

# Nonlinear Dirac equations and stability of solitary waves in one spatial dimension

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The nonlinear Dirac equations in one spatial dimension,

$$\begin{cases} i(u_t + u_x) + v = \partial_{\bar{u}} W(u, v), \\ i(v_t - v_x) + u = \partial_{\bar{v}} W(u, v), \end{cases}$$

where  $W(u, v) : \mathbb{C}^2 \rightarrow \mathbb{R}$  satisfies the following three conditions:

- symmetry  $W(u, v) = W(v, u)$ ;
- gauge invariance  $W(e^{i\theta} u, e^{i\theta} v) = W(u, v)$  for any  $\theta \in \mathbb{R}$ ;
- polynomial in  $(u, v)$  and  $(\bar{u}, \bar{v})$ .

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- polynomial in  $(u, v)$  and  $(\bar{u}, \bar{v})$ .

Compare with quantum relativistic physics in three spatial dimensions,

$$i \left( u_t + \sum_{j=1}^3 \alpha_j u_{x_j} \right) - m\beta u + g(u\bar{u})\beta u = 0,$$

where  $\bar{u} = \beta u^*$ ,  $m \in \mathbb{R}$  is Dirac mass,  $g(\cdot)$  is a nonlinear function, and

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix},$$

and  $\sigma_1, \sigma_2, \sigma_3$  are Pauli matrices.

- Coupled-mode equations for Bragg resonance (photonic crystals)

$$W = \alpha(|u|^2 + |v|^2)^2 + 2\alpha|u|^2|v|^2, \quad \alpha \in \mathbb{R}.$$

- Periodic modulations of Kerr nonlinearity (nonlinear optics)

$$W = \alpha(\bar{u}v + u\bar{v})(|u|^2 + |v|^2) + \beta(\bar{u}^2v^2 + u^2\bar{v}^2),$$

- Gross–Neveu model (general relativity)

$$W = \alpha(\bar{u}v + u\bar{v})^2,$$

- Massive Thirring model (integrable systems)

$$W = \alpha|u|^2|v|^2,$$

- Feshbach resonance in optical lattices (Bose–Einstein condensation)

$$W = \alpha(|u|^2 + |v|^2)|u|^2|v|^2.$$

## Theorem

Assume  $\mathbf{u}_0 \in H^s(\mathbb{R})$  for any fixed  $s > \frac{1}{2}$ . There exists  $T > 0$  such that the nonlinear Dirac equations

$$\begin{cases} i(u_t + u_x) + v = \partial_{\bar{u}} W(u, v), \\ i(v_t - v_x) + u = \partial_v W(u, v), \end{cases}$$

admit a unique solution

$$\mathbf{u}(t) \in C([0, T], H^s(\mathbb{R})) \cap C^1([0, T], H^{s-1}(\mathbb{R})) : \quad \mathbf{u}(0) = \mathbf{u}_0,$$

which depends continuously on the initial data.

- Global existence in  $H^1(\mathbb{R})$  or even in  $L^2(\mathbb{R})$ :  
How does it depend on the nonlinearity  $W$ ?
- Spectral stability of solitary waves:  
Can we control isolated eigenvalues inducing instabilities?
- Asymptotic stability of solitary waves:  
Is the linear dispersion sufficient for decay of perturbations?

Theorem (Delgado, 1978; Goodman, Weinstein and Holmes, 2001)

*Assume that  $W$  is a polynomial in variables  $|u|^2$  and  $|v|^2$ . A local solution is extended globally as  $\mathbf{u}(t) \in C(\mathbb{R}_+, H^1(\mathbb{R}))$ .*

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- $L^2$  conservation gives  $\|\mathbf{u}(t)\|_{L^2} = \|\mathbf{u}(0)\|_{L^2}$
- To obtain a priori energy estimates, cancellation of  $W$  is used in

$$\partial_t (|u|^{2p+2} + |v|^{2p+2}) + \partial_x (|u|^{2p+2} - |v|^{2p+2}) = i(p+1)(v\bar{u} - \bar{v}u)(|u|^{2p} - |v|^{2p}).$$



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- By Gronwall's inequality, we have

$$\|\mathbf{u}(t)\|_{L^{2p+2}} \leq e^{2|t|} \|\mathbf{u}(0)\|_{L^{2p+2}}, \quad t \in [0, T],$$

which holds for any  $p \geq 0$  including  $p \rightarrow \infty$ .

