Stationary states and nonlinear dynamics in PT-symmetric dNLS lattices

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PT-symmetric quantum mechanics

Consider the evolution problem

$$i\frac{du}{dt} = Hu, \quad u(\cdot, t) \in L^2, \quad t \in \mathbb{R},$$

where H is a linear operator with a domain in L^2. If H is Hermitian, then $\sigma(H) \subset \mathbb{R}$ and e^{-itH} is a unitary group on L^2.

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Let us now assume that H is not Hermitian but PT-symmetric, where

- P stands for parity transformation
- T stands for time reversion and complex conjugation.

In other words, there is an operator $P : L^2 \to L^2$ such that $P^2 = \mathrm{Id}$ and

$$\bar{H} = PHP, \quad \text{or} \quad THT = PHP, \quad \text{or} \quad PTH = HPT,$$

where $Tu(t) = \bar{u}(-t)$ [C.M. Bender, 2007]

Properties of PT-symmetric systems

If $u(t)$ is a solution of the evolution equation, then

$$v(t) = PTu(t) = P\bar{u}(-t)$$

is also a solution of the same system

$$i\frac{dv}{dt} = Hv \;\Rightarrow\; -iP\frac{d\bar{u}}{dt} = HP\bar{u} \;\Rightarrow\; -i\frac{d\bar{u}}{dt} = \bar{H}\bar{u} \;\Rightarrow\; i\frac{du}{dt} = Hu.$$

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If E is an eigenvalue and U is an eigenfunction, then $\bar{E}$ is also an eigenvalue with the eigenfunction $P\bar{U}$, because

$$u(t) = Ue^{-iEt} \quad \Rightarrow \quad v(t) = P\bar{U}e^{-i\bar{E}t}.$$

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Bender's Conjecture: For many physically relevant PT-symmetric operators H, all eigenvalues are real and all eigenfunctions are PT-symmetric.

Examples of PT-symmetric operators

Consider a Schrödinger operator

$$H := -\partial_x^2 + V(x), \quad \text{where} \quad \bar{V}(-x) = V(x).$$

This operator is PT-symmetric w.r.t. space reflection: $Pu(x) := u(-x)$.

- a harmonic oscillator with a linear damping term

$$V(x) = x^2 + i\gamma x = \left(x + \frac{i\gamma}{2}\right)^2 + \frac{\gamma^2}{4}$$

The spectrum of H is purely discrete and real

$$\sigma(H) = \left\{ \frac{\gamma^2}{4} + 1 + 2m, \quad m \in \mathbb{N}_0 \right\}.$$

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- an unharmonic oscillator

$$V(x) = x^2(-ix)^\gamma.$$

  The spectrum of H is purely discrete and real for $\gamma > 0$ (Bender C. M.; Boettcher S.; PRL **80** (1998) 5243).

Goals of our studies

We consider the PT-symmetric discrete nonlinear Schrödinger equation

$$i\frac{du_n}{dt} = u_{n+1} - 2u_n + u_{n-1} + V_n u_n + |u_n|^2 u_n, \quad n \in S \subset \mathbb{Z},$$

where $\bar{V}_{-n} = V_n$.

- If $S = \mathbb{Z}$ and V_n is spatially extended, we show that the spectrum of the linear Schrödinger operators is not real (P., Kevrekidis, Franzeskakis, EPL **101** (2013), 11002).
- If $S = \mathbb{Z}$ and V_n is compactly supported, we prove existence of localized states (Kevrekidis, P., Tyugin, SIAD (2013), accepted).
- If $S = \{1, 2, ..., 2N\}$ and $V_n = i\gamma(-1)^n$, we study nonlinear dynamics of PT-symmetric oscillators (Kevrekidis, P., Tyugin, JPA (2013), submitted).

