# Instability of peaked waves

**Dmitry Pelinovsky**

Department of Mathematics, McMaster University, Canada
http://dmpeli.math.mcmaster.ca

Joint work with
**Anna Geyer** (Delft University of Technology, Netherlands)
**Fabio Natali** (University of Maringa, Brazil)

# Long-wave models

The *Korteweg–De Vries equation* (1895) governs dynamics of small-amplitude long waves in a fluid:

$$u_t + uu_x + \beta u_{xxx} = 0.$$

It arises from the dispersion relation for linear waves $e^{i(kx-\omega t)}$:

$$\omega^2 = c^2k^2 + \beta k^4 + \mathcal{O}(k^6) \quad \Rightarrow \quad \omega - ck = \frac{1}{2c}\beta k^3 + \mathcal{O}(k^5).$$

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The *Ostrovsky equation* (1978) models rotation effects:

$$(u_t + uu_x + \beta u_{xxx})_x = \gamma^2 u,$$

as follows from:

$$\omega^2 = \gamma^2 + c^2k^2 + \beta k^4 + \cdots \quad \Rightarrow \quad \omega - ck = \frac{\beta}{2c}k^3 + \frac{\gamma^2}{2ck} + \cdots$$

As $\beta \to 0$, we obtain *the reduced Ostrovsky equation*.

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The *Whitham equation* (1967) models full-dispersion effects:

$$u_t + uu_x + K * u_x = 0,$$

where the Fourier transform of the convolution kernel:

$$\hat{K}(k) = \sqrt{gh\frac{\tanh(kh)}{kh}} = \sqrt{gh}\left(1 - \frac{1}{6}k^2h^2 + \cdots\right)$$

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The *Camassa–Holm equation* (1994) models dispersion-modified nonlinear effects:

$$u_t + 3uu_x - u_{txx} = 2u_xu_{xx} + uu_{xxx}.$$

# Traveling wave solutions

Traveling wave solutions are solutions of the form

$$u(x,t) = U(x-ct),$$

where $z = x - ct$ is the travelling wave coordinate and $c$ is the wave speed. For fixed $c$, the wave profile $U$ is either $2T$-periodic or decaying to 0 at infinity.

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For the KdV equation, $U$ satisfies

$$\beta \frac{d^2 U}{dz^2} - cU + U^2 = 0.$$

All solutions are smooth.
[ODE textbooks]

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For the reduced Ostrovsky equation, $U$ satisfies

$$\frac{d}{dz}\left((c-U)\frac{dU}{dz}\right) + U(z) = 0.$$

Solutions are smooth if $c - U(z) > 0$ for all $z$.
[A.Geyer, D.P., 2017]

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For the Whitham equation, $U$ satisfies

$$K * U = (c - U)U.$$

Solutions are smooth if $c - U(z) > 0$ for all $z$.
[M. Ehrnström, H. Kalisch, 2013] [M. Ehrnström, E.Wahlén, 2015]

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For the Camassa-Holm equation, $U$ satisfies

$$(c-U)\left[\frac{d^2U}{dz^2} - U\right] = 0.$$

All solutions are peaked with $U(z_0) = c$ for some $z_0 \in \mathbb{R}$.
[R. Camassa, D. Holm, J. Hyman, 1994]

# Stability of smooth and peaked periodic waves

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  [J.Carter & H.Kalisch, 2014]
- Camassa-Holm, Degasperis–Procesi, Novikov: peaked waves are orbitally and asymptotically stable in energy space.
  [A.Constantin & W.Strauss, 2000], [J.Lenells, 2005], [Z.Lin, Y.Liu, 2006], ...
  but they are unstable w.r.t. piecewise smooth perturbations
  [F.Natali & D.P. 2019]

# Plan of my talk

1. Instability of peaked waves in the reduced Ostrovsky equation

$$(u_t + uu_x)_x = u$$

   - Cauchy problem in Sobolev spaces
   - Existence of peaked periodic waves
   - Linear instability of the peaked wave

