Nonlinear dynamics in PT-symmetric dNLS lattices

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PT-symmetric quantum mechanics

Consider the evolution problem

$$i\frac{du}{dt} = Hu, \quad u(\cdot, t) \in L^2, \quad t \in \mathbb{R},$$

where H is a linear operator with a domain in L^2. If H is Hermitian, then $\sigma(H) \subset \mathbb{R}$ and e^{-itH} is a unitary group on L^2.

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Let us now assume that H is not Hermitian but PT-symmetric, where

- P stands for parity transformation
- T stands for time reversion and complex conjugation.

In other words, there is an operator $P : L^2 \to L^2$ such that $P^2 = \mathrm{Id}$ and

$$\bar{H} = PHP, \quad \text{or} \quad THT = PHP, \quad \text{or} \quad PTH = HPT,$$

where $Tu(t) = \bar{u}(-t)$ [C.M. Bender, 2007]

Properties of PT-symmetric systems

If $u(t)$ is a solution of the evolution equation, then

$$v(t) = PTu(t) = P\bar{u}(-t)$$

is also a solution of the same system

$$i\frac{dv}{dt} = Hv \;\Rightarrow\; -iP\frac{d\bar{u}}{dt} = HP\bar{u} \;\Rightarrow\; -i\frac{d\bar{u}}{dt} = \bar{H}\bar{u} \;\Rightarrow\; i\frac{du}{dt} = Hu.$$

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If E is an eigenvalue and U is an eigenfunction, then $\bar{E}$ is also an eigenvalue with the eigenfunction $P\bar{U}$, because

$$u(t) = Ue^{-iEt} \quad \Rightarrow \quad v(t) = P\bar{U}e^{-i\bar{E}t}.$$

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Bender's Conjecture: For many physically relevant PT-symmetric operators H, all eigenvalues are real and all eigenfunctions are PT-symmetric.

Examples of PT-symmetric operators

Consider a Schrödinger operator

$$H := -\partial_x^2 + V(x), \quad \text{where} \quad \bar{V}(-x) = V(x).$$

This operator is PT-symmetric w.r.t. space reflection: $Pu(x) := u(-x)$.

- a harmonic oscillator with a linear damping term

$$V(x) = x^2 + i\gamma x = \left(x + \frac{i\gamma}{2}\right)^2 + \frac{\gamma^2}{4}$$

  The spectrum of H is purely discrete and real

$$\sigma(H) = \left\{\frac{\gamma^2}{4} + 1 + 2m, \quad m \in \mathbb{N}_0\right\}.$$

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- an unharmonic oscillator

  $$V(x) = x^2(-ix)^\gamma.$$

  The spectrum of H is purely discrete and real for $\gamma > 0$ (Bender C. M.; Boettcher S.; PRL **80** (1998) 5243).

PT-dNLS equation: main results

We consider the PT-symmetric discrete nonlinear Schrödinger equation

$$i\frac{du_n}{dt} = u_{n+1} + u_{n-1} + V_n u_n + |u_n|^2 u_n, \quad n \in S \subset \mathbb{Z},$$

where $\bar{V}_{-n} = V_n$.

1. If $S = \mathbb{Z}$ and V_n is spatially extended: spectrum of the linear operators is not real (P., Kevrekidis, Franzeskakis, EPL **101** (2013), 11002).
2. If $S = \{1, 2, ..., 2N\}$: classification of stationary states in the "anti-continuum" limit (P., Konotop, Zezyulin, JPA **47** (2014), 085204).
3. If $S = \{1, 2, ..., 2N\}$ and $V_n = i\gamma(-1)^n$: nonlinear dynamics of oscillators (Kevrekidis, P., Tyugin, JPA **46** (2013), 365201).
4. If $S = \mathbb{Z}$ and V_n is compactly supported: discrete solitons exist (Kevrekidis, P., Tyugin, SIAD **12** (2013), 1210).