Spectrum of the linear DNLS equation

Consider the spatially extended PT-symmetric potential,

$$Eu_n = -\left(u_{n+1} + u_{n-1} - 2u_n\right) + \left(n^2 + i\gamma n\right) u_n, \quad n \in \mathbb{Z}.$$

The spectrum is purely discrete for any $\gamma \in \mathbb{R}$ because

$$\mathrm{Re}(V_n) = n^2 \to \infty \quad \text{as} \quad n \to \infty.$$

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Using the discrete Fourier transform:

$$u_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{u}(k) e^{-ikn} dk,$$

we convert the spectral problem to the differential equation

$$\frac{d^2\hat{u}}{dk^2} + \gamma \frac{d\hat{u}}{dk} + [E - 2 + 2\cos(k)]\, \hat{u}(k) = 0,$$

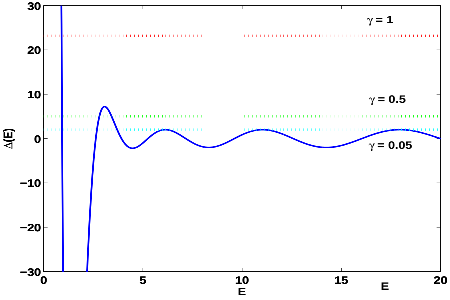
subject to the 2π-periodicity of $\hat{u}(k)$.

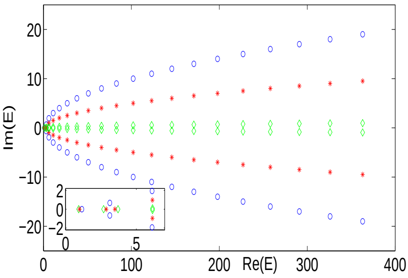
Complex spectrum

If $\hat{v}(k) = \hat{u}(k)e^{\gamma k/2}$, then $\hat{v}(k)$ satisfies the Mathieu equation:

$$\frac{d^2\hat{v}}{dk^2} + \left[E - 2 - \frac{\gamma^2}{4} + 2\cos(k)\right]\hat{v} = 0,$$

subject to the condition $\hat{v}(k+2\pi) = e^{\pi\gamma}\hat{v}(k)$. We are looking for the Floquet multiplier $\mu_* = e^{\pi\gamma}$ of the monodromy matrix associated with the Mathieu equation.





Spectrum of another linear DNLS equation

Consider the spatially extended potential without real part,

$$Eu_n = -\left(u_{n+1} + u_{n-1} - 2u_n\right) + i\gamma n u_n, \quad n \in \mathbb{Z}.$$

- If $\gamma = 0$, the spectrum is purely continuous.
- If γ is large enough, the spectrum is purely discrete.

Spectrum of another linear DNLS equation

Consider the spatially extended potential without real part,

$$Eu_n = -(u_{n+1} + u_{n-1} - 2u_n) + i\gamma n u_n, \quad n \in \mathbb{Z}.$$

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The equivalent differential equation

$$\gamma \frac{d\hat{u}}{dk} + [E - 2 + 2\cos(k)]\,\hat{u} = 0,$$

has the exact solution

$$\hat{u}(k) = \hat{u}(0) e^{\gamma^{-1}[(2-E)k - 2\sin(k)]}. \quad (1)$$

The 2π-periodicity of the discrete Fourier transform $\hat{u}(k)$ gives now the eigenvalues $E = 2 + i\gamma m$, where m is an arbitrary integer.

Numerically obtained spectrum

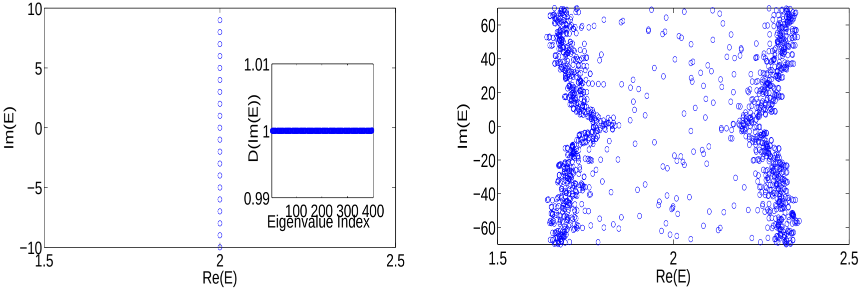

Figure : Eigenvalues for $\gamma = 1$ (left) and $\gamma = 0.1$ (right).