2. Instability of peaked waves in the Camassa–Holm equation

$$u_t + 3uu_x - u_{txx} = 2u_x u_{xx} + uu_{xxx}.$$

   - Cauchy problem in Sobolev spaces
   - Orbital stability of peakons in $H^1$
   - Nonlinear instability of peakons in $H^1 \cap W^{1,\infty}$

# Cauchy problem in Sobolev spaces

Consider Cauchy problem for *the reduced Ostrovsky equation*

$$\begin{cases} (u_t + uu_x)_x = u, \\ u|_{t=0} = u_0. \end{cases}$$

- Local well-posedness for $u_0 \in H^s$ with $s > 3/2$
  [A.Stefanov et. al., 2010]
- Zero mass constraint is necessary in the periodic domain: $\int_{-\pi}^{\pi} u_0(x)dx = 0$.

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- Zero mass constraint is necessary in the periodic domain: $\int_{-\pi}^{\pi} u_0(x)dx = 0$.
- Local solutions break in finite time for large initial data.
  [Y.Liu & D.P. & A.Sakovich 2010]
- Global solutions exist for small initial data.
  [R.Grimshaw & D.P. 2014]

# Global solutions for small initial data

## Theorem (R.Grimshaw & D.P., 2014)

*Let* $u_0 \in H^3$ *such that* $1 - 3u_0''(x) > 0$ *for all* $x$. *There exists a unique solution* $u(t) \in C(\mathbb{R}, H^3)$ *with* $u(0) = u_0$.

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The quantity $1 - 3u_{xx}$ appears in the Lax pair [A. Hone & M. Wang (2003)]

$$\left\{ \begin{array}{c} 3\lambda\psi_{xxx} + (1 - 3u_{xx})\psi = 0, \\ \psi_t + \lambda\psi_{xx} + u\psi_x - u_x\psi = 0, \end{array} \right.$$

and in the conserved quantities [J. Brunelli & S.Sakovich (2013)]

$$\begin{aligned} E_0 &= \int_{\mathbb{R}} u^2 dx \\ E_1 &= \int_{\mathbb{R}} \left[ (1 - 3u_{xx})^{1/3} - 1 \right] dx, \\ E_2 &= \int_{\mathbb{R}} \frac{(u_{xxx})^2}{(1 - 3u_{xx})^{7/3}} dx \end{aligned}$$

# Wave breaking for large initial data

## Lemma

*Let $u_0 \in H^2_{\rm per}$. The local solution $u \in C([0,T), H^2_{\rm per})$ blows up in a finite time $T < \infty$ in the sense $\lim_{t\uparrow T} \|u(\cdot,t)\|_{H^2} = \infty$ if and only if*

$$\lim_{t\uparrow T} \inf_{x} u_x(t,x) = -\infty, \quad \textit{while} \quad \lim_{t\uparrow T} \sup_{x} |u(t,x)| < \infty.$$

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## Theorem (J.Hunter, 1990)

*Let $u_0 \in C^1_{\rm per}$ and define*

$$\inf_{x \in \mathbb{S}} u_0'(x) = -m \quad \text{and} \quad \sup_{x \in \mathbb{S}} |u_0(x)| = M.$$

*If $m^3 > 4M(4+m)$, a smooth solution $u(t,x)$ breaks in a finite time.*

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## Theorem (Y.Liu, D.P. & A.Sakovich, 2010)

*Assume that $u_0 \in H^2_{\rm per}$. The solution breaks if*

$$\text{either} \quad \int_{\mathbb{S}} (u_0'(x))^3 \, dx < -\left(\frac{3}{2}\|u_0\|_{L^2}\right)^{3/2},$$

$$\text{or} \quad \exists x_0 : \quad u_0'(x_0) < -\left(\|u_0\|_{L^\infty} + T_1 \|u_0\|_{L^2}\right)^{\frac{1}{2}}.$$

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*Let* $u_0 \in H^2_{\rm per}$. *The local solution* $u \in C([0,T), H^2_{\rm per})$ *blows up in a finite time* $T < \infty$ *in the sense* $\lim_{t\uparrow T} \|u(\cdot,t)\|_{H^2} = \infty$ *if and only if*

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Conjecture on sharp wave breaking:

Smooth solutions break in a finite time if $u_0 \in H^3$ yields sign-indefinite $1 - 3u_0''(x)$.

