Nonlinearity management in time-periodic NLS systems

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References:

Phys. Rev. Lett. 91, 240201 (2003)

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Phys. Rev. E 70, 047604 (2004)

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J. Diff. Eqs. 220, 85 (2006)

Background and motivations

Time-periodic NLS equation

$$iu_t = -\Delta u + \gamma(t)|u|^2 u + V(x)u,$$

where

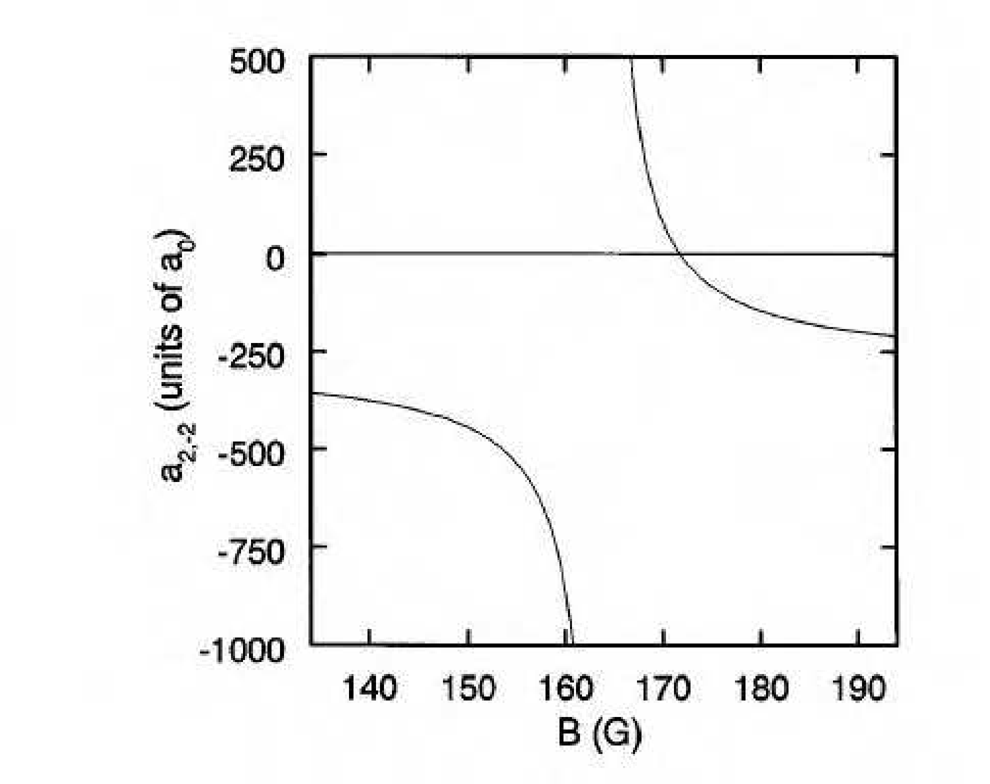
- $u(x,t) : \mathbb{R}^d \times \mathbb{R}_+ \mapsto \mathbb{C}$ is a classical solution
- $\gamma(t + t_0) = \gamma(t)$ is a periodic coefficient
- $V(x) \geq 0$ is a (decaying, parabolic, and/or periodic) potential

Applications:

- Feshbach resonance in Bose-Einstein condensates
- optical pulse propagation in layered optical media

Physical experiments in BECs (1998)

Scattering length versus magnetic field



Feshbach resonance in ^{85}Rb

Mathematical problems

Time-periodic NLS equation

$$iu_t = -\Delta u + \gamma(t)|u|^2 u + V(x)u,$$

- homogenization in the limit of short and large-amplitude variations of $\gamma(t)$
  $\Rightarrow$ derivation of the averaged NLS equation
- arrest of blowup in dimensions $d \geq 2$
  $\Rightarrow$ local and global well-posedness of the averaged equation
- stability of gap solitons in periodic potentials
  $\Rightarrow$ computations of eigenvalues of linearized equations
- radiative decay of small-amplitude localized solutions
  $\Rightarrow$ decay law of the amplitude of localized solutions

