Instability of peaked waves in the Camassa-Holm equation

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Joint work with
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Long-wave models

The *Korteweg–De Vries equation* (1895) governs dynamics of small-amplitude long waves in a fluid:

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The *Camassa–Holm equation* (1994) models dispersion-modified nonlinear effects:

$$u_t + 3uu_x - u_{txx} = 2u_x u_{xx} + uu_{xxx},$$

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$$u_t + uu_x + K * u_x = 0, \qquad \hat{K}(k) = \sqrt{gh\frac{\tanh(kh)}{kh}}.$$

The *Camassa–Holm equation* (1994) in a weaker form:

$$u_t + uu_x + (1 - \partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2) = 0.$$

Traveling wave solutions

Traveling wave solutions are solutions of the form

$$u(x,t) = U(x-ct),$$

where $z = x - ct$ is the travelling wave coordinate and c is the wave speed. For fixed c, the wave profile U is either 2π-periodic or decaying to 0 at infinity.

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For the KdV equation, U satisfies

$$\beta \frac{d^2 U}{dz^2} - cU + U^2 = 0.$$

All solutions are smooth.
[ODE textbooks]

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For the Whitham equation, U satisfies

$$K * U = (c - U)U.$$

Solutions are smooth if $c - U(z) > 0$ for all z.

[M. Ehrnström, H. Kalisch, 2013] [M. Ehrnström, E.Wahlén, 2015]

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For the Camassa-Holm equation, U satisfies

$$(c-U)^2 \left[\frac{d^2U}{dz^2} - U \right] = a.$$

There are smooth, peaked, and cusped solutions:
smooth if $c - U(z) > 0$, peaked and cusped if $c - U(z) \geq 0$
[J. Lenells, 2005]

Stability of smooth and peaked periodic waves

- KdV equation: smooth waves are linearly and orbitally stable
  [B. Deconinck et.al. 2009,2010]

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- Camassa-Holm, Degasperis–Procesi, Novikov: peaked waves are orbitally and asymptotically stable in energy space.
  [A.Constantin & W.Strauss, 2000], [A.Constantin & L.Molinet, 2001], [J.Lenells, 2004], [Z.Lin, Y.Liu, 2006], [X. Liu, Y. Liu, C. Qu, 2014]

Plan of my talk

Instability of peaked waves in the Camassa–Holm equation

$$u_t + uu_x + (1 - \partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2) = 0.$$

- Cauchy problem in Sobolev spaces
- Orbital stability of peakons in H^1
- Linear instability of peakons in $H^1 \cap W^{1,\infty}$
- Nonlinear instability of peakons in $H^1 \cap W^{1,\infty}$

Global existence versus wave breaking for large initial data

Definition

We say that the Cauchy problem is locally well-posed in Banach space X if for every initial data $u_0 \in X$, there are $T > 0$ and a unique solution $u \in C((-T,T),X)$ such that $u|_{t=0} = u_0$ and the solution depends continuosly on u_0 in X.

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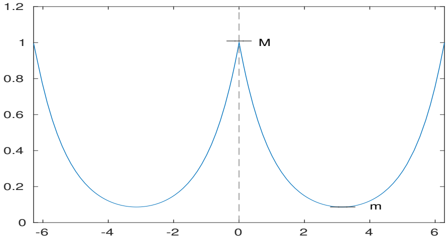
- If the solution can be continued for every $T > 0$ so that $u \in C(\mathbb{R},X)$, the solution exists globally.
- If there is $T < \infty$ such that $\|u(t,\cdot)\|_X \to \infty$ as $t \to T^-$, the solution blows up in a finite time.
- The finite time blow-up is called **wave breaking** if $\|u(t,\cdot)\|_{L^\infty} < \infty$ and $\|\partial_x u(t,\cdot)\|_{L^\infty} \to \infty$ as $t \to T^-$.