1. Spectrum of the linear DNLS equation

Consider the spatially extended PT-symmetric potential,

$$Eu_n = -(u_{n+1} + u_{n-1}) + \left(n^2 + i\gamma n\right) u_n, \quad n \in \mathbb{Z}.$$

The spectrum is purely discrete for any $\gamma \in \mathbb{R}$ because

$$\mathrm{Re}(V_n) = n^2 \to \infty \quad \text{as} \quad n \to \infty.$$

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Using the discrete Fourier transform:

$$u_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{u}(k) e^{-ikn} dk,$$

we convert the spectral problem to the differential equation

$$\frac{d^2 \hat{u}}{dk^2} + \gamma \frac{d\hat{u}}{dk} + [E + 2\cos(k)]\, \hat{u}(k) = 0,$$

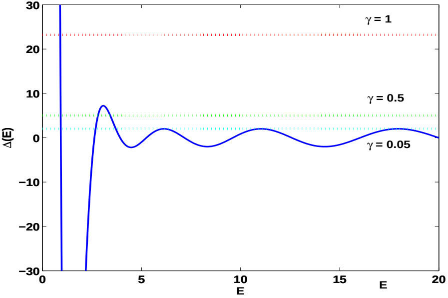
subject to the 2π-periodicity of $\hat{u}(k)$.

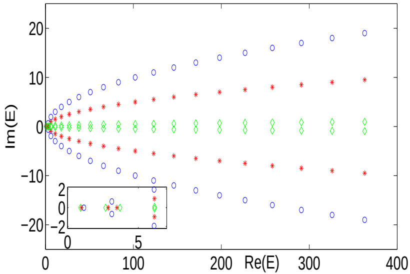
Complex spectrum

If $\hat{v}(k) = \hat{u}(k) e^{\gamma k/2}$, then $\hat{v}(k)$ satisfies the Mathieu equation:

$$\frac{d^2 \hat{v}}{dk^2} + \left[E - \frac{\gamma^2}{4} + 2\cos(k) \right] \hat{v} = 0,$$

subject to the condition $\hat{v}(k+2\pi) = e^{\pi\gamma} \hat{v}(k)$. Hence we look for the Floquet multiplier $\mu_* = e^{\pi\gamma}$ of the monodromy matrix.





Spectrum of another linear DNLS equation

Consider the spatially extended potential without real part,

$$Eu_n = -(u_{n+1} + u_{n-1}) + i\gamma n u_n, \quad n \in \mathbb{Z}.$$

- If $\gamma = 0$, the spectrum is purely continuous.
- If γ is large enough, the spectrum is purely discrete.

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The equivalent differential equation

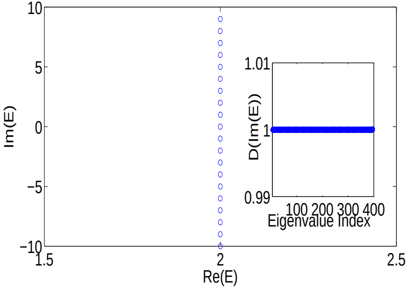
$$\gamma \frac{d\hat{u}}{dk} + [E + 2\cos(k)]\, \hat{u} = 0,$$

has the exact solution

$$\hat{u}(k) = \hat{u}(0) e^{-\gamma^{-1}(Ek + 2\sin(k))}.$$

The 2π-periodicity of the discrete Fourier transform $\hat{u}(k)$ gives now the eigenvalues $E = i\gamma m$, where m is an arbitrary integer.

Numerically obtained spectrum



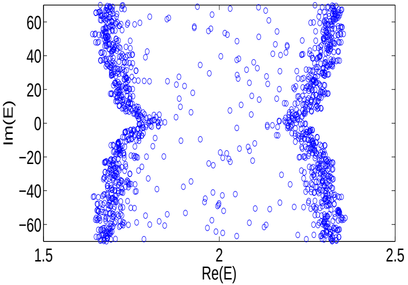


Figure : Eigenvalues for $\gamma = 1$ (left) and $\gamma = 0.1$ (right).

Conjecture: There exists $\gamma_0 \in (0, \infty)$ such that the spectrum is purely discrete for $\gamma > \gamma_0$. It is a union of the set of simple eigenvalues embedded into a vertical strip of the continuous spectrum for $\gamma \in (0, \gamma_0)$.