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- This allows to control

$$\frac{d}{dt} \|\mathbf{u}_x(t)\|_{L^2}^2 \leq C_W e^{4(N-1)|t|} \|\mathbf{u}_x(t)\|_{L^2}^2,$$

where  $N$  is the degree of  $W$  in variables  $|u|^2$  and  $|v|^2$ .

- For the nonlinear Schrödinger equation,

$$iu_t + u_{xx} \pm |u|^{2N}u = 0,$$

global existence in  $H^1(\mathbb{R})$  is known for  $-|u|^{2N}u$  with any  $N \geq 0$  and for  $|u|^{2N}u$  with  $0 \leq N < 2$ . Blowup in a finite time is known for  $N \geq 2$ .

For the nonlinear Dirac equation, the result does not depend on the power of nonlinearity.

- The energy conservation is crucial in the proof of global existence for the NLS equation and plays no role for the nonlinear Dirac equations, because the energy

$$H = \frac{i}{2} \int_{\mathbb{R}} (u_x \bar{u} - u \bar{u}_x - v_x \bar{v} + v \bar{v}_x) dx + \int_{\mathbb{R}} (v \bar{u} + u \bar{v} - W(u, v)) dx$$

is not sign-definite near the zero equilibrium.

# Global existence and scattering in Strichartz spaces

Strichartz spaces  $L_t^p L_x^q$  and  $L_x^q L_t^p$  are defined for  $1 \leq p, q \leq \infty$  by

$$\|f\|_{L_t^p L_x^q} := \left( \int_0^T \|f(\cdot, t)\|_{L_x^q}^p dt \right)^{1/p}, \quad \|f\|_{L_x^q L_t^p} := \left( \int_{\mathbb{R}} \|f(x, \cdot)\|_{L_t^p}^q dx \right)^{1/q},$$

We say that a pair  $(q, r)$  is Strichartz admissible for the nonlinear Dirac equations in Strichartz space  $L_t^q L_x^r$  if

$$q \geq 2, \quad r \geq 2 \quad \text{and} \quad \frac{2}{q} + \frac{1}{r} \leq \frac{1}{2}.$$

In particular,  $(q, r) = (4, \infty)$  and  $(q, r) = (\infty, 2)$  are end-point Strichartz pairs.

**Lemma (Nakanishi, 1999; P, Stefanov, 2012)**

*Let  $(q, r)$  be a Strichartz admissible pair. There are constants  $C > 0$  such that*

$$\begin{aligned} \|e^{-it\mathcal{H}}\mathbf{f}\|_{L_t^4 L_x^\infty} &\leq C\|\mathbf{f}\|_{H_x^1}, \\ \|e^{-it\mathcal{H}}\mathbf{f}\|_{L_t^\infty H_x^1} &\leq C\|\mathbf{f}\|_{H_x^1}, \\ \left\| \int_0^t e^{-i(t-\tau)\mathcal{H}}\mathbf{F}(\tau, \cdot)d\tau \right\|_{L_t^4 L_x^\infty \cap L_t^\infty H_x^1} &\leq C\|\mathbf{F}\|_{L_t^1 H_x^1}. \end{aligned}$$

## Theorem

*Assume  $W$  be a homogeneous polynomial in  $\mathbf{u}$  of degree  $2n + 2$  for  $n \geq 2$ . Assume  $\mathbf{u}(0) \in H^1(\mathbb{R})$  and  $\|\mathbf{u}(0)\|_{H^1}$  be sufficiently small. There exists a global solution*

$$\mathbf{u}(t) \in C(\mathbb{R}_+, H^1(\mathbb{R})) \cap L^4(\mathbb{R}_+, L^\infty(\mathbb{R})).$$

