx.$$

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Bright soliton is energetically stable in a Banach space X if

$$\Lambda_{\omega=1}(u_C+v+iw)-\Lambda_{\omega=1}(u_C) \geq K(\|u\|_X^2+\|v\|_X^2)-K(\|u\|_X^3+\|v\|_X^3).$$

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$$\Lambda_{\omega=1}(u_C + v + iw) = \Lambda_{\omega=1}(u_C) + Q_+(v) + Q_-(w) + R(v, w),$$

$R(v, w)$ is cubic with respect to perturbation:

$$|R(v,w)| \leq K\|(1-u_C^2)^{-1}v\|^3_{L^2\cap L^\infty} + K\|(1-u_C^2)^{-1}w\|^3_{L^2\cap L^\infty},$$

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$$\Lambda_{\omega=1}(u_C + v + iw) = \Lambda_{\omega=1}(u_C) + Q_+(v) + Q_-(w) + R(v,w),$$

whereas Q_+ and Q_- are the quadratic forms:

$$Q_+(v) = \int_{\mathbb{R}} \left[(v_x)^2 + \frac{(1+u_C^2)v^2}{(1-u_C^2)^2} \right] dx, Q_-(w) = \int_{\mathbb{R}} \left[(w_x)^2 + \frac{w^2}{1-u_C^2} \right] dx,$$

The quadratic forms are coercive and bounded as

$$Q_\pm(v) \geq \|v\|^2_{H^1}, \quad Q_\pm(v) \leq K_\pm \left(\|v'\|^2_{L^2} + \|(1-u_C^2)^{-1} v\|^2_{L^2} \right)$$

Hence $u_{C_{\mu L}}$ is a minimizer of $Q(u)$ in X_L for fixed $L > 0$ and $\mu > 0$.

Numerical illustrations

Solving NLS-IDD numerically

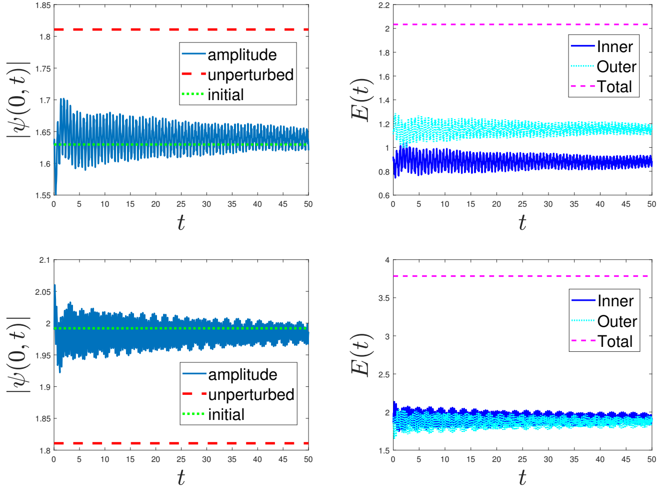
$$i\psi_t + (1 - |\psi|^2)\psi_{xx} = 0,$$

from the initial data:

$$\psi(x,0) = \begin{cases} Pu_{C,\text{head}}(x), & |x| < \ell_C \\ u_{\text{cusp}}(|x| - \ell_C), & |x| \geq \ell_C \end{cases}$$

where $P \neq 1$ is the perturbation factor.

Numerical illustrations



Summary on bright solitons

We have considered NLS-IDD model

$$i\psi_t + (1 - |\psi|^2)\psi_{xx} = 0.$$

- Continuum of singular solitary waves exists $\psi(x,t) = u_C(x)e^{it}$.
- Each solitary wave can be characterized as a minimizer of mass for fixed energy and fixed distance between two singularities.
- Numerical computations show stability of the bright solitons.

Section 3

Dark solitons in NLS-IDD

Another NLS-IDD model

For another NLS-IDD,

$$i(1 - |\psi|^2)\psi_t + \psi_{xx} = 0,$$

transformation $\psi(t,x) = u(t,x)e^{2it}$ recovers the defocusing NLS

$$i(1 - |u|^2)u_t + u_{xx} + 2(1 - |u|^2)u = 0,$$

which admit the black soliton in the form $u(x) = \tanh(x)$.