2. Stationary states for PT-symmetric dNLS

Consider the stationary PT-symmetric DNLS equation

$$Ew_n = w_{n+1} + w_{n-1} + i\gamma(-1)^n w_n + |w_n|^2 w_n, \quad n \in S_N := \{1, 2, ..., 2N\},$$

subject to the Dirichlet end-point conditions $w_0 = w_{2N+1} = 0$.

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Eigenvalues of the spectral problem are found explicitly:

$$\gamma^2 + E^2 = 4\cos^2\left(\frac{\pi j}{1+2N}\right), \quad 1 \leq j \leq N.$$

In particular, all eigenvalues are simple and real for $\gamma \in (-\gamma_N, \gamma_N)$, where

$$\gamma_N := 2\cos\left(\frac{\pi N}{1+2N}\right).$$

Spectrum of the linear DNLS equation

Substitution $x_k = w_{2k-1}$, $y_k = w_{2k}$ takes the spectral problem to the form

$$\left\{ \begin{array}{l} Ex_k = y_{k-1} + y_k - i\gamma x_k, \\ Ey_k = x_k + x_{k+1} + i\gamma y_k, \end{array} \right. \quad \Rightarrow \quad (\gamma^2 + E^2) x_k = x_{k-1} + 2x_k + x_{k+1},$$

subject to the boundary condition $x_0 = -x_1$ ($y_0 = 0$) and $x_{N+1} = 0$. Then, the spectrum is found explicitly by using the discrete Fourier transform.

In particular,

$$\gamma_1 = 1, \quad \gamma_2 \approx 0.618, \quad \gamma_3 \approx 0.445,$$

and $\gamma_N \to 0$ as $N \to \infty$. The infinite PT-symmetric DNLS lattice is unstable for any $\gamma \neq 0$.

Local bifurcation from a simple real eigenvalue E_0

Theorem

Let $(E_0, \mathbf{w}_0)$ be the eigenvalue-eigenvector pair for linear Schrödinger operator. There exists a unique (up to a gauge transformation) PT-symmetric solution $\mathbf{w} = P\bar{\mathbf{w}}$ of the stationary PT-dNLS equation for real $E > E_0$. Moreover, the solution branch is parameterized by a small real parameter a such that the map $\mathbb{R} \ni a \to (E, \mathbf{w}) \in \mathbb{R} \times \mathbb{C}^{2N}$ is C^∞ and for sufficiently small real a, there is a positive constant C such that

$$\|\mathbf{w}\|^2 + |E - E_0| \leq Ca^2.$$

The proof is achieved by Lyapunov–Schmidt reductions

$$E = E_0 + \Delta, \quad \mathbf{w} = a\mathbf{w}_0 + \mathbf{u}, \quad \langle P\mathbf{w}_0, \mathbf{u} \rangle = 0,$$

and the symmetry constraints that yield real positive Δ and PT-symmetric $\mathbf{u}$.

Bifurcation from infinity for large E

Theorem

For any $\gamma \in (-1,1)$, the stationary PT-dNLS equation in the limit of large real E admits 2^N PT-symmetric solutions $\mathbf{w} = P\bar{\mathbf{w}}$ (unique up to a gauge transformation) such that, for sufficiently large real E, the map $E \to \mathbf{w}$ is C^∞ at each solution and there is an E-independent constant C such that

$$\left| \sum_{n \in S_N} |w_n|^2 - 2NE \right| \leq C.$$

The difficulty in the proof arises due to the fact that, although the algebraic equations decouple as $E \to \infty$ with the N independent solutions

$$\mathbf{W}_k = e^{-i\theta_k}\mathbf{e}_k + e^{i\theta_k}\mathbf{e}_{2N+1-k}, \quad 1 \leq k \leq N,$$

the nonlinear system does not enjoy the linear superposition principle.