Conjecture:

There exists $\gamma_0 \in (0, \infty)$ such that the spectrum is purely discrete for $\gamma > \gamma_0$. It is a union of the set of simple eigenvalues embedded into a vertical strip of the continuous spectrum for $\gamma \in (0, \gamma_0)$.

Stationary states for PT-symmetric dNLS

Consider the stationary PT-symmetric DNLS equation

$$Ew_n = w_{n+1} + w_{n-1} + i\gamma(-1)^n w_n + |w_n|^2 w_n, \quad n \in S_N := \{1, 2, ..., 2N\},$$

subject to the Dirichlet end-point conditions $w_0 = w_{2N+1} = 0$.

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Eigenvalues of the spectral problem are found explicitly:

$$\gamma^2 + E^2 = 4\cos^2\left(\frac{\pi j}{1+2N}\right), \quad 1 \leq j \leq N.$$

In particular, all eigenvalues are simple and real for $\gamma \in (-\gamma_N, \gamma_N)$, where

$$\gamma_N := 2\cos\left(\frac{\pi N}{1+2N}\right).$$

Local bifurcation from a simple real eigenvalue E_0

Theorem

There exists a unique (up to a gauge transformation) PT-symmetric solution $\mathbf{w} = P\bar{\mathbf{w}}$ *of the stationary PT-dNLS equation for real* $E > E_0$*. Moreover, the solution branch is parameterized by a small real parameter* a *such that the map* $\mathbb{R} \ni a \to (E, \mathbf{w}) \in \mathbb{R} \times \mathbb{C}^{2N}$ *is* C^∞ *and for sufficiently small real* a*, there is a positive constant* C *such that*

$$\|\mathbf{w}\|^2 + |E - E_0| \leq Ca^2.$$

The proof is achieved by Lyapunov–Schmidt reductions

$$E = E_0 + \Delta, \quad \mathbf{w} = a\mathbf{w}_0 + \mathbf{u}, \quad \langle P\mathbf{w}_0, \mathbf{u} \rangle = 0,$$

and the symmetry constraints that yield real Δ and PT-symmetric $\mathbf{u}$.

Bifurcation from infinity for large E

Theorem

For any $\gamma \in (-1,1)$, the stationary PT-dNLS equation in the limit of large real E admits 2^N PT-symmetric solutions $\mathbf{w} = P\bar{\mathbf{w}}$ (unique up to a gauge transformation) such that, for sufficiently large real E, the map $E \to \mathbf{w}$ is C^∞ at each solution and there is an E-independent constant C such that

$$\left| \sum_{n \in S_N} |w_n|^2 - 2NE \right| \leq C.$$

The difficulty in the proof arises due to the fact that, although the algebraic equations decouple as $E \to \infty$ with the N independent solutions

$$\mathbf{W}_k = e^{-i\varphi_k}\mathbf{e}_k + e^{i\varphi_k}\mathbf{e}_{2N+1-k}, \quad 1 \leq k \leq N,$$

where $\varphi_k \in \mathbb{R}$ is arbitrary, the nonlinear system does not enjoy the superposition principle.

Bifurcation from infinity for large E

Setting $E = \frac{1}{\delta}$ and $\mathbf{w} = \frac{\mathbf{W}}{\sqrt{\delta}}$, we write the stationary DNLS equation:

$$(1 - |W_n|^2) W_n = \delta \left[W_{n+1} + W_{n-1} + i\gamma(-1)^n W_n \right], \quad n \in S_N,$$

subject to the boundary condition $W_0 = 0$ and $W_{2N} = 0$.

For $N = 1$, we have

$$(1 - |W_1|^2) W_1 = \delta \left[\bar{W}_1 - i\gamma W_1 \right].$$

Setting $W_1 = A_1^{1/2} e^{i\varphi_1}$, we obtain

$$A_1 = 1 - \delta \cos(2\varphi_1), \quad -\sin(2\varphi_1) - \gamma = 0,$$

which yields two branches by the two solutions of $\sin(2\varphi_1) = -\gamma$.