Cauchy problem in Sobolev spaces

Let $\varphi(x) = e^{-|x|}$ be the Greens function satisfying $(1 - \partial_x^2)\varphi = 2\delta$. The Cauchy problem for *the Camassa–Holm equation* can be written in the convolution form:

$$\begin{cases} u_t + uu_x + \frac{1}{2}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right) = 0, \\ u|_{t=0} = u_0. \end{cases}$$

The quantity $m := (1 - \partial_x^2)u$ is referred as the momentum density.



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- ▷ Local well-posedness for $u_0 \in H^s$ with $s > 3/2$.
  [A. Constantin, J. Escher (1998)] [Y.Li-P.Olver (2000)] [G.Rodriguez (2001)] [R. Danchin (2001)] [A.Himonas, G. Misiolek (2001)] [G. Misiolek (2002)]
- ▷ Global existence for $u_0 \in H^3$ if $m_0 \geq 0$
  [A.Constantin (2000)]
- ▷ Wave breaking for $u_0 \in H^3$ if $\exists x_0$: $(x - x_0)m_0(x) \leq 0$.
  [A.Constantin, J. Escher (1998)] [L. Brandolese (2014)]

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The quantity $m := (1 - \partial_x^2)u$ is referred as the momentum density.

- No continuous dependence (norm inflation) for $u_0 \in H^{s \leq 3/2}$.
  [P. Byers (2006)] [A. Himonas, G. Misiolek, G. Ponce (2007)] [A. Himonas, K. Grayshan, C. Holliman (2016)] [Z.Guo, X.Liu, L. Molinet, Z.Yin (2018)]
- Global existence of weak solutions $u_0 \in H^1$ with $m_0 \geq 0$.
  [A.Constantin, L. Molinet (2000)]
- Global existence of weak solutions $u_0 \in H^1$.
  [A. Bressan, A.Constantin (2006)] [H. Holden, X. Raynaud (2007)]

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The quantity $m := (1 - \partial_x^2)u$ is referred as the momentum density.

- ▷ Uniqueness of weak global solutions $u_0 \in H^1$.

  [A. Bressan, G. Chen, Q. Zhang (2015)]
- ▷ Continuous dependence for $u_0 \in H^1 \cap W^{1,\infty}$.

  [C. De Lellis, T. Kappeler, P. Topalov (2007)]

  [F. Linares, G. Ponce, T. Sideris (2019)]
- ▷ Local solutions may break in a finite time with $u_x(t,x) \to -\infty$ at some $x \in \mathbb{R}$ as $t \to T^-$.

Existence and stability of peakons

For every $c \in \mathbb{R}$, $u(t,x) = c\varphi(x-ct)$ is a solution to

$$u_t + uu_x + \frac{1}{2}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right) = 0.$$

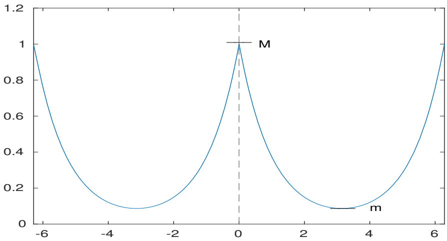

Figure: The graph of φ on $[-2\pi, 2\pi]$.

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There exist two conserved quantities:

$$E(u) = \int_{\mathbb{R}} (u^2 + u_x^2)dx, \quad F(u) = \int_{\mathbb{R}} u(u^2 + u_x^2)dx.$$

such that $\|u(t,\cdot)\|_{H^1} = \|u_0\|_{H^1}$ for almost every $t \in \mathbb{R}$.

Theorem (A. Constantin–L.Molinet (2001))

φ is a unique (up to translation) minimizer of $E(u)$ in H^1 subject to $3F(u) = 2E(u)$. Consequently, global solutions with $u_0 \in H^3$ with $m_0 \geq 0$ close to φ in H^1 stay close to $\{\varphi(\cdot - a)\}_{a\in\mathbb{R}}$ in H^1 for all t.