Averaging theory

Time-periodic NLS equation

$$iu_t = -u_{xx} + \gamma_0 |u|^2 u + \frac{1}{\epsilon}\gamma\left(\frac{t}{\epsilon}\right)|u|^2 u,$$

where

- $V(x) \equiv 0$ for simplicity
- $d = 1$ without loss of generality
- $\epsilon \to 0$ is the limit of short and large-amplitude variations of $\gamma(t)$, such that

$$\gamma(\tau + 1) = \gamma(\tau), \qquad \int_0^1 \gamma(\tau) d\tau = 0.$$

Equivalent transformations

Local transformation

$$u(x,t) = v(x,t)\ \exp\left(-i\gamma_{-1}(\tau)|v|^2(x,t)\right)$$

where $\gamma_{-1}(\tau)$ is the mean-zero antiderivative of $\gamma(\tau)$.

Equivalent NLS equation

$$iv_t = -v_{xx} + \gamma_0|v|^2 v + 2i\gamma_{-1}(\tau)\left(v^2\bar{v}_{xx} + 2|v_x|^2 v + v_x^2\bar{v}\right)$$
$$-\gamma_{-1}^2(\tau)\left(\left(|v|_x^2\right)^2 + 2|v|_{xx}^2|v|^2\right)v.$$

Methods of averaging:

- canonical transformations of the Hamiltonian
- near-identity transformations
- asymptotic multiscale expansions

Asymptotic multi-scale expansions

Asymptotic expansion

$$v(x,t) = w(x,t) + \epsilon v_1(x,t,\tau) + \mathrm{O}(\epsilon^2)$$

where τ is fast time and t is slow time.

The averaged NLS equation

$$i w_t = -w_{xx} + \gamma_0 |w|^2 w - \sigma^2 \left(\left(|w|_x^2\right)^2 + 2|w|_{xx}^2 |w|^2 \right) w,$$

where σ^2 is the mean value of $\gamma_{-1}^2(\tau)$

The first-order correction

$$v_1 = 2(\gamma_{-1})_{-1} \left(w^2 \bar{w}_{xx} + 2|w_x|^2 w + w_x^2 \bar{w} \right)$$
$$-i(\gamma_{-1}^2 - \sigma^2)_{-1} \left(\left(|w|_x^2\right)^2 + 2|w|_{xx}^2 |w|^2 \right) w,$$

Properties of averaged NLS equation

Hamiltonian form of the time-periodic NLS equation

$$H = \int_{\mathbb{R}} \left(|u_x|^2 + \frac{1}{2}\gamma_0 |u|^4 + \frac{1}{2\epsilon}\gamma\left(\frac{t}{\epsilon}\right)|u|^4 \right) dx.$$

Hamiltonian form of the averaged NLS equation

$$H = \int_{\mathbb{R}} \left(|w_x|^2 + \frac{1}{2}\gamma_0 |w|^4 + \sigma^2 |w|^2 \left(|w|_x^2\right)^2 \right) dx.$$

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Hamiltonian form of the averaged NLS equation

$$H = \int_{\mathbb{R}} \left(|w_x|^2 + \frac{1}{2}\gamma_0 |w|^4 + \sigma^2 |w|^2 \left(|w|^2_x\right)^2 \right) dx.$$

Local well-posedness in $H^\infty = \cap_{n\geq 0} H^n(\mathbb{R})$ ($d = 1$):

Let $w(x,0) \in H^\infty$. There exists $T > 0$ such that the averaged NLS equation possess a unique solution $w(x,t) \in C^1([0,T], H^\infty)$.

M. Poppenberg, Nonlinear Anal. Theory 45, 723 (2001)

Nonlinear bound states

ODE reductions for nonlinear bound states

$$w(x,t) = \Phi(x)e^{i\omega t}, \qquad \left(\frac{d\Phi}{dx}\right)^2 = \frac{(2\omega + \gamma_0\Phi^2)}{2(1+4\sigma^2\Phi^4)}\Phi^2$$