Numerical illustration: $N = 2$

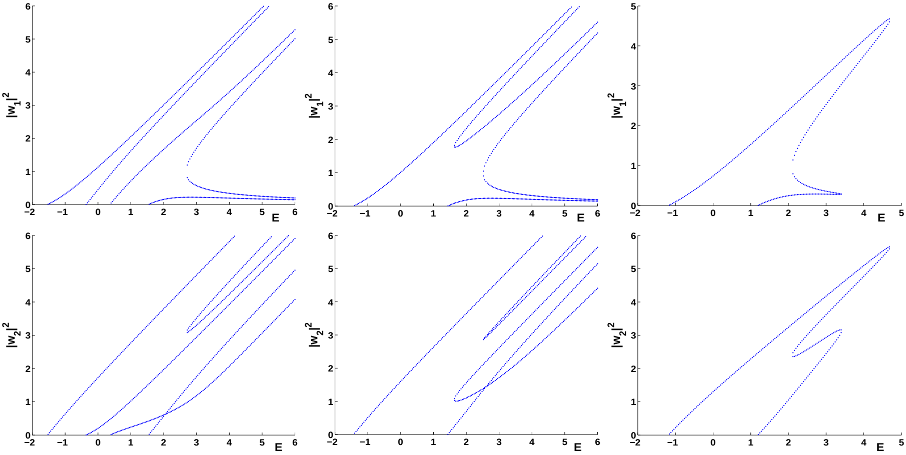

Figure : Nonlinear stationary states for $\gamma = 0.5$ (left), $\gamma = 0.75$ (middle), and $\gamma = 1.1$ (right). The top and bottom rows show components $|w_1|^2$ and $|w_2|^2$.

Numerical illustration: $N = 3$

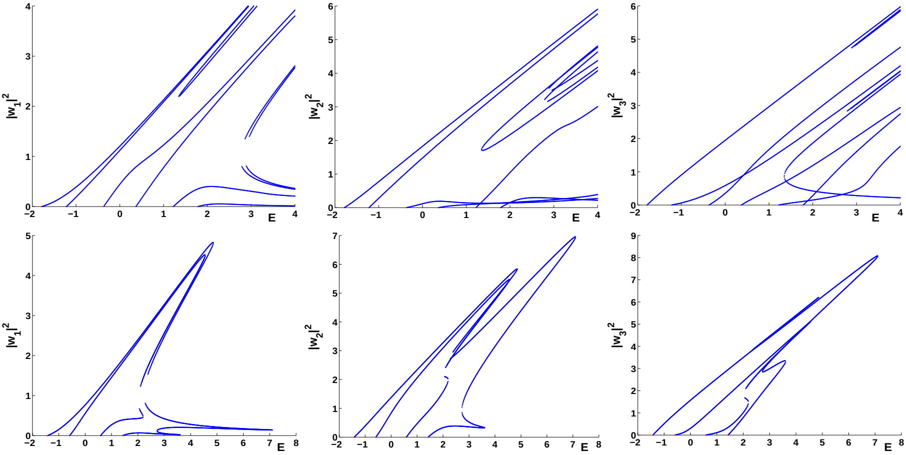

Figure : Nonlinear stationary states for $\gamma = 0.25$ (top) and $\gamma = 1.1$ (bottom). The left, middle, and right columns show components $|w_1|^2$, $|w_2|^2$, and $|w_3|^2$.

Discrete solitons for PT-symmetric dNLS

Consider the stationary PT-symmetric DNLS equation

$$Ew_n = w_{n+1} + w_{n-1} + i\gamma(-1)^n \chi_{n\in S_N} w_n + |w_n|^2 w_n, \quad n \in \mathbb{Z},$$

where $\chi_{n\in S_N}$ is a characteristic function for the set $S_N := \{1, 2, ..., 2N\}$ (PT-symmetric defects).

Theorem

For any $\gamma \in (-1, 1)$, the nonlinear stationary PT-dNLS equation admits 2^N PT-symmetric solutions $\mathbf{W} = P\bar{\mathbf{W}} \in l^2(\mathbb{Z})$ (unique up to a gauge transformation) such that, for sufficiently small positive δ, the map $\delta \to \mathbf{W}$ is C^∞ at each solution and there is a positive δ-independent constant C:

$$|\mathbf{W} - \mathbf{W}_0| \leq C\delta,$$

where $\mathbf{W}_0$ is the PT-symmetric state on the set S_N.