# Travelling periodic waves

Let $c > 0$ and consider a periodic solution $U$ of

$$\frac{d}{dz}\left((c-U)\frac{dU}{dz}\right) + U = 0. \tag{ODE}$$

The solution $U$ is smooth if and only if $(u, v) = (U, U')$ is a periodic orbit $\gamma_E$ of the planar system

$$\begin{cases} u' = v, \\ v' = \dfrac{-u + v^2}{c-u}, \end{cases}$$

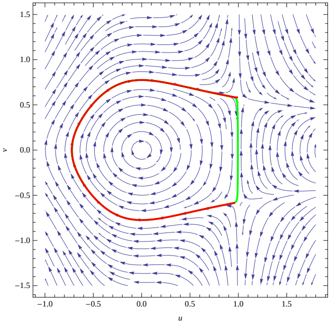
which has the first integral

$$E(u,v) = \frac{1}{2}(c-u)^2 v^2 + \frac{c}{2}u^2 - \frac{1}{3}u^3.$$

The solution $U$ is smooth if and only if $c - U(z) > 0$ for every $z$.

Let $c > 0$. The first integral is

$$E(u,v) = \frac{1}{2}(c-u)^2 v^2 + \frac{c}{2}u^2 - \frac{1}{3}u^3.$$

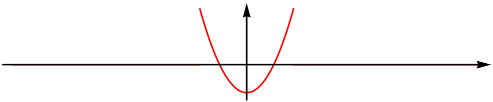



There exists a smooth family of periodic solutions parametrized by the energy $E \in (0, E_c)$, where $2T$ depends on $E$.

# Peaked periodic wave

For $c = c_* := \pi^2/9$ there exists a solution with parabolic profile

$$U_*(z) := \frac{3z^2 - \pi^2}{18}, \quad z \in [-\pi, \pi],$$



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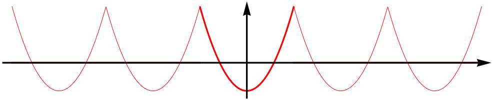

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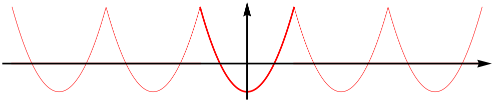

The peaked periodic wave $U_* \in H^s_{\rm per}(-\pi, \pi)$ for $s < 3/2$:

$$U_*(z) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{3n^2} \cos(nz),$$

with $U_*(\pm\pi) = c_*$ and $U_*'(\pm\pi) = \pm\pi/3$.

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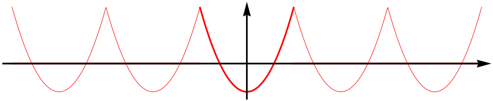

The peaked wave satisfies the border case:
$1 - 3U_*''(z) = 0$ for $z \in (-\pi, \pi)$.