No exact solutions exist generally if $V(x) \neq 0$

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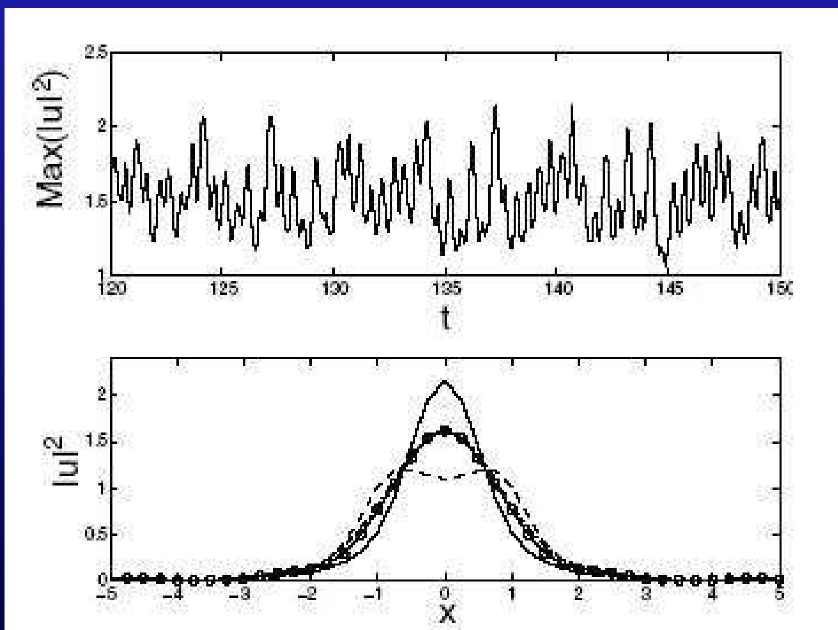
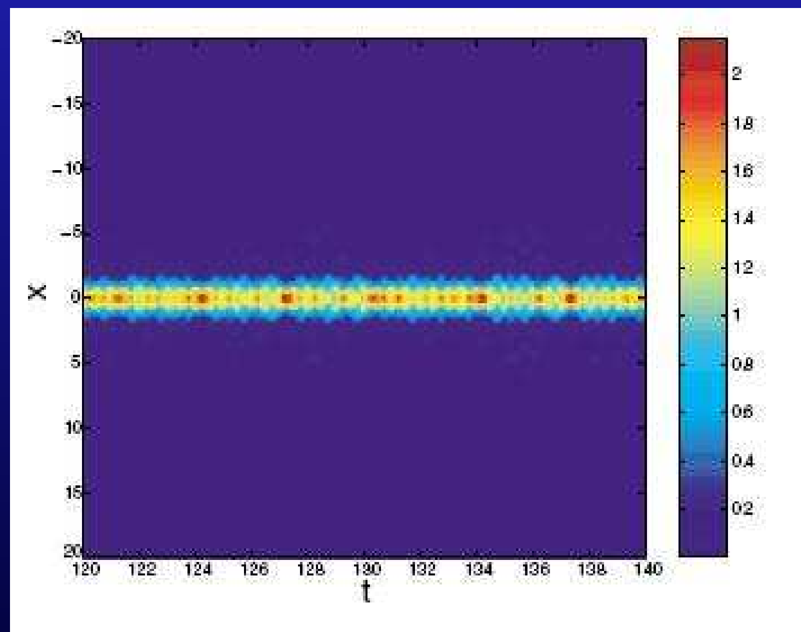
No exact solutions exist generally if $V(x) \neq 0$

Temporal evolution of the bound state if $V(x) \sim x^2$





Arrest of blowup

Hamiltonian of the averaged NLS equation

$$H = H_1(w) + \gamma_0 H_2(w),$$

where $d \geq 2$, $\gamma_0 < 0$, and

$$H_1 = \int_{\mathbb{R}^d} \left(|\nabla w|^2 + \sigma^2 |w|^2 \left(\nabla |w|^2 \right)^2 \right) dx, \qquad H_2 = \frac{1}{2} \int_{\mathbb{R}^d} |w|^4 dx.$$

- Blowup occurs at $\sigma = 0$ (no nonlinearity management)
- Blowup may occur in the time-periodic NLS equation
  V. Konotop and P. Pacciani, Phys. Rev. Lett. 94, 240405 (2005)
- We show that blowup never occurs in the averaged NLS equation with $\sigma \neq 0$ (strong nonlinearity management)