Bifurcation from infinity for large E

Setting $E = \frac{1}{\delta}$ and $\mathbf{w} = \frac{\mathbf{W}}{\sqrt{\delta}}$, we write the stationary DNLS equation:

$$(1 - |W_n|^2) W_n = \delta \left[W_{n+1} + W_{n-1} + i\gamma(-1)^n W_n \right], \quad n \in S_N,$$

subject to the boundary condition $W_0 = 0$ and $W_{2N+1} = 0$ and the PT-symmetry condition $W_n = \bar{W}_{2N+1-n}$.

For $N = 1$, we have

$$(1 - |W_1|^2) W_1 = \delta \left[\bar{W}_1 - i\gamma W_1 \right].$$

Setting $W_1 = A_1^{1/2} e^{i\varphi_1}$, we obtain

$$A_1 = 1 - \delta \cos(2\varphi_1), \quad -\sin(2\varphi_1) - \gamma = 0,$$

which yields two branches by the two solutions of $\sin(2\varphi_1) = -\gamma$.

Bifurcation from infinity for large E

For $N \geq 2$, the degeneracy is unfolded by using the transformation

$$\left\{ \begin{array}{l} W_1 = A_1^{1/2} e^{i\varphi_1}, \\ W_2 = (A_1 A_2)^{1/2} e^{i\varphi_1 + i\varphi_2}, \\ \vdots \\ W_N = (A_1 A_2 \cdots A_N)^{1/2} e^{i\varphi_1 + i\varphi_2 + \cdots + i\varphi_N}, \end{array} \right.$$

which yields two sets of equations

$$\left\{ \begin{array}{l} A_2^{1/2} \sin(\varphi_2) - \gamma = 0, \\ A_3^{1/2} \sin(\varphi_3) - A_2^{-1/2} \sin(\varphi_2) + \gamma = 0, \\ \vdots \\ -\sin 2(\varphi_1 + \varphi_2 + \cdots + \varphi_N) - A_N^{-1/2} \sin(\varphi_N) + (-1)^N \gamma = 0, \end{array} \right.$$

and

$$
\begin{cases}
1 - A_1 = \delta A_2^{1/2} \cos(\varphi_2), \\
1 - A_1 A_2 = \delta (A_3^{1/2} \cos(\varphi_3) + A_2^{-1/2} \cos(\varphi_2)), \\
\vdots \\
1 - A_1 A_2 \cdots A_N = \delta (\cos 2(\varphi_1 + \varphi_2 + \cdots + \varphi_N) + A_N^{-1/2} \cos(\varphi_N)).
\end{cases}
$$

2^N solutions are obtained as $\delta \to 0$ by the Implicit Function Theorem:

- For all $(\varphi_1, \varphi_2, \ldots, \varphi_N) \in \mathbb{T}^N$ and small $\delta \in \mathbb{R}$, there is a unique solution for $(A_1, A_2, \ldots, A_N)$ and

$$
|A_1 - 1| + |A_2 - 1| + \cdots + |A_N - 1| \leq C|\delta|.
$$

- For $\delta = 0$ and $\gamma \in (-1, 1)$, there are 2^N solutions of the system

$$
\begin{cases}
\sin(\varphi_2) = \gamma, \\
\sin(\varphi_3) = -\gamma + \sin(\varphi_2) \equiv 0, \\
\vdots \\
\sin 2(\varphi_1 + \varphi_2 + \cdots + \varphi_N) = (-1)^N \gamma - \sin(\varphi_N),
\end{cases}
$$

and the Jacobian matrix is invertible.