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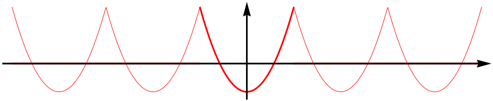

## Theorem (A.Geyer & D.P, 2019)

*The peaked periodic wave $U_*$ is the* unique *peaked solution with the jump at $z = \pm\pi$.*

See also [Bruell & Dhara, 2019]

# Linear instability of the peaked periodic wave

We consider *co-periodic* perturbations of the traveling waves, that is, *perturbations with the same period 2T and zero mean.*

Using $u(t,x) = U_*(z) + v(t,z)$, where $z = x - ct$ yields the linearized evolution:

$$\left\{ \begin{array}{l} v_t + \partial_z \left[ (U_*(z) - c_*) v \right] = \partial_z^{-1} v, \quad t > 0, \\ v|_{t=0} = v_0. \end{array} \right. \qquad \text{(linO)}$$

## Definition

The travelling wave $U$ is *linearly unstable* if there exists $v_0 \in \mathrm{dom}(\partial_z L)$ such that the unique global solution $v \in C(\mathbb{R}, \mathrm{dom}(\partial_z L))$ satisfies $\lim_{t \to \infty} \|v(t)\|_{L^2} = \infty$, where

$$\mathrm{dom}(\partial_z L) = \left\{ v \in \dot{L}^2_{\mathrm{per}} : \quad \partial_z \left[ (c_* - U_*) v \right] \in \dot{L}^2_{\mathrm{per}} \right\}.$$

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## Theorem (Geyer & P., 2019)

*The peaked travelling wave U is linearly unstable with*

$$\|v(t)\|_{L^2} \geq C_0 e^{\pi t/6} \|v_0\|_{L^2}, \quad t > 0$$

*for some $C_0 > 0$.*

# Linear instability of the peaked periodic wave

▷ **Step 1**: The *truncated problem*

$$\begin{cases} v_t + \frac{1}{6}\partial_z \left[ (z^2 - \pi^2) v \right] = 0, \quad t > 0, \\ v|_{t=0} = v_0. \end{cases} \tag{truncO}$$

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**Method of characteristics.** The characteristic curves $z = Z(s,t)$ are found explicitly and the solution of $V(s,t) := v(Z(s,t),t)$ is

$$V(s,t) = \frac{1}{\pi^2}[\pi\cosh(\pi t/6) - s\sinh(\pi t/6)]^2 v_0(s), \quad s \in [-\pi,\pi], \quad t \in \mathbb{R}.$$

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This yields the linear instability result for the truncated problem:

**Lemma**

*For every $v_0 \in \mathrm{dom}(\partial_z L)$ $\exists!$ global solution $v \in C(\mathbb{R}, \mathrm{dom}(\partial_z L))$. If $v_0$ is odd, then the global solution satisfies*

$$\frac{1}{2}\|v_0\|_{L^2} e^{\pi t/6} \le \|v(t)\|_{L^2} \le \|v_0\|_{L^2} e^{\pi t/6}, \quad t > 0.$$

# Linear instability of the peaked periodic wave

▷ **Step 2**: The *full evolution problem*

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**Generalized Meth. of Char.** Treat $\partial_z^{-1} v$ as a *source term* in (linO).

- ▷ truncated problem $v_t = A_0 v$ has a unique global solution in $\dot{L}^2_{\text{per}}$
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  $A := A_0 + \partial_z^{-1}$ is the generator of $C^0$-semigroup on $\dot{L}^2_{\text{per}}$

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## Lemma

*For every* $v_0 \in \text{dom}(\partial_z L)$ $\exists!$ *global solution* $v \in C(\mathbb{R}, \text{dom}(\partial_z L))$. *If* $v_0$ *is odd and satisfies some constraints, then the solution satisfies*

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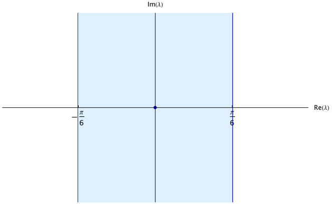
The peaked periodic wave is *linearly unstable*.