Local and global solutions

- Local solutions of the averaged NLS equation in $H^\infty(\mathbb{R}^d)$
  C.E. Kenig, *The Cauchy problem for the Quasilinear Schroödinger Equation* (2002)

- Local solutions of the time-periodic NLS equation in $H^1(\mathbb{R}^d)$
  T. Cazenave, *Semilinear Schrödinger equations* (2003)

- Difficulty: no local existence of the averaged NLS equation is proved in $H^1(\mathbb{R}^d)$
- Assuming the local existence for the averaged NLS equation in $H^1(\mathbb{R}^d)$, we show that the solution remains globally in $H^1(\mathbb{R}^d)$, so that the standard blow-up mechanism for the focusing NLS equation with $d \geq 2$ does not occur.

Proof of arrest of blow-up

Gagliardo–Nirenberg inequality

$$\|w\|_{L^4} \leq \|w\|_{L^6}^{3/4} \|w\|_{L^2}^{1/4}.$$

Poincare's inequality

$$\|f\|_{L^2} \leq C \left(\|\nabla f\|_{L^2} + \|f\|_{L^1}\right),$$

results in the inequality

$$\|w\|_{L^6}^6 \leq C \left(H_1(w) + \|w\|_{L^3}^6\right)$$

Proof of arrest of blow-up

Another Gagliardo–Nirenberg inequality

$$\|w\|_{L^3} \leq \|w\|_{L^6}^{1/2} \|w\|_{L^2}^{1/2},$$

results in the inequality

$$\|w\|_{L^6}^6 \leq C \left(H_1(w) + \frac{1}{4\mu} \|w\|_{L^2}^6 \right),$$

for any $0 < \mu < 1/(2C)$.

Therefore,

$$H_2(w) \leq \mu H_1(w) + C \left(P(w) + P^2(w) \right)$$
$$H_1(w) \leq C \left(H(w) + P(w) + P^2(w) \right),$$

where $H(w)$ and $P(w) = \|w\|_{L^2}^2$ are constant in the time evolution.

Strong versus weak managements

Weak nonlinearity management

$$\frac{1}{\epsilon}\gamma\left(\frac{t}{\epsilon}\right) \mapsto \gamma\left(\frac{t}{\epsilon}\right)$$

reduces the averaged NLS equation to the form

$$iw_t = -\Delta w + \gamma_0 |w|^2 w - \epsilon^2 \sigma^2 \left(\left|\nabla |w|^2\right|^2 + 2|w|^2 \Delta |w|^2 \right) w,$$

where ϵ^2 is small.

F. Abdullaev, J. Caputo, R. Kraenkel and B. Malomed, Phys. Rev. A 67, 013605 (2003)

Strong versus weak managements

Contradiction:

- No blow-up occurs in the averaged NLS equation for any $\epsilon^2\sigma^2 \neq 0$
- Blow-up *may* occur in the time-periodic NLS equation for small ϵ^2

Strong versus weak managements

Contradiction:

- No blow-up occurs in the averaged NLS equation for any $\epsilon^2\sigma^2 \neq 0$
- Blow-up *may* occur in the time-periodic NLS equation for small ϵ^2

Let us consider this contradiction under the simplifications:

- $d = 2$ (critical blow-up)
- exact ODE reduction by using the method of moments

$$\ddot{R}(t) = \frac{\alpha + \beta\gamma(t/\epsilon)}{R^3},$$

where $\alpha, \beta = \mathrm{O}(1)$ as $\epsilon \to 0$ and $\beta > 0$.

ODE analysis

ODE with $\beta > 0$:

$$\ddot{R}(t) = \frac{\alpha + \beta\gamma(t/\epsilon)}{R^3}.$$

Montesinos, Perez-Garcia, Torres, Physica D 191, 193 (2004)

- Sufficient condition for blow-up

$$\alpha < -\beta \max_{0 \le \tau \le 1} (\gamma).$$

- Necessary condition for bounded oscillations

$$\alpha > -\beta \max_{0 \le \tau \le 1} (\gamma).$$

Strong versus weak managements

Contradiction:

- Strong management $\beta \gg |\alpha|$ results in the blow-up arrest
- Weak management $\beta \sim |\alpha|$ *may* result in blow-up