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Theorem (A. Constantin–W. Strauss (2000))

For every small $\varepsilon > 0$, if the initial data satisfies

$$\|u_0 - \varphi\|_{H^1} < \left(\frac{\varepsilon}{3}\right)^4,$$

then the solution satisfies

$$\|u(t,\cdot) - \varphi(\cdot - \xi(t))\|_{H^1} < \varepsilon, \quad t \in (0,T),$$

where $\xi(t)$ is a point of maximum for $u(t,\cdot)$.

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$$u_t + uu_x + \frac{1}{2}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right) = 0.$$

- Asymptotic stability of peakons for $u_0 \in H^1$ with $m_0 \geq 0$.
  [L. Molinet (2018)]
- Asymptotic stability of trains of peakons and anti-peakons.
  [L. Molinet (2019)]
- Inverse scattering for weak global solutions with multi-peakons.
  [L.Li (2009)] [J. Eckhardt, A. Kostenko (2014)] [J. Eckhardt (2018)]

Instability of peakons

Consider solutions of the Cauchy problem:

$$\begin{cases} u_t + uu_x + Q[u] = 0, \\ u|_{t=0} = u_0 \in H^1 \cap W^{1,\infty}, \end{cases} \qquad Q[u] := \frac{1}{2}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right).$$

Assume that u_0 is piecewise C^1 with a single peak.

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Theorem (F. Natali–D. Pelinovsky (2020))

For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H^1 \cap W^{1,\infty}$ satisfying

$$\|u_0 - \varphi\|_{H^1} + \|u_0' - \varphi'\|_{L^\infty} < \delta,$$

such that the unique solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ with $T > t_0$ satisfies

$$\|u_x(t_0,\cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty} > 1,$$

where $\xi(t)$ is a point of peak of $u(t,\cdot)$ for $t \in [0,T)$.
Moreover, there exists u_0 such that T is finite.

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Weak formulation of the unique global conservative solution:

$$\int_0^\infty \int_{\mathbb{R}} \left(u\psi_t + \frac{1}{2}u^2\psi_x - Q[u]\psi\right) dxdt + \int_{\mathbb{R}} u_0(x)\psi(0,x)dx = 0,$$

where $\psi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$.

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Assume that u_0 is piecewise C^1 with a single peak.

- ▷ If $u \in H^1(\mathbb{R})$, then $Q[u] \in C(\mathbb{R})$.
- ▷ If $u \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, then $Q[u]$ is Lipschitz continuous.

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Assume that u_0 is piecewise C^1 with a single peak.

If $u(t,\cdot) \in H^1(\mathbb{R}) \cap C^1(\mathbb{R} \backslash \{\xi(t)\})$ for $t \in [0,T)$. Then,
$\xi(t) \in C^1(0,T)$ and

$$\frac{d\xi}{dt} = u(t,\xi(t)), \quad t \in (0,T).$$

Decomposition near a single peakon

Consider a decomposition:

$$u(t,x) = \varphi(x-t-a(t)) + v(t,x-t-a(t)), \quad t \in [0,T), \quad x \in \mathbb{R},$$

with the peak at $\xi(t) = t + a(t)$ for $v(t,\cdot) \in H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{\xi(t)\})$.

Then,

$$(\varphi - 1)\varphi' + Q(\varphi) = 0,$$

$$\frac{da}{dt} = v(t,0),$$

and

$$v_t = (1-\varphi)v_x + (v|_{x=0} - v)\varphi' + (v|_{x=0} - v)v_x - \varphi' * (\varphi v + \frac{1}{2}\varphi' v_x) - Q[v].$$

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Due to

$$[v(0) - v(x)]\varphi'(x) - \varphi' * \varphi v - \frac{1}{2}\varphi' * \varphi' v_x = \varphi(x) \int_0^x v(y)dy,$$

the evolution of $v(t,x)$ simplifies to

$$v_t = (1-\varphi)v_x + \varphi w + (v|_{x=0} - v)v_x - Q[v],$$

where $w(t,x) = \int_0^x v(t,y)dy$.