# Spectral instability of the peaked periodic wave




## Theorem (Geyer & P., 2020)

$$\sigma(\partial_z L) = \left\{ \lambda \in \mathbb{C} : \quad -\frac{\pi}{6} \leq \text{Re}(\lambda) \leq \frac{\pi}{6} \right\},$$

*where* $\partial_z L v := \partial_z \left[ (c_* - U_*) v \right] + \partial_z^{-1} v$ *with with*

$$\text{dom}(\partial_z L) = \left\{ v \in \dot{L}^2_{\text{per}} : \quad \partial_z \left[ (c_* - U_*) v \right] \in \dot{L}^2_{\text{per}} \right\}.$$

# Nonlinear instability ???

Consider Cauchy problem for *the reduced Ostrovsky equation*

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Does linear instability imply nonlinear instability?

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Does linear instability imply nonlinear instability?

- Lack of well-posedness results for $u_0 \in H^s_{\rm per}$ with $s < 3/2$.
- Lack of information on dynamics of peaked perturbations to the peaked periodic wave.

# Plan of part II

1. Instability of peaked waves in the reduced Ostrovsky equation

$$(u_t + uu_x)_x = u$$

   - Cauchy problem in Sobolev spaces
   - Existence of peaked periodic waves
   - Linear instability of the peaked wave

2. Instability of peaked waves in the Camassa–Holm equation

$$u_t + 3uu_x - u_{txx} = 2u_x u_{xx} + uu_{xxx}.$$

   - Cauchy problem in Sobolev spaces
   - Orbital stability of peakons in $H^1$
   - Nonlinear instability of peakons in $H^1 \cap W^{1,\infty}$

# Cauchy problem in Sobolev spaces

Let $\varphi(x) = e^{-|x|}$ be the Greens function satisfying $(1 - \partial_x^2)\varphi = 2\delta$. The Cauchy problem for *the Camassa–Holm equation* can be written in the convolution form:

$$\begin{cases} u_t + uu_x + \frac{1}{2}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right) = 0, \\ u|_{t=0} = u_0. \end{cases}$$

The quantity $m := (1 - \partial_x^2)u$ is referred as the momentum density.

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- Local well-posedness for $u_0 \in H^s$ with $s > 3/2$.
  [Y.Li-P.Olver (2000)] [Rodriguez (2001)]
- Local and global well-posedness for $u_0 \in H^3$ if $m_0 \geq 0$
  [A.Constantin (2000)]
- Wave breaking for $u_0 \in H^3$ if $\exists x_0$: $(x - x_0)m_0(x) \leq 0$.
  [A.Constantin, J. Escher (1998)]

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- Ill-posedness and norm inflation for $u_0 \in H^s$ with $s \leq 3/2$.
  [P. Byers (2006)] [Z.Guo et al. (2018)]
- Global existence of weak solutions $u_0 \in H^1$ with $m_0 \geq 0$.
  [A.Constantin, L. Molinet (2000)]
- Global existence of weak solutions $u_0 \in H^1$.
  [A. Bressan, A.Constantin (2006)] [H. Holden, X. Raynaud (2007)]

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- Uniqueness of weak global solutions $u_0 \in H^1$.

  [A. Bressan, G. Chen, Q. Zhang (2015)

- Continuous dependence for $u_0 \in H^1 \cap W^{1,\infty}$ but no global existence in $H^1 \cap W^{1,\infty}$.

  [F. Linares, G. Ponce, and T. Sideris (2019)]

- Local solutions may break in a finite time with $u_x(t,x) \to -\infty$ at some $x \in \mathbb{R}$ as $t \nearrow T$.

# Existence and stability of peakons

For every $c \in \mathbb{R}$, $u(t,x) = c\varphi(x-ct)$ is a solution to

$$u_t + uu_x + \frac{1}{2}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right) = 0.$$

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There exist two conserved quantities:

$$E(u) = \int_{\mathbb{R}} (u^2 + u_x^2)dx, \quad F(u) = \int_{\mathbb{R}} u(u^2 + u_x^2)dx.$$

such that $\|u(t,\cdot)\|_{H^1} = \|u_0\|_{H^1}$ for almost every $t \in \mathbb{R}$.