Dark solitons $u(t,x) = U_c(x - 2ct)$ are found from

$$U_c'' - 2ic(1 - |U_c|^2)U_c' + 2(1 - |U_c|^2)U_c = 0,$$

for any $c \in \mathbb{R}$.

Time evolution

Solutions are to be considered in the set $\mathcal{F}$,

$$\mathcal{F} := \{f \in L^\infty(\mathbb{R}) : \quad |f(x)| < 1, \;\; x \in \mathbb{R}, \;\; |f(x)| \to 1 \;\; \text{as} \;\; |x| \to \infty\}.$$

Dark solitons exist with $U_c \in \mathcal{F}$.

Conjecture: the set $\mathcal{F}$ is invariant under the time evolution of the NLS-IDD for solutions satisfying $u(t, \cdot) - U_c \in H^\infty(\mathbb{R})$, $t \in [0, \tau_0)$.

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Dark solitons exist with $U_c \in \mathcal{F}$.

NLS-IDD admits conserved mass and energy

$$M(\psi) = \int (1 - |\psi|^2)^2 dx, \quad E(\psi) = \int |\psi_x|^2 dx$$

as well as momentum

$$P(\psi) = \frac{1}{2i} \int \frac{(1 - |\psi|^2)^2}{|\psi|^2} (\bar{\psi}\psi_x - \bar{\psi}_x \psi) dx.$$

Their conservation is proven for smooth solutions satisfying $\psi(t,x) = e^{i\theta_\pm}(1 + \mathcal{O}(e^{-\alpha_\pm |x|}))$ as $x \to \pm\infty$.

Linearization and spectral stability of the black soliton

Using the decomposition $\psi(t,x) = e^{-2it}[\varphi(x) + u(t,x) + iv(t,x)]$, where $\varphi(x) = \tanh(x)$ and $u + iv$ is the perturbation, we obtain the linearized equations of motion

$$(1-\varphi^2)u_t = L_- v, \quad (1-\varphi^2)v_t = -L_+ u,$$

where $L_+ = -\partial_x^2 + 4 - 6\text{sech}^2(x)$ and $L_- = -\partial_x^2 - 2\text{sech}^2(x)$ are the same as in the NLS equation.

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The spectral problem

$$L_- v = \lambda(1-\varphi^2)u, \quad L_+ u = -\lambda(1-\varphi^2)v$$

is defined in the Hilbert space $\mathcal{H}$ with the inner product

$$(f,g)_{\mathcal{H}} := \int (1-\varphi^2)\bar{f} g dx = \int \text{sech}^2(x)\bar{f}(x)g(x)dx.$$

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$$(f,g)_{\mathcal{H}} := \int (1-\varphi^2)\bar{f} g dx = \int \text{sech}^2(x)\bar{f}(x)g(x)dx.$$

Theorem (P–Plum, SIAM J. Math. Anal. (2023), in print)

- *The spectrum of L_+ in $\mathcal{H}$ consists of simple eigenvalues* $\mu_n = n(n+5)$, $n \geq 0$.
- *The spectrum of L_- in $\mathcal{H}$ consists of simple eigenvalues* $\nu_n = n(n+1) - 2$, $n \geq 0$.
- *The spectrum of the stability problem in $\mathcal{H} \times \mathcal{H}$ consists of pairs of isolated eigenvalues* $\{\pm i\omega_1, \pm i\omega_2, \cdots\}$ *and zero eigenvalue.*

Energetic stability of the black soliton

Expanding the energy functional

$$\Lambda(\psi) := \int [|\psi_x|^2 + (1 - |\psi|^2)^2] dx$$

at the black soliton $\varphi(x) = \tanh(x)$ yields

$$\Lambda(\psi = \varphi + u + iv) - \Lambda(\varphi) = Q_+(u) + Q_-(v) + R(u, v),$$

where $Q_+(u) = (L_+ u, u)_{L^2}$, $Q_-(v) = (L_- v, v)_{L^2}$, and

$$R(u, v) = \int [(2\varphi u + u^2 + v^2)^2 - 4\varphi^2 u^2] dx$$

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Black soliton is energetically stable in a Banach space X if

$$\Lambda(\psi) - \Lambda(\varphi) \geq C(\|u\|_X^2 + \|v\|_X^2) - C(\|u\|_X^3 + \|v\|_X^3).$$

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However, two obstacles arise due to nonzero boundary conditions

- $L_- = -\partial_x^2 - 2\text{sech}^2(x)$ is not coercive in $H^1(\mathbb{R})$
- $R(u, v)$ is not cubic if $(u, v) \notin H^1(\mathbb{R})$.