Linearized evolution

Truncation of the quadratic terms yields the linearized problem:

$$\begin{cases} v_t = (1-\varphi)v_x + \varphi w, & t > 0, \\ v|_{t=0} = v_0(x), \end{cases}$$

where $w(t,x) = \int_0^x v(t,y)dy$.

Solution with the method of characteristic curves:

$$x = X(t,s), \qquad v(t,X(t,s)) = V(t,s), \quad w(t,X(t,s)) = W(t,s).$$

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The evolution problem splits into

$$\begin{cases} \frac{dX}{dt} = \varphi(X) - 1, \\ X|_{t=0} = s, \end{cases} \qquad \begin{cases} \frac{dW}{dt} = \varphi'(X)W, \\ W|_{t=0} = w_0(s), \end{cases} \qquad \begin{cases} \frac{dV}{dt} = \varphi(X)W, \\ V|_{t=0} = v_0(s). \end{cases}$$

Since φ is Lipschitz, there exists unique characteristic function $X(t,s)$ for each $s \in \mathbb{R}$. The peak location $X(t,0) = 0$ is invariant in the time evolution.

Properties of the linearized evolution

Assume $v_0 \in H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{0\})$. For every $t > 0$, we have:

- $\exists C_0 > 0$: $\|v(t,\cdot)\|_{L^\infty} \leq C_0$.
- $\lim_{x\to 0^+} v_x(t,x) = v_0'(0^+)e^t$, $\lim_{x\to 0^-} v_x(t,x) = v_0'(0^-)e^{-t}$.
- $\|v(t,\cdot)\|_{H^1}^2 = C_+e^t + C_0 + C_-e^{-t}$ for some C_+, C_0, C_-.

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- $\lim_{x\to 0^+} v_x(t,x) = v_0'(0^+)e^t$, $\lim_{x\to 0^-} v_x(t,x) = v_0'(0^-)e^{-t}$.
- $\|v(t,\cdot)\|^2_{H^1} = C_+e^t + C_0 + C_-e^{-t}$ for some C_+, C_0, C_-.

Growth of $\|v(t,\cdot)\|^2_{H^1}$ contradicts to H^1 orbital stability of peakons.
Both properties are related to the existence of conserved quantities:

$$E(u) = \int_{\mathbb{R}} (u^2 + u_x^2)dx, \quad F(u) = \int_{\mathbb{R}} u(u^2 + u_x^2)dx.$$

Illustration of the linear instability

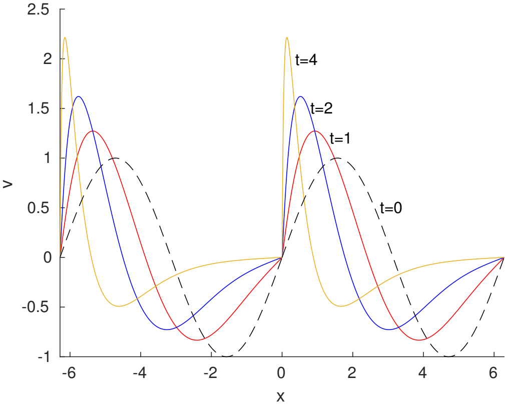

Figure: The plots of $v(t,x)$ versus x on $[-2\pi, 2\pi]$ for different values of t in the case $v_0(x) = \sin(x)$.

Nonlinear evolution

Recall the evolution problem:

$$\begin{cases} v_t = (1-\varphi)v_x + \varphi w + (v|_{x=0} - v)v_x - Q[v], & t \in (0,T), \\ v|_{t=0} = v_0(x), \end{cases}$$

where $w(t,x) = \int_0^x v(t,y)dy$ and $Q[v] := \frac{1}{2}\varphi' * \left(v^2 + \frac{1}{2}v_x^2\right)$.