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- Because  $\|\mathbf{u}(t)\|_{L^\infty}$  is a continuous function of  $t \in \mathbb{R}_+$  and  $\|\mathbf{u}(t)\|_{L^\infty} \in L^4(\mathbb{R}_+)$ , we have

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- Although cubic nonlinearity (quartic  $W$  with  $n = 1$ ) are excluded, analysis of Hayashi & Naumkin (2008,2009) relying on properties of  $e^{-t\langle i\partial_x \rangle}$ , where  $\langle i\partial_x \rangle \equiv \sqrt{1 - \partial_x^2}$ , show scattering to zero with

$$\|\langle i\partial_x \rangle \mathbf{u}(t)\|_{L^\infty} \leq C\epsilon(1+t)^{-1/2}, \quad t \in \mathbb{R}_+,$$

if  $\|\langle x \rangle \langle i\partial_x \rangle^4 \mathbf{u}(0)\|_{L^2} \leq \epsilon$  sufficiently small.

- By Duhamel's principle, we have

$$\mathbf{u}(t) = e^{-it\mathcal{H}}\mathbf{u}(0) + \int_0^t e^{-i(t-s)\mathcal{H}}\nabla W(\mathbf{u}(s))ds,$$

where  $\mathcal{H}$  is the Dirac operator in one dimension.



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- By the preceding Lemma, we have

$$\|\mathbf{u}\|_{L_t^4 L_x^\infty \cap L_t^\infty H_x^1} \leq C\|\mathbf{u}_0\|_{H^1} + C\|\nabla W(\mathbf{u})\|_{L_t^1 H_x^1},$$

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- By the assumption on nonlinearity,

$$\|\nabla W(\mathbf{u})\|_{L_t^1 H_x^1} \leq C\left(\|\mathbf{u}\| + \|\mathbf{u}_x\|\right)\|\mathbf{u}\|^{2n} \leq C\|\mathbf{u}\|_{L_t^\infty H_x^1} \|\mathbf{u}\|_{L_t^{2n} L_x^\infty}^{2n}.$$

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- For a Strichartz admissible pair (when  $n \geq 2$ ),

$$\|\mathbf{u}\|_{L_t^{2n} L_x^\infty} \leq \|\mathbf{u}\|_{L_t^4 L_x^\infty}^{2/n} \|\mathbf{u}\|_{L_t^\infty L_x^\infty}^{1-2/n} \leq C\|\mathbf{u}\|_{L_t^4 L_x^\infty \cap L_t^\infty H_x^1}.$$

The fixed point argument is closed for small  $\mathbf{u}(0) \in H^1(\mathbb{R})$ .

Time-periodic space-localized solutions

$$u(x, t) = U(x)e^{-i\omega t}, \quad v(x, t) = V(x)e^{-i\omega t}$$

satisfy a system of stationary Dirac equations

$$(\mathcal{H} - \omega I)\mathbf{U} + \nabla W(\mathbf{U}) = \mathbf{0}.$$

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- Translations in  $x$  and  $t$  can be added as free parameters.
- Constraint  $\omega \in (-1, 1)$  exists because spectrum of linear waves is located for  $(-\infty, -1] \cup [1, \infty)$ .
- If  $|U|, |V| \rightarrow 0$  as  $|x| \rightarrow \infty$ , then  $U(x) = \bar{V}(x)$  for all  $x \in \mathbb{R}$ .
- Analytical expressions are available for homogeneous polynomials  $W$ ,

$$U(x) = \frac{\sqrt{1-\omega}}{\sqrt{1-\omega} \cosh(\sqrt{1-\omega^2}x) + i\sqrt{1+\omega} \sinh(\sqrt{1-\omega^2}x)}.$$

Given a time-periodic space-localized solution, the stability can be considered in three senses: (a) spectral, (b) orbital, and (c) asymptotic.

**Spectral stability:** We say that the gap soliton is spectrally unstable if the spectral problem for the linearized operator in  $L^2(\mathbb{R})$  has at least one eigenvalue  $\lambda$  with  $\operatorname{Re}\lambda > 0$ . Otherwise, it is (weakly) spectrally stable.