Numerical illustration: $N = 2$

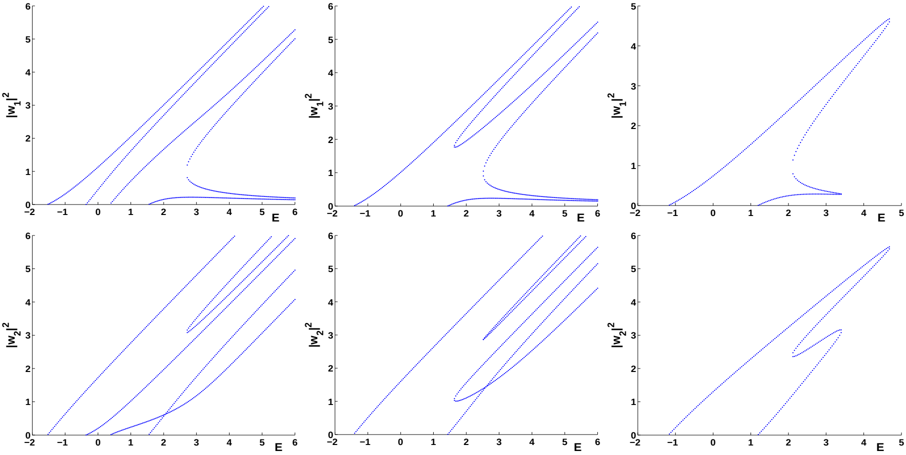

Figure : Nonlinear stationary states for $\gamma = 0.5$ (left), $\gamma = 0.75$ (middle), and $\gamma = 1.1$ (right). The top and bottom rows show components $|w_1|^2$ and $|w_2|^2$.

Numerical illustration: $N = 3$

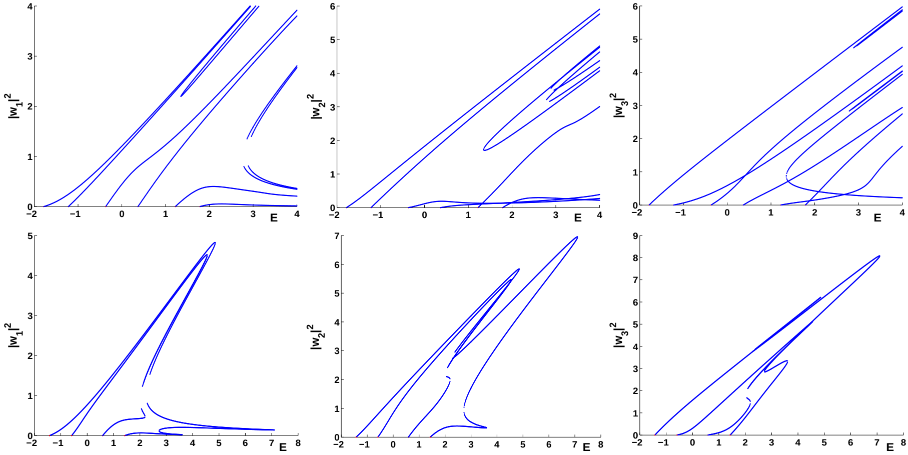

Figure : Nonlinear stationary states for $\gamma = 0.25$ (top) and $\gamma = 1.1$ (bottom). The left, middle, and right columns show components $|w_1|^2$, $|w_2|^2$, and $|w_3|^2$.

3. Spectral stability in PT-symmetric dNLS

Substitute $\mathbf{u}(t) = e^{-iEt}(\mathbf{w} + \mathbf{U}(t))$ to the PT-dNLS equation with $w_n = \bar{w}_{2N+1-n}$ and obtain the linearized time-evolution problem

$$i\frac{dU_n}{dt} + EU_n = U_{n+1} + U_{n-1} + i(-1)^n\gamma U_n + 2|w_n|^2 U_n + w_n^2 \bar{U}_n.$$

Then, use

$$\mathbf{U}(t) = \phi e^{-\lambda t} \quad \text{and} \quad \bar{\mathbf{U}}(t) = \psi e^{-\lambda t},$$

to obtain the spectral problem with the eigenvalue parameter λ

$$\left\{ \begin{array}{l} (E - i\lambda)\phi_n = \phi_{n+1} + \phi_{n-1} + i(-1)^n\gamma\phi_n + 2|w_n|^2\phi_n + w_n^2\psi_n, \\ (E + i\lambda)\psi_n = \psi_{n+1} + \psi_{n-1} - i(-1)^n\gamma\psi_n + \bar{w}_n^2\phi_n + 2|w_n|^2\psi_n. \end{array} \right.$$

Unless λ is real, ϕ_n and ψ_n are not complex-conjugate to each other.