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The characteristic coordinates $X(t,s)$ satisfies

$$\begin{cases} \frac{dX}{dt} = \varphi(X) - 1 + v(t,X) - v(t,0), \quad t \in (0,T), \\ X|_{t=0} = s. \end{cases}$$

Since φ is Lipschitz, there exists the unique characteristic function $X(t,s)$ for each $s \in \mathbb{R}$ if $v(t,\cdot)$ remains in $H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{0\})$
The peak location $X(t,0) = 0$ is invariant in the time evolution.

Local existence in class $H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{0\})$

We introduce on the characteristic curves:

$$v(t, X(t,s)) = V(t,s), \quad w(t, X(t,s)) = W(t,s), \quad v_x(t, X(t,s)) = U(t,s).$$

and write the dynamical system:

$$\frac{d}{dt}\begin{bmatrix} X \\ V \\ W \\ U \end{bmatrix} = \begin{bmatrix} \varphi(X) - \varphi(0) + V - V|_{s=0} \\ \varphi(X)W - Q[v](X) \\ \varphi'(X)W + \frac{1}{2}[V^2 - (V|_{s=0})^2] - P[v](X) + P[v]|_{s=0} \\ \varphi'(X)[W - U] + \varphi(X)V - \frac{1}{2}U^2 + V^2 - P[v](X) \end{bmatrix}$$

subject to the initial and boundary condition

$$\begin{bmatrix} X \\ V \\ W \\ U \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} s \\ v_0(s) \\ w_0(s) \\ v_0'(s) \end{bmatrix} \qquad \left\{ \begin{array}{l} X(t,0) = 0, \\ V(t,0) = V|_{s=0}, \\ W(t,0) = 0. \end{array} \right.$$

Local existence in class $H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{0\})$

Theorem

For every $v_0 \in H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{0\})$, there exists the maximal existence time $T > 0$ (finite or infinite) and the unique solution $v \in C^1([0,T), H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{0\})$ to the evolution problem that depends continuously on v_0.

Moreover, if $T < \infty$, there $\|v_x(t,\cdot)\|_{L^\infty} \to \infty$ as $t \to T^-$.

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Moreover, if $T < \infty$, there $\|v_x(t,\cdot)\|_{L^\infty} \to \infty$ as $t \to T^-$.

Remark: The result is similar to the local well-posedness theory in $H^1 \cap W^{1,\infty}$ but the method of the proof is very different.

[C. De Lellis, T. Kappeler, P. Topalov (2007)]

[F. Linares, G. Ponce, T. Sideris (2019)]

Instability theorem

Theorem (F. Natali–D. Pelinovsky (2020))

For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H^1 \cap W^{1,\infty}$ satisfying

$$\|u_0 - \varphi\|_{H^1} + \|u_0' - \varphi'\|_{L^\infty} < \delta,$$

such that the unique solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ with $T > t_0$ satisfies

$$\|u_x(t_0, \cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty} > 1,$$

where $\xi(t)$ is a point of peak of $u(t, \cdot)$ for $t \in [0, T)$.

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From the right side of the peak, $V_0(t) = V(t,0)$, $U_0(t) = U(t,0^+)$:

$$\frac{dU_0}{dt} = U_0 + V_0 + V_0^2 - \frac{1}{2}U_0^2 - P[v](0), \quad P[v] := \frac{1}{2}\varphi * \left(v^2 + \frac{1}{2}v_x^2\right).$$

Proof of instability

From orbital stability in H^1 [A. Constant, W. Strauss (2000)]
If $\|v_0\|_{H^1} < (\varepsilon/3)^4$, then

$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

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From the equation on the right side of the peak:

$$\frac{dU_0}{dt} = U_0 + V_0 + V_0^2 - \frac{1}{2}U_0^2 - P[v](0) \leq U_0 + C\varepsilon$$