### Theorem (A. Constantin–L.Molinet (2001))

*$\varphi$ is a unique (up to translation) minimizer of $E(u)$ in $H^1$ subject to $3F(u) = 2E(u)$. Consequently, global solutions with $u_0 \in H^3$ with $m_0 \geq 0$ close to $\varphi$ in $H^1$ stay close to $\{\varphi(\cdot - a)\}_{a \in \mathbb{R}}$ in $H^1$ for all $t$.*

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## Theorem (A. Constantin–W. Strauss (2000))

*For every small $\varepsilon > 0$, if the initial data satisfies*

$$\|u_0 - \varphi\|_{H^1} < \left(\frac{\varepsilon}{3}\right)^4,$$

*then the solution satisfies*

$$\|u(t,\cdot) - \varphi(\cdot - \xi(t))\|_{H^1} < \varepsilon, \quad t \in (0,T),$$

*where $\xi(t)$ is a point of maximum for $u(t,\cdot)$ and the maximal existence time $T > 0$ may be finite.*

# Existence and stability of peakons

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$$u_t + uu_x + \frac{1}{2}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right) = 0.$$

- Asymptotic stability of peakons for $u_0 \in H^1$ with $m_0 \geq 0$.
  [L. Molinet (2018)]
- Asymptotic stability of trains of peakons and anti-peakons.
  [L. Molinet (2019)]
- Inverse scattering for weak global solutions with multi-peakons.
  [L.Li (2009)] [J. Eckhardt, A. Kostenko (2014)] [J. Eckhardt (2018)]

# Instability of peakons

Consider solutions of the Cauchy problem:

$$\begin{cases} u_t + uu_x + Q[u] = 0, \\ u|_{t=0} = u_0 \in H^1 \cap W^{1,\infty}, \end{cases}$$

where $Q[u] := \frac{1}{2}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right)$. Moreover, assume that $u_0$ is piecewise $C^1$ with a single peak.

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**Theorem (F. Natali–D.P. (2019))**

*For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H^1 \cap W^{1,\infty}$ satisfying*

$$\|u_0 - \varphi\|_{H^1} + \|u_0' - \varphi'\|_{L^\infty} < \delta,$$

*such that the global conservative solution satisfies*

$$\|u_x(t_0, \cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty} > 1,$$

*where $\xi(t)$ is a point of peak of $u(t, \cdot)$ for $t \in [0, t_0]$.*

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Weak formulation of the unique global conservative solution:

$$\int_0^\infty \int_{\mathbb{R}} \left( u\psi_t + \frac{1}{2}u^2\psi_x - Q[u]\psi \right) dxdt + \int_{\mathbb{R}} u_0(x)\psi(0,x)dx = 0,$$

where $\psi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$.

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- ▷ If $u \in H^1(\mathbb{R})$, then $Q[u] \in C(\mathbb{R})$.
- ▷ If $u \in H^1(\mathbb{R}) \cap C^1(-\infty, 0) \cap C^1(0, \infty)$, then $Q[u] \in C(\mathbb{R}) \cap C^1(-\infty, 0) \cap C^1(0, \infty)$.

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where $Q[u] := \frac{1}{2}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right)$. Moreover, assume that $u_0$ is piecewise $C^1$ with a single peak.

If $u(t, \cdot + \xi(t)) \in H^1(\mathbb{R}) \cap C^1(-\infty, 0) \cap C^1(0, \infty)$ for $t \in (0, T)$ with $\xi(t) \in C^1(0, T)$, then

$$\frac{d\xi}{dt} = u(t, \xi(t)), \quad t \in (0, T).$$

# Decomposition near a single peakon

Consider a decomposition:

$$u(t,x) = \varphi(x - t - a(t)) + v(t, x - t - a(t)), \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R},$$

where $a'(t) = v(t,0)$. Then $v(t,x)$ satisfies the Cauchy problem:

$$\begin{cases} v_t = (1-\varphi)v_x + \varphi w + (v|_{x=0} - v)v_x - Q[v], \quad t \in (0,T), \\ v|_{t=0} = v_0, \end{cases}$$

where $w(t,x) = \int_0^x v(t,y)dy$.