Consider the averaging method for $\gamma = \sin(2\pi\tau)$:

$$R = r(t) + \epsilon^2 R_2(\tau, r) + \epsilon^4 R_4(\tau, r) + O(\epsilon^6),$$

where the mean-value term $r(t)$ satisfies the averaged equation

$$\ddot{r} = \frac{\alpha}{r^3} + \epsilon^2 \frac{3\beta^2}{2r^7} + \epsilon^4 \frac{15\alpha\beta^2}{2r^{11}} + O(\epsilon^6),$$

where $\alpha < 0$ and $\beta > 0$

Failure of the averaged equation

Effective potential

$$U(r) = \frac{\alpha}{2r^2} + \epsilon^2 \frac{\beta^2}{4r^6} + \epsilon^4 \frac{3\alpha\beta^2}{4r^{10}} + \mathrm{O}(\epsilon^6)$$

with $\alpha < 0$ and $\beta > 0$.

- $\epsilon = 0$: blow-up in a finite time
- $\mathrm{O}(\epsilon^2)$: blow-up is arrested
- $\mathrm{O}(\epsilon^4)$: blow-up may occur depending on the ratio between parameters α and β

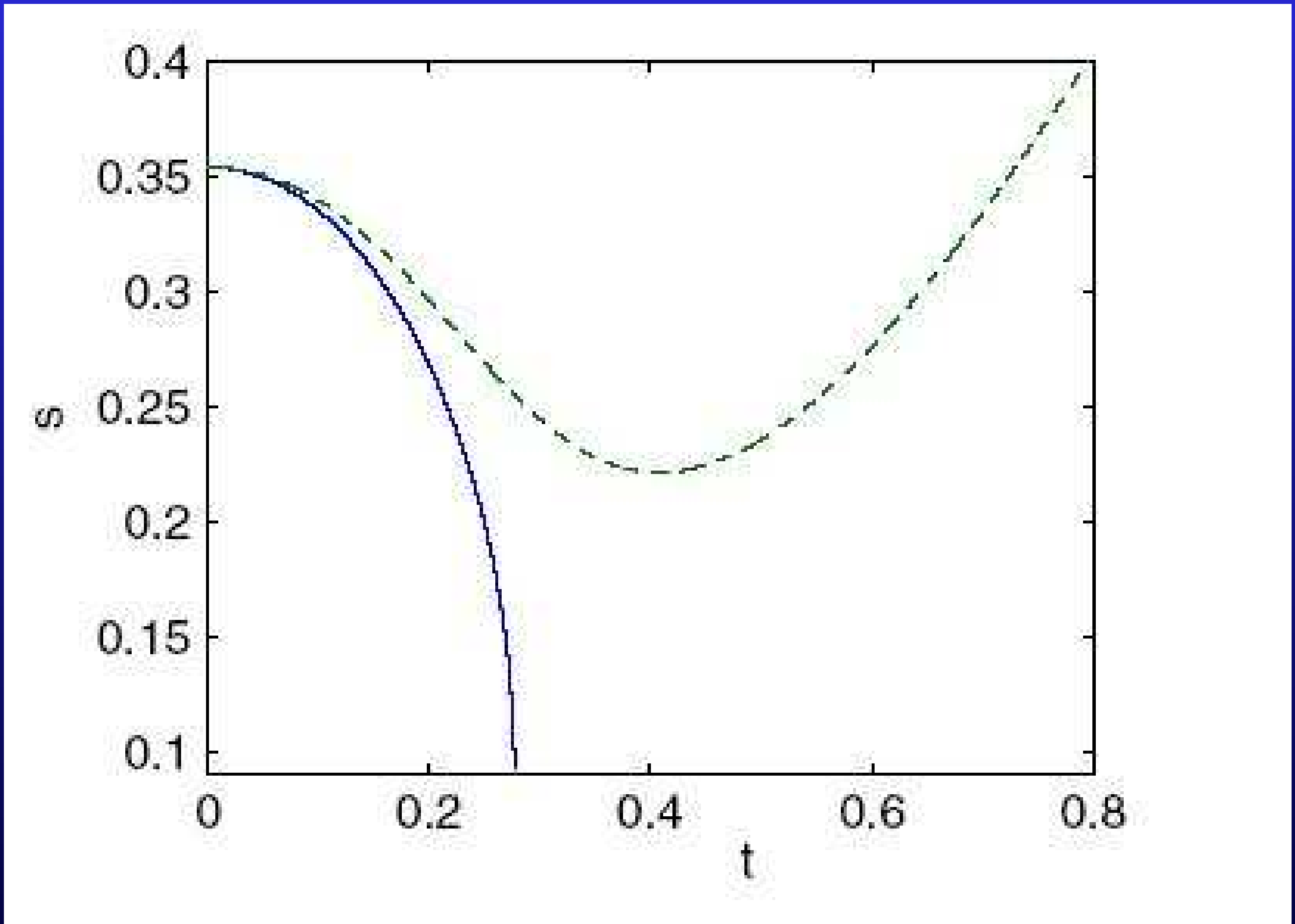
The exact threshold $\alpha = -\beta$ can not be found from the truncated averaged equation!