Energetic stability of the black soliton

For the cubic NLS, these issues were handled in [Gravejat–Smets, 2015] by using the revised decomposition

$$\Lambda(\psi = \varphi + u + iv) - \Lambda(\varphi) = Q_-(u) + Q_-(v) + \|\eta\|_{L^2}^2$$

where $Q_-(v) = (L_- v, v)_{L^2}$ and $\eta := |\psi|^2 - \varphi^2 = 2\varphi u + u^2 + v^2$. The distance for perturbations in Banach space X was chosen to be

$$\mathcal{D}_X(\psi_1, \psi_2) := \sqrt{\|\psi_1' - \psi_2'\|_{L^2}^2 + \||\psi_1|^2 - |\psi_2|^2\|_{L^2}^2 + \|\psi_1 - \psi_2\|_{\mathcal{H}}^2}.$$

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For the NLS–IDD, we have several advantages:

- $\mathcal{H}$ appears naturally in the time evolution
- $Q_-(u)$ and $Q_-(v)$ are coercive in $\mathcal{H}$ if
  - $u \in \mathcal{H}$ satisfies orthogonality $(\varphi', u)_{\mathcal{H}} = (\varphi, u)_{\mathcal{H}} = 0$
  - $v \in \mathcal{H}$ satisfies orthogonality $(\varphi', v)_{\mathcal{H}} = (\varphi, v)_{\mathcal{H}} = 0$

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For the four orthogonality conditions, we use the decomposition

$$\psi(t,x) = e^{i\theta(t)} \left[U_{c(t),\omega(t)}(x + \zeta(t)) + u(t, x + \zeta(t)) + iv(t, x + \zeta(t)) \right],$$

where the additional parameter ω is due to the scaling invariance $\psi(t,x) \mapsto \psi(\omega^2 t, \omega x)$ of the NLS equation $i(1 - |\psi|^2)\psi_t + \psi_{xx} = 0$.

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Theorem (P–Plum, SIAM J. Math. Anal. (2023), in print)

Assume that the initial-value problem is well-posed in $\mathcal{F} \subset X$ with the distance $\mathcal{D}_X$ and the values of $M(\psi)$, $E(\psi)$, and $P(\psi)$ are conserved in the time evolution. Then, the black soliton is orbitally stable in X.

Summary on the black soliton

We considered NLS-IDD model

$$i(1-|\psi|^2)\psi_t + \psi_{xx} = 0.$$

- Linearization at the black soliton consists of isolated eigenvalues
- Perturbations near the black soliton are controlled by the conserved energy, mass, and momentum.
- Local existence of solutions in the set $\mathcal{F} \subset X$ needs to be addressed in future.

Section 4

More NLS models?

Further emerging NLS models

The regularized NLS equation

$$i(1-\epsilon^2\partial_x^2)u_t + u_{xx} + 2(1-|u|^2)u = 0.$$

M. Colin, D. Lannes SIMA **41** (2009) 708–732

D. Lannes, Proc. R. Soc. Edinburgh Ser A **141** (2011) 253–286

- Local well-posedness can be shown for $u = \varphi + v$, where $\varphi(x) = \tanh(x)$ and $v \in H^s(\mathbb{R})$ with $s > \frac{1}{2}$.

- Theorem [P–Plum, Proceeding of AMS (2023), in print]. The black soliton φ is spectrally stable for $\epsilon \leq \epsilon_0 := (5/8)^{1/4}$ and spectrally unstable for $\epsilon > \epsilon_0$.

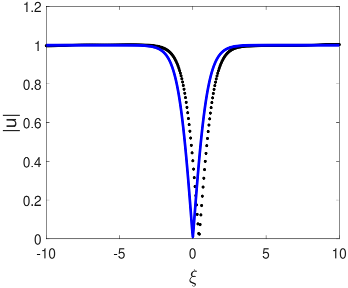
- However, the energy is defined for $v \in H^1$ and the momentum is defined for $v \in H^2(\mathbb{R})$. Orbital stability of black soliton is open.