Stability Theorem

Theorem

Consider 2^N stationary solutions in the limit $E \to \infty$. There exists exactly one spectrally stable stationary mode among the 2^N solutions for sufficiently large E.

Using the rescaling $E = 1/\delta$, $\mathbf{w} = \mathbf{W}/\delta^{1/2}$, and $\lambda = \Lambda/\delta$, we obtain

$$\begin{cases} (1-2|W_n|^2)\phi_n - W_n^2\psi_n = i\Lambda\phi_n + \delta\left(\phi_{n+1} + \phi_{n-1} + i(-1)^n\gamma\phi_n\right), \\ -\bar{W}_n^2\phi_n + (1-2|W_n|^2)\psi_n = -i\Lambda\psi_n + \delta\left(\psi_{n+1} + \psi_{n-1} - i(-1)^n\gamma\psi_n\right). \end{cases}$$

where we recall $W_n = e^{i\theta_n}(1 + \mathcal{O}(\delta))$.

For $\delta = 0$, there exists only one eigenvalue $\Lambda = 0$ of algebraic multiplicity $4N$. When $\delta \neq 0$, the multiple zero eigenvalue splits into the complex plane.

Reduction of the stability problem

For $\delta = 0$, the spectral problem has $2N$ eigenvectors

$$\begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix} = i c_n \begin{bmatrix} e^{i\theta_n} \\ -e^{-i\theta_n} \end{bmatrix}, \quad 1 \leq n \leq 2N,$$

and $2N$ generalized eigenvectors

$$\begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix} = \frac{1}{2} c_n \begin{bmatrix} e^{i\theta_n} \\ e^{-i\theta_n} \end{bmatrix}, \quad 1 \leq n \leq 2N.$$

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Using the perturbative expansion $\Lambda = \mu\delta^{1/2} + \mathcal{O}(\delta^{3/2})$ and

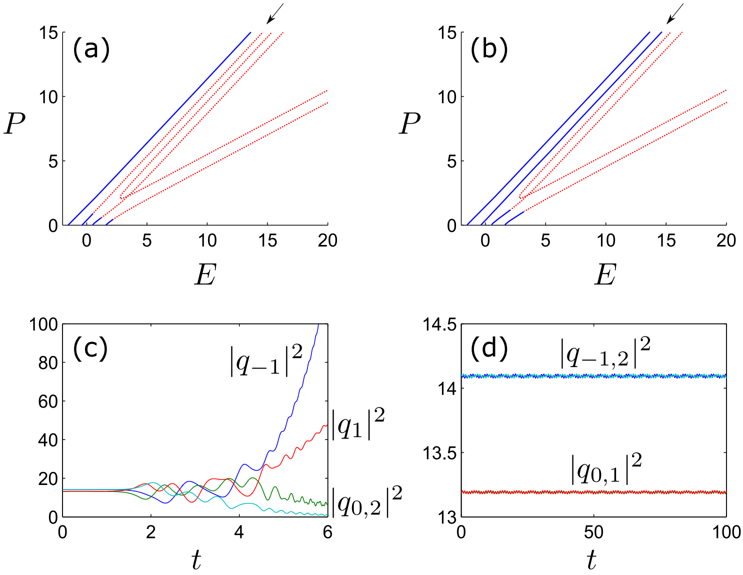
$$\boldsymbol{\phi} = \boldsymbol{\phi}^{(0)} + \mu\delta^{1/2}\boldsymbol{\phi}^{(1)} + \delta\boldsymbol{\phi}^{(2)} + \mathcal{O}(\delta^{3/2})$$

we obtain the reduced eigenvalue problem

$$\mu^2 c_n = 2\cos(\theta_{n+1} - \theta_n)(c_n - c_{n+1}) + 2\cos(\theta_{n-1} - \theta_n)(c_n - c_{n-1}).$$

The only spectrally stable stationary solution has the out-of-phase configuration for all phase differences in the sequence $\theta_{n+1} - \theta_n$.