**Orbital stability:** We say that the gap soliton  $e^{-i\omega t}\mathbf{U}$  is orbitally stable if for any  $\epsilon > 0$  there is a  $\delta(\epsilon) > 0$ , such that if  $\|\mathbf{u}(0) - \mathbf{U}\|_{H^1} \leq \delta(\epsilon)$  then

$$\inf_{\theta \in \mathbb{R}} \|\mathbf{u}(t) - e^{-i\theta}\mathbf{U}\|_{H^1} \leq \epsilon,$$

for all  $t > 0$ .

**Asymptotic stability:** We say that the gap soliton is asymptotically stable if it is orbitally stable and for any  $\mathbf{u}(0)$  near  $\mathbf{U}$ , there is  $\mathbf{U}_\infty$  near  $\mathbf{U}$  such that

$$\lim_{t \rightarrow \infty} \inf_{\theta \in \mathbb{R}} \|\mathbf{u}(t) - e^{-i\theta}\mathbf{U}_\infty\|_{L^\infty} = 0.$$

Stability depends on  $W$  and  $\omega$ . Linearization with the decomposition

$$\begin{cases} u(x, t) = e^{-i\omega t} [U(x) + U_1(x)e^{\lambda t}], \\ v(x, t) = e^{-i\omega t} [V(x) + V_1(x)e^{\lambda t}], \end{cases}$$

yields the linear eigenvalue problem

$$\begin{cases} i\lambda \mathbf{U}_1 = (\mathcal{H} - \omega I)\mathbf{U}_1 + V_{11}\mathbf{U}_1 + V_{12}\mathbf{U}_2, \\ -i\lambda \mathbf{U}_2 = (\bar{\mathcal{H}} - \omega I)\mathbf{U}_2 + \bar{V}_{12}\mathbf{U}_1 + \bar{V}_{11}\mathbf{U}_2, \end{cases}$$

where  $\mathbf{U}_{1,2} = [U_{1,2}, V_{1,2}]^T \in \mathbb{C}^2$ , or compactly

$$\lambda \sigma \mathbf{Z} = H_\omega \mathbf{Z},$$

where  $\mathbf{Z} = [\mathbf{U}_1, \mathbf{U}_2]^T \in \mathbb{C}^4$ .

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## Lemma (Chugunova, P, 2006)

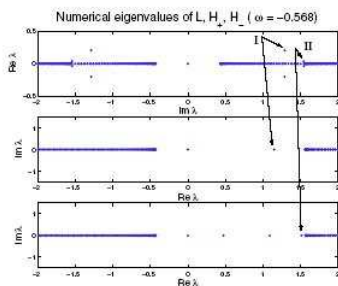
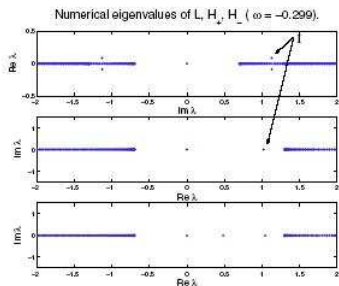
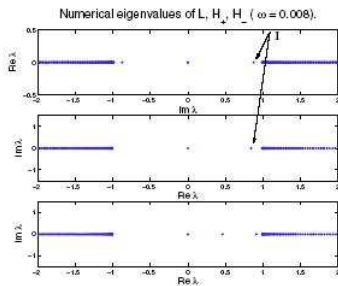
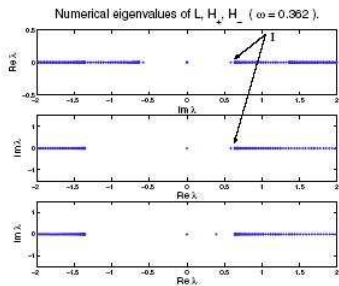
*There exists an orthogonal similarity transformation  $S$  in  $\mathbb{C}^4$  such that*

$$S^{-1} H_\omega S = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad S^{-1} \sigma H_\omega S = \sigma \begin{pmatrix} 0 & H_- \\ H_+ & 0 \end{pmatrix}$$

*where  $H_\pm$  are Dirac operators.*



# An example for cubic nonlinearity



Numerical eigenvalues of  $L, H_+, H_-$  ( $\omega = -0.888$ ).