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where $w(t,x) = \int_0^x v(t,y)dy$.

The characteristic coordinates $X(t,s)$ satisfies the IVP:

$$\begin{cases} \frac{dX}{dt} = \varphi(X) - 1 + v(t,X) - v(t,0), & t \in (0,T), \\ X|_{t=0} = s, \end{cases}$$

which has a unique solution since $\varphi$ and $v$ is Lipschitz continuous.
$\Rightarrow X(t,0) = 0$ is invariant in $t$.

# Evolution near a single peakon

On characteristic curves, $V(t,s) := v(t,X(t,s))$ satisfies:

$$\left\{ \begin{array}{l} \frac{dV}{dt} = \varphi(X) w(t,X) - Q[v](X), \\ V|_{t=0} = v_0(s). \end{array} \right.$$

whereas $V'(t,s) := v_x(t,X(t,s))$ satisfies

$$\left\{ \begin{array}{l} \frac{dV'}{dt} = -\varphi'(X) V' + \varphi(X) V + \varphi'(X) w(t,X) - \frac{1}{2}(V')^2 + V^2 - P[v](X), \\ V'|_{t=0} = v_0'(s). \end{array} \right.$$

where $P[v](x) := \frac{1}{2} \int_{\mathbb{R}} \varphi(x-y) \left( [v(y)]^2 + \frac{1}{2}[v'(y)]^2 \right) dy.$

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where $P[v](x) := \frac{1}{2}\int_{\mathbb{R}} \varphi(x-y)\left([v(y)]^2 + \frac{1}{2}[v'(y)]^2\right) dy.$

From one side of the peak, $V_0(t) = V(t,0)$, $V_0'(t) = V'(t,+0)$:

$$\frac{d}{dt}(V_0 + V_0') = (V_0 + V_0') + V_0^2 - \frac{1}{2}(V_0')^2 - Q[v](0) - P[v](0).$$

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Integrating with the integrating factors,

$$\frac{d}{dt} \left[ e^{-t} (V_0 + V_0') \right] = e^{-t} \left[ V_0^2 - \frac{1}{2} (V_0')^2 - Q[v](0) - P[v](0) \right] \leq e^{-t} V_0^2.$$

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This yields the bound

$$V_0(t) + V_0'(t) \leq e^t \left[ V_0(0) + V_0'(0) + \int_0^t e^{-\tau} V_0^2(\tau) d\tau \right], \quad t \in [0, T).$$

# Proof of instability

- From orbital stability in $H^1$ [A. Constant, W. Strauss (2000)]
  If $\|v_0\|_{H^1} < (\varepsilon/3)^4$, then

$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

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- Let $\lim_{x\to 0^+} v_0'(x) = -\|v_0'\|_{L^\infty} = -2\varepsilon^2$. If $v_0 \in H^1 \cap W^{1,\infty}$ satisfies $\|v_0\|_{L^\infty} + \|v_0'\|_{L^\infty} < \delta$, then $\forall \delta > 0$, $\exists \varepsilon > 0$ such that
  $$\left(\frac{\varepsilon}{3}\right)^4 + 2\varepsilon^2 < \delta,$$

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- From the bound above, we have

$$V_0(t) + V_0'(t) \leq -\varepsilon^2 e^t,$$

  hence $|V_0(t_0) + V_0'(t_0)| \geq 2$ for $t_0 := \log(2) - 2\log(\varepsilon)$
  $\Rightarrow |V_0'(t_0)| > 1$.

# Remarks

1. Instability of peakons with respect to peaked perturbations is