Numerical computations

$\alpha = -20, \beta = 8$

Solid - time-periodic NLS equation

Dashed - averaged NLS equation



Gap solitons in periodic potentials

The averaged NLS equation with $V = V_0 \cos(2\omega x)$:

$$iw_t = -w_{xx} + V_0 \cos(2\omega x) w - \sigma^2 \left((|w|_x^2)^2 + 2|w|^2(|w|^2)_{xx} \right] w,$$

where $d = 1$ and $\gamma_0 = 0$.

Coupled-mode theory in the limit $V_0 \to 0$:

$$i(A_T + 2\omega A_X) = V_0 B + 8\sigma^2\omega^2(2|A|^2 + |B|^2)|B|^2 A,$$
$$i(B_T - 2\omega B_X) = V_0 A + 8\sigma^2\omega^2(|A|^2 + 2|B|^2)|A|^2 B,$$

where $X = \epsilon x$, $T = \epsilon t$, and

$$w(x,t) = \sqrt{\epsilon}\left(A(X,T) e^{i\omega x - i\omega_0 t} + B(X,T) e^{-i\omega x - i\omega^2 t} + \mathrm{O}(\epsilon) \right).$$

M. Chugunova, D.P., SIAM J. Appl. Dyn. Syst. 5, 66 (2006)

Existence of gap solitons

Exact solution for gap solitons

$$A(X,T) = a(X)e^{-i\Omega T}, \quad B(X,T) = \bar{a}(X)e^{-i\Omega T},$$

where

$$a(X) = \frac{\sqrt[4]{\gamma(\cosh(4\beta X) - \Omega)}}{\sqrt{\sigma}[\cosh(2\beta X) + i\sqrt{\gamma}\sinh(2\beta X)]},$$

$\gamma = \frac{1+\Omega}{1-\Omega}$ and $\beta = \sqrt{1-\Omega^2}$.

The family has the threshold in the power $P \geq P_0$, where

$$P = \int_{-\infty}^{\infty} \left(|A|^2 + |B|^2\right) dX, \quad P_0 = \frac{\pi}{\sigma\sqrt{2}}.$$

The threshold was discovered numerically in the full problem in A. Gubeskys, B. Malomed, I. Merhasin, Stud. Appl. Math. 115,

Eigenvalues of stability problem

Standard linearization, e.g. $A(X,T) = e^{-i\Omega t}\left(a(x) + U_1(x)e^{\lambda t}\right)$, results in the eigenvalue problem

$$H_\omega \mathbf{U} = i\lambda\sigma\mathbf{U}, \quad \mathbf{U} \in \mathbb{C}^4,$$