Numerical illustrations of discrete solitons for $N = 1$

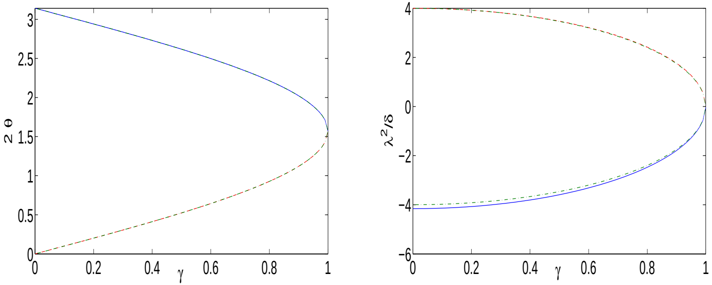

Figure : The left panel shows the relative phase 2θ between the two central sites obtained by the numerical computation and the solvability condition $\sin(2\theta) = \gamma$ (a green dash-dotted line). The right panel shows the squared eigenvalue of the linearized PT-dNLS equation at the discrete soliton versus γ for each branch.

Numerical instabilities of in-phase discrete solitons

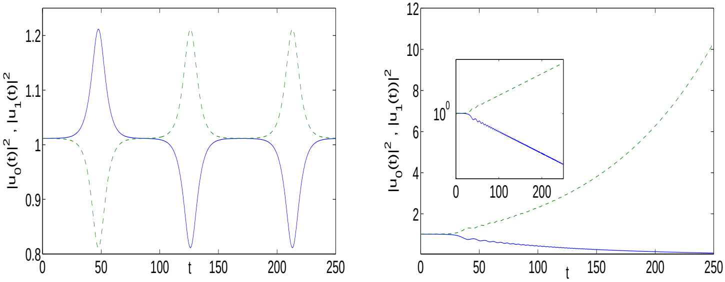

Figure : The evolution of the unstable in-phase solution for $\gamma = 0$ (left) and $\gamma = 0.5$ (right).

Nonlinear dynamics of a dimer

Consider a PT-symmetric dimer ($N = 1$) with two complex amplitude:

$$\left\{ \begin{array}{l} i\frac{da}{dt} = b - i\gamma a + |a|^2 a, \\ i\frac{db}{dt} = a + i\gamma b + |b|^2 b. \end{array} \right.$$

For $\gamma \in (0,1)$, the zero equilibrium is stable.

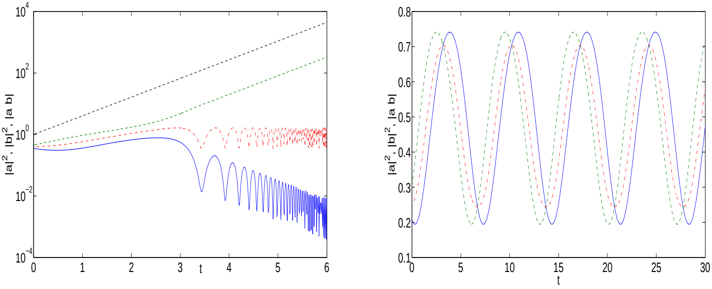

Figure : Left: unbounded growth of the amplitude $|b(t)|$ for the gained oscillator. Right: bounded oscillations of the amplitudes.

Main claims of new work

For the nonlinear dynamics of a finite PT-dNLS chain, we prove the following results:

- Solutions of the PT-dNLS equation do not blow up in a finite time.
- For values of the gain and loss coefficient γ when the zero equilibrium state is neutrally stable, the solutions of the finite PT-dNLS equation starting with small initial data remain bounded for all times.
- For the same values of γ, there exist time evolutions of the finite PT-dNLS equation starting with sufficiently large initial data which grow exponentially fast for larger times.

Discussions: next goals

- General theorems that linear PT-dNLS equation with spatially extended potentials has unstable spectrum for any $\gamma \neq 0$.
- Nonlinear stability of spectrally stable stationary states, e.g. the spectrally stable fundamental discrete soliton for $N = 1$.
- Sharp conditions on the initial data to distinguish bounded oscillations and exponentially growing trajectories.
- Extensions to multi-dimensional and continuous settings of the PT-dNLS equation.