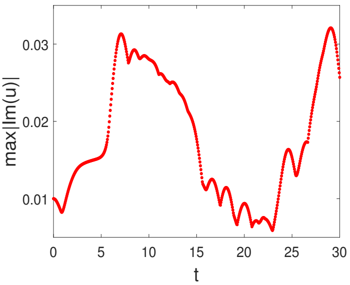
Numerical illustration

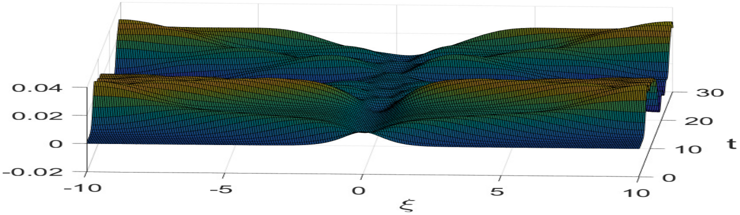
Initial data: $u_0(x) = \tanh(x) + ia\,\mathrm{sech}^2(x)$ with $a = 0.01$

Numerical illustration

$\epsilon = 0.5$

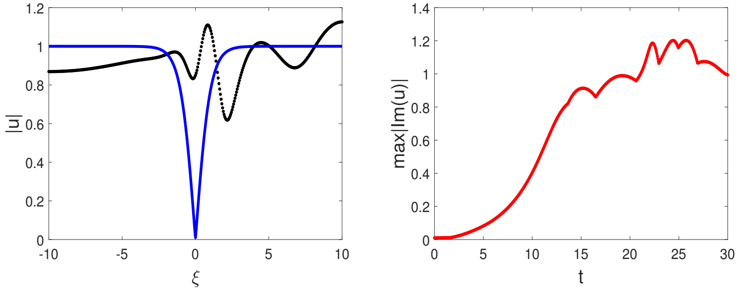


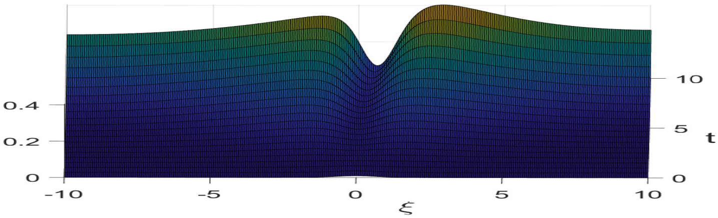




Numerical illustration

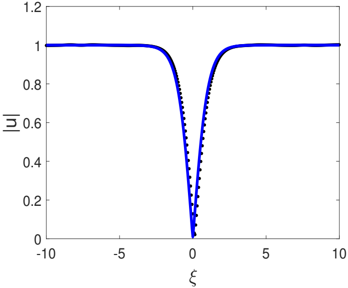
$\epsilon = 1$

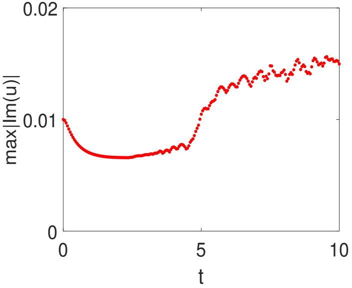


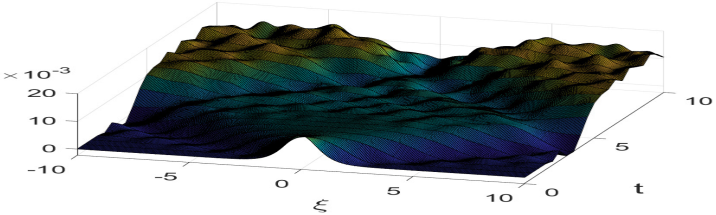


Numerical illustration

$\epsilon = 0$







Conclusion

- We have considered new variations of the classical NLS model with intensity-dependent dispersion or a regularized dispersion.

- Bright and black solitons are energeticaly stable in the energy space but singularities of the NLS models with intensity-dependent dispersion may suggest problems in the existence of time-dependent solutions in the energy space.

- Regularized NLS models are well-posed in the energy space but the energy space does not coincide with the momentum space.

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MANY THANKS FOR YOUR ATTENTION!