Numerical eigenvalues of  $L, H_+, H_-$  ( $\omega = -0.986$ ).

- Berkolaiko & Comech (Math. Model. Nat. Phenom. **7**, 13–31 (2012)) showed more examples of instabilities and tried to capture unstable eigenvalues.
- Comech (math-ph/1107.1763) constructed examples of neutrally stable eigenvalues and discussed the instability bifurcations (via famous Vakhitov-Kolokolov/Grillakis-Shatah-Strauss criterion)
- Comech (math-ph/1203.3859) studied bifurcations of eigenvalues from resonances of the nonlinear Schrödinger equations, which is an asymptotic reduction of the nonlinear Dirac equations
- Boussaid & Cuccagna (Comm. PDEs **37**, 1001–1056 (2012)) introduced a concept of Krein signature for eigenvalues of the Dirac operator with sign-indefinite energy and used it for asymptotic stability in three spatial dimensions.

The nonlinear Dirac equations with a potential,

$$\begin{cases} i(u_t + u_x) + v = \partial_{\bar{u}}W(u, v), \\ i(v_t - v_x) + u = \partial_{\bar{v}}W(u, v), \end{cases}$$

where  $W = \beta(x)(|u|^2 + |v|^2) + \gamma(x)(\bar{u}v + u\bar{v}) + W_{\text{nl}}(u, v)$ .

## Assumptions:

- $\beta, \gamma \in L^\infty(\mathbb{R})$  and there is  $C > 0$  and  $\kappa > 0$  such that

$$|\beta(x)| + |\gamma(x)| \leq Ce^{-\kappa|x|}, \quad x \in \mathbb{R}.$$

- $\sigma(\mathcal{H}) \setminus \sigma_c(\mathcal{H}) = \{\omega_0\}$ , where  $\omega_0 \in (-1, 1)$  is a simple eigenvalue of  $\mathcal{H}$  with the  $L^2$ -normalized eigenfunction  $\mathbf{u}_0 \in H^1(\mathbb{R})$ .
- No resonances occur at the end points  $\pm 1$  of  $\sigma_c(\mathcal{H})$  in the sense that no solutions of  $\mathcal{H}\mathbf{u} = \pm\mathbf{u}$  exist in  $L^\infty(\mathbb{R})$ .
- The nonlinearity is homogeneous,

$$\nabla W_{\text{nl}}(a\mathbf{U}) = a^{2n+1} \nabla W_{\text{nl}}(\mathbf{U}), \quad a \in \mathbb{R}.$$

## Lemma

Let Assumptions be satisfied and

$$\langle \mathbf{u}_0, \nabla W_{\text{nl}}(\mathbf{u}_0) \rangle_{L^2} > 0.$$

For sufficiently small  $\epsilon > 0$ , there is a family of solutions  $\mathbf{U} \in H^1(\mathbb{R})$  of the nonlinear Dirac equations for any  $\omega \in (\omega_0, \omega_0 + \epsilon)$  such that the map  $(\omega_0, \omega_0 + \epsilon) \ni \omega \mapsto \mathbf{U} \in H^1(\mathbb{R})$  is defined implicitly by small parameter  $a \in \mathbb{R}$  and by the asymptotic expansion,

$$\begin{aligned} \|\mathbf{U} - a\mathbf{u}_0\|_{H^1} &= \mathcal{O}(a^{2n+1}), \\ |\omega - \omega_0 - a^{2n} \langle \mathbf{u}_0, \mathbf{N}(\mathbf{u}_0) \rangle_{L^2}| &= \mathcal{O}(a^{4n}), \end{aligned}$$

as  $a \rightarrow 0$ .