where H is a four-component Dirac operator

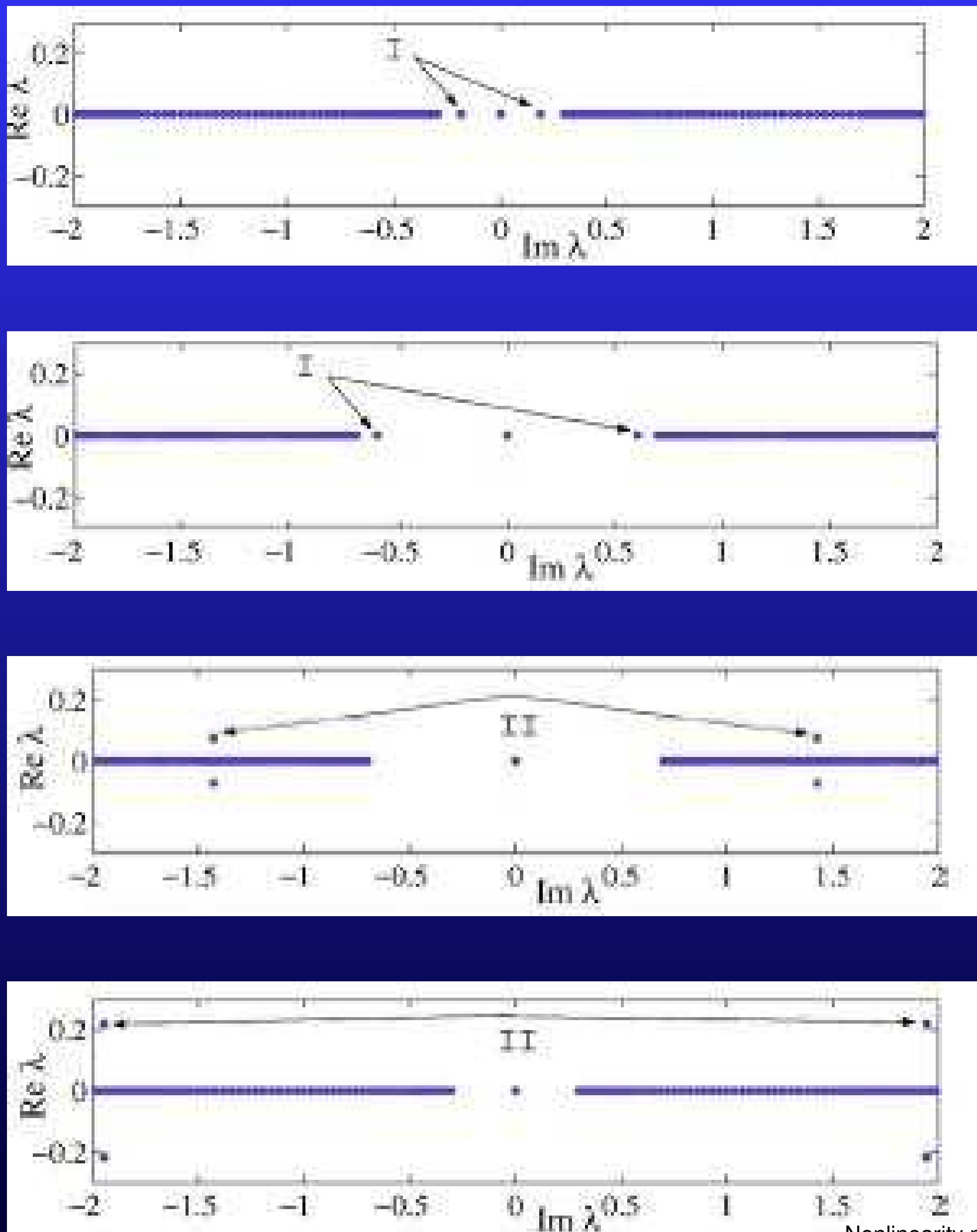
If the coupled-mode system is symmetric with respect to $a \longleftrightarrow b$, there exists an orthogonal similarity transformation S in $\mathbb{C}^4$:

$$S^{-1}\sigma H_\omega S = \sigma \begin{pmatrix} 0 & H_- \\ H_+ & 0 \end{pmatrix},$$