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Assume $\lim_{x\to 0^+} v_0'(x) = -\|v_0'\|_{L^\infty} = -2C\varepsilon$.
The initial constraint $\|v_0\|_{L^\infty} + \|v_0'\|_{L^\infty} < \delta$, is satisfied if $\forall\delta > 0$, $\exists\varepsilon > 0$ such that

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From the ODE comparison theory, we obtain

$$U_0(t) \leq -C\varepsilon e^t,$$

hence $|U_0(t_0)| \geq 1$ for $t_0 := -\log(C\varepsilon)$.

Strong instability theorem

Theorem (F. Natali–D.Pelinovsky (2020))

For every $\delta > 0$*, there exist* $u_0 \in H^1 \cap W^{1,\infty}$ *satisfying*

$$\|u_0 - \varphi\|_{H^1} + \|u_0' - \varphi'\|_{L^\infty} < \delta,$$

such that the maximal existence time of the unique solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ *is finite.*

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From the right side of the peak, $V_0(t) = V(t,0)$, $U_0(t) = U(t,+0)$:

$$\frac{dU_0}{dt} = U_0 + V_0 + V_0^2 - \frac{1}{2}U_0^2 - P[v](0) \leq U_0 - \frac{1}{2}U_0^2 + C\varepsilon.$$

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such that the maximal existence time of the unique solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ *is finite.*

By the ODE comparison theory, $U_0(t) \leq \overline{U}(t)$, where the supersolution satisfies

$$\frac{d\overline{U}}{dt} = \overline{U} - \frac{1}{2}\overline{U}^2 + C\varepsilon$$

with $U_0(0) = \overline{U}(0) = -C\varepsilon$.

Concluding remarks

1. Instability of peakons with respect to peaked perturbations is consistent with local well-posedness for $u_0 \in H^1 \cap W^{1,\infty}$ and wave breaking in a finite time: $u_x(t,x) \to -\infty$ at some $x \in \mathbb{R}$.
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2. It follows from the method of characteristics that if $v_0 \in C^1(\mathbb{R})$, then $v(t,\cdot) \notin C^1(\mathbb{R})$ for $t > 0$ due to the single peak at $x = \xi(t)$:

$$u(t,x) = \varphi(x-t-a(t)) + v(t,x-t-a(t)), \quad t \in [0,T), \quad x \in \mathbb{R}.$$

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3. The H^1 orbital stability results on peakons are misleading as the perturbations near the peakon are growing in $W^{1,\infty}$ norm and may blow up in a finite time.

Other investigations

- Much of the theory applies to the instability of perturbations to the peaked periodic waves in the Camassa–Holm equation

$$u_t + 3uu_x - u_{txx} = 2u_x u_{xx} + uu_{xxx}.$$

[A. Madiyeva and D. Pelinovsky (2020)]

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- Instability of peakons was also discovered in the Novikov equation

$$u_t + 4u^2 u_x - u_{txx} = 3uu_x u_{xx} + u^2 u_{xxx},$$

where the unique global weak solution exists in $H^1 \cap W^{1,4}$. Nevertheless, the peakons are strongly unstable in $H^1 \cap W^{1,\infty}$.

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- An interesting difference is that the peakons are linearly unstable in H^1 for Camassa-Holm and linearly stable in H^1 for Novikov. Linear stability theory for peakons in the energy space does not imply anything for the nonlinear stability theory.

Summary

- Global solutions and breaking in the Camassa–Holm equation

$$u_t + 3uu_x - u_{txx} = 2u_x u_{xx} + uu_{xxx}.$$

- Unique global solutions exist in H^1 but continuous dependence only holds in $H^1 \cap W^{1,\infty}$.
- *Peakons* are orbitally *stable* in H^1.
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Thank you! Questions ???