The proof holds by the Lyapunov–Schmidt decomposition,

$$\mathbf{U} = a\mathbf{u}_0 + \mathbf{V}, \quad a \in \mathbb{R}, \quad \langle \mathbf{u}_0, \mathbf{V} \rangle_{L^2} = 0.$$

Let us consider a local solution near the gap solitons,

$$\begin{cases} u(x, t) = e^{-i\theta(t)} [U(x; \omega(t)) + U_1(x, t)], \\ v(x, t) = e^{-i\theta(t)} [V(x; \omega(t)) + V_1(x, t)]. \end{cases}$$

If  $(\omega, \theta) \in C^1(\mathbb{R}_+, \mathbb{R}^2)$ , then  $\mathbf{U}_1 = [U_1, V_1]^T$  satisfies the time evolution,

$$i \frac{d\mathbf{U}_1}{dt} = (\mathcal{H} - \omega I)\mathbf{U}_1 - i\dot{\omega}\partial_\omega \mathbf{U} - (\dot{\theta} - \omega)(\mathbf{U} + \mathbf{U}_1) + \mathbf{N}(\mathbf{U} + \mathbf{U}_1) - \mathbf{N}(\mathbf{U}),$$

where  $\mathbf{N}(\mathbf{U}) = \nabla W_{\text{nl}}(\mathbf{U})$ .

**Question:** How to ensure that the decomposition is unique and to define evolutions of  $(\omega, \theta)$ ?

Let us consider a local solution near the gap solitons,

$$\begin{cases} u(x, t) = e^{-i\theta(t)} [U(x; \omega(t)) + U_1(x, t)], \\ v(x, t) = e^{-i\theta(t)} [V(x; \omega(t)) + V_1(x, t)]. \end{cases}$$

If  $(\omega, \theta) \in C^1(\mathbb{R}_+, \mathbb{R}^2)$ , then  $\mathbf{U}_1 = [U_1, V_1]^T$  satisfies the time evolution,

$$i \frac{d\mathbf{U}_1}{dt} = (\mathcal{H} - \omega I)\mathbf{U}_1 - i\dot{\omega}\partial_\omega \mathbf{U} - (\dot{\theta} - \omega)(\mathbf{U} + \mathbf{U}_1) + \mathbf{N}(\mathbf{U} + \mathbf{U}_1) - \mathbf{N}(\mathbf{U}),$$

where  $\mathbf{N}(\mathbf{U}) = \nabla W_{\text{nl}}(\mathbf{U})$ .

**Question:** How to ensure that the decomposition is unique and to define evolutions of  $(\omega, \theta)$ ?

**Answer:**  $\mathbf{U}_1$  is required to satisfy the symplectic orthogonality conditions to the two-dimensional generalized null space of the linearized operator.

# Generalized null space of the linearized operator

The linearized operator

$$\lambda\sigma\mathbf{Z} = H_\omega\mathbf{Z},$$

where  $\mathbf{Z} = [\mathbf{U}_1, \bar{\mathbf{U}}_1]^T \in \mathbb{C}^4$ .

The kernel of the linearized operator:

$$\text{Ker}(H_\omega) = \text{span}\{\mathbf{F}\}, \quad \mathbf{F} = i \begin{bmatrix} \mathbf{U} \\ -\bar{\mathbf{U}} \end{bmatrix}, \quad H_\omega\mathbf{F} = \mathbf{0}.$$

The generalized kernel of the linearized operator

$$N_g(L_\omega) = \text{span}\{\mathbf{F}, \mathbf{G}\}, \quad \mathbf{G} = -\partial_\omega \begin{bmatrix} \mathbf{U} \\ \bar{\mathbf{U}} \end{bmatrix}, \quad H_\omega\mathbf{G} = \sigma\mathbf{F}.$$

The symplectic orthogonality conditions are

$$\langle \sigma\mathbf{F}, \mathbf{Z} \rangle_{L^2} = 0, \quad \langle \sigma\mathbf{G}, \mathbf{Z} \rangle_{L^2} = 0,$$

or equivalently

$$\text{Re}\langle \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} = 0, \quad \text{Im}\langle \partial_\omega \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} = 0.$$