where $H_\pm$ are two-by-two Dirac operators, such that

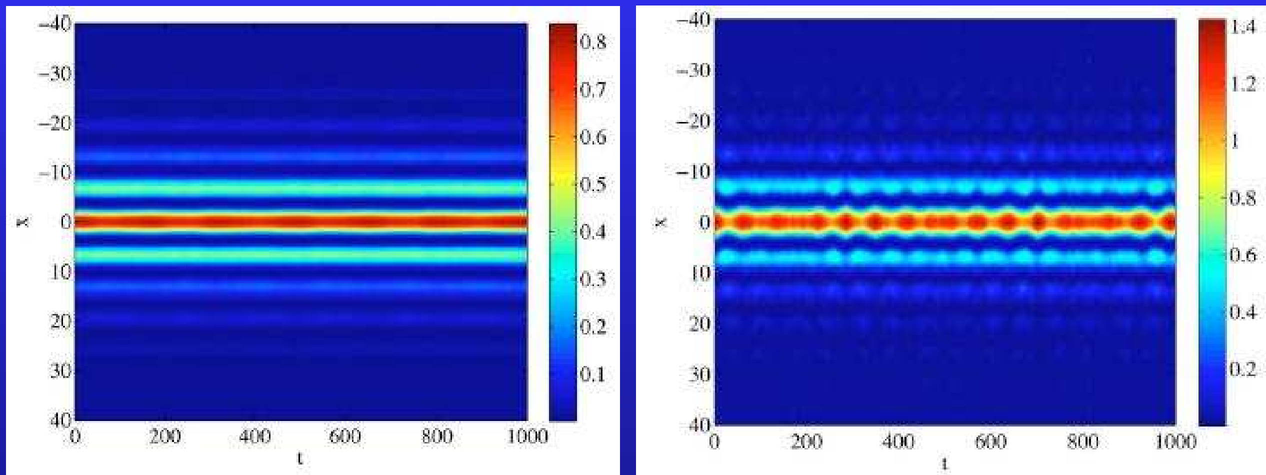
$$\sigma_3 H_- \sigma_3 H_+ \mathbf{V}_1 = -\lambda^2 \mathbf{V}_1, \quad \sigma_3 H_+ \sigma_3 H_- \mathbf{V}_2 = -\lambda^2 \mathbf{V}_2, \quad \mathbf{V}_1, \mathbf{V}_2 \in \mathbb{C}^2$$

Numerical approximations of eigenvalues

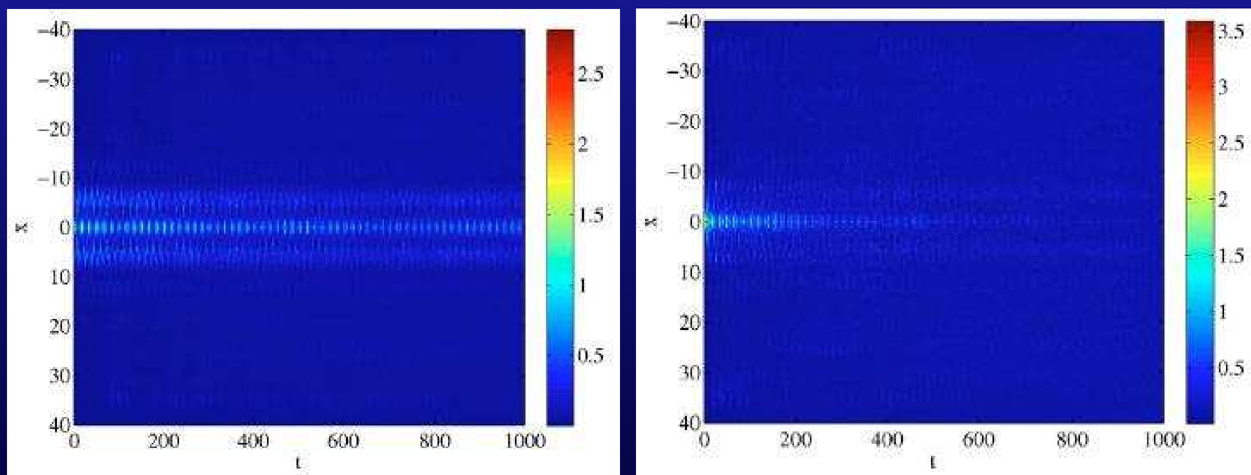


Numerical evolution of gap solitons

Simulations of the averaged NLS equation



Simulations of the time-periodic NLS equation



Open problems

- Local and global well-posedness of the averaged NLS equation in $H^1(\mathbb{R}^d)$
- Error bounds on the distance between time-periodic and averaged NLS equations
- Decay rate on radiative damping of localized solutions in the time-periodic NLS equation
- Sharp bounds on the initial data for blow-up in the time-periodic NLS equation
- Dynamics of dark solitons under the time-periodic NLS equation