Numerical illustration: $N = 2$



4. Nonlinear dynamics of a dimer: $N = 1$

Consider a PT-symmetric dimer with two complex amplitude:

$$\begin{cases} i\frac{da}{dt} = b - i\gamma a + |a|^2 a, \\ i\frac{db}{dt} = a + i\gamma b + |b|^2 b. \end{cases}$$

For $\gamma \in (-1, 1)$, the zero equilibrium is stable.

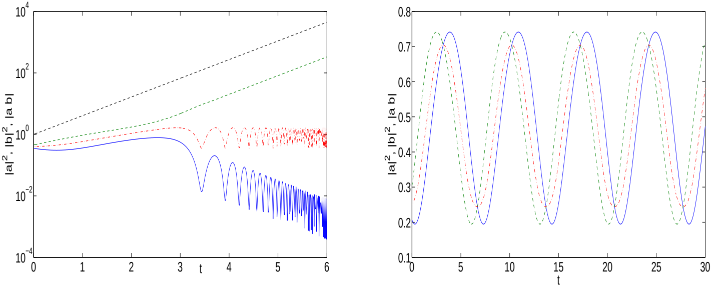


Figure : Left: unbounded growth of the amplitude $|b(t)|$ for the gained oscillator. Right: bounded oscillations of the amplitudes.

Main results on the dimer

1. Solutions of the PT-dNLS equation do not blow up in a finite time.
2. If $\gamma \in (-1, 1)$ (when the zero equilibrium state is neutrally stable), all trajectories starting from small initial data remain bounded for all times.
3. If $\gamma \in (-1, 1)$, there exist trajectories starting from sufficiently large initial data which grow exponentially fast for larger times.

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Remark: These results have been generalized for any finite PT-dNLS network with $N < \infty$ for $\gamma \in (-\gamma_N, \gamma_N)$.

Nonlinear dynamics of a quadrimer: $N = 2$

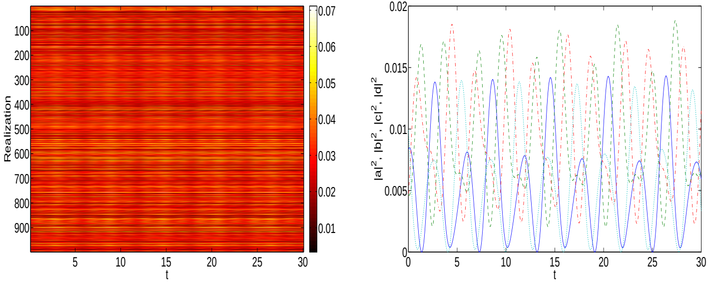

Figure : Dynamics of the quadrimer with small initial data. For the contour plot evolution of the squared l^2 norm, it is clear that all orbits remain bounded (left panel). A typical example of the resulting bounded orbit is shown in the right panel with the blue solid and red dash-dotted lines denoting the gain sites, while the green dashed and cyan dotted lines correspond to the lossy ones.

Nonlinear dynamics of a quadrimer: $N = 2$

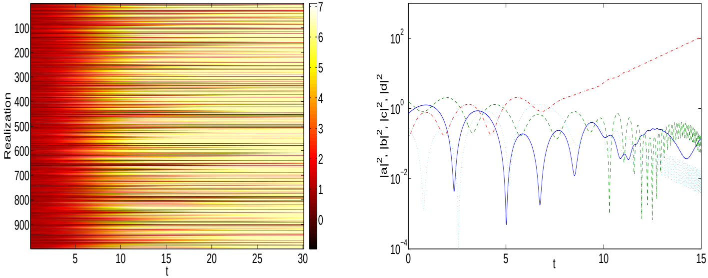

Figure : Dynamics of the quadrimer with large initial data. From the saturation of the left panel's logarithmic scale, it is clear that most trajectories lead to indefinite growth. The right panel illustrates the unbounded dynamics, when one of the gain sites grows.

Open problems

- General theorems on the (unstable) spectrum of the linear PT-dNLS equation with spatially extended potentials.
- Nonlinear stability of spectrally stable stationary states.
- Sharp conditions on the initial data to distinguish bounded oscillations and exponentially growing trajectories.
- Extensions to multi-dimensional setting of the PT-dNLS equation.