Thanks to symplectic orthogonality conditions, we obtain the modulation equations on  $\omega(t)$  and  $\theta(t)$ :

$$\begin{cases} \dot{\omega} \operatorname{Re} \langle \partial_{\omega} \mathbf{U}, \mathbf{U} - \mathbf{U}_1 \rangle_{L^2} + (\dot{\theta} - \omega) \operatorname{Im} \langle \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} = \Omega_1, \\ \dot{\omega} \operatorname{Im} \langle \partial_{\omega}^2 \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} + (\dot{\theta} - \omega) \operatorname{Re} \langle \partial_{\omega} \mathbf{U}, \mathbf{U} + \mathbf{U}_1 \rangle_{L^2} = \Omega_2, \end{cases}$$

where

$$\begin{aligned} \Omega_1 &= \operatorname{Im} \left[ \langle \mathbf{U}, \mathbf{N}(\mathbf{U} + \mathbf{U}_1) - \mathbf{N}(\mathbf{U}) \rangle_{L^2} + \langle \bar{V}_{12} \bar{\mathbf{U}} - V_{11} \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} \right], \\ \Omega_2 &= \operatorname{Re} \left[ \langle \partial_{\omega} \mathbf{U}, \mathbf{N}(\mathbf{U} + \mathbf{U}_1) - \mathbf{N}(\mathbf{U}) \rangle_{L^2} - \langle V_{12} \partial_{\omega} \bar{\mathbf{U}} + V_{11} \partial_{\omega} \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} \right] \end{aligned}$$

Modulation equations determine uniquely the time evolution of  $\mathbf{U}_1$ .



## Theorem (P. & Stefanov, 2012)

Fix  $\epsilon > 0$  and  $\delta > 0$  sufficiently small such that  $\theta(0) = 0$ ,  $\omega(0) \in (\omega_0, \omega_0 + \epsilon)$ , and  $\mathbf{U}_1(0) \in B_\delta(H^1)$ . There exist  $\epsilon_0 > \epsilon$ ,  $\theta_\infty \in \mathbb{R}$ ,  $\omega_\infty \in (\omega_0, \omega_0 + \epsilon_0)$ ,  $(\omega, \theta) \in C^1(\mathbb{R}_+, \mathbb{R}^2)$ , and

$$\mathbf{U}_1(t) \in C(\mathbb{R}_+, H^1) \cap L^4(\mathbb{R}_+, L^\infty)$$

such that  $(\omega, \theta)(t)$  solve the modulation equations,  $\mathbf{U}_1(t)$  solves the time evolution equation, and

$$\lim_{t \rightarrow \infty} \left( \theta(t) - \int_0^t \omega(s) ds \right) = \theta_\infty, \quad \lim_{t \rightarrow \infty} \omega(t) = \omega_\infty, \quad \lim_{t \rightarrow \infty} \|\mathbf{U}_1(t)\|_{L^\infty} = 0.$$

The proof of this theorem brings together Strichartz estimates for nonlinear terms and Mizumachi estimates for quadratic, exponentially decaying terms.

- Global existence in  $H^1(\mathbb{R})$  or even in  $L^2(\mathbb{R})$ :  
Can the proof be extended for  $W$  that depend on  $(\bar{u}v + u\bar{v})$ ?
- Spectral stability of solitary waves:  
Can we use the new ideas of Krein signature to control isolated eigenvalues inducing instabilities?
- Asymptotic stability of solitary waves:  
Can we prove asymptotic stability for the cubic nonlinearity  $n = 1$ ?
- Integrable system: the massive Thirring model

$$\begin{cases} i(u_t + u_x) + v = |v|^2 u, \\ i(v_t - v_x) + u = |u|^2 v. \end{cases}$$

Candy (2011) proved local and global well-posedness in  $L^2(\mathbb{R})$ .

Can we use a Bäcklund transformation to control nonlinear perturbations to solitary waves in  $L^2$ ?

See T. Mizumachi and D. Pelinovsky, "Bäcklund transformation and  $L^2$ -stability of NLS solitons", IMRN **2012**, 2034–2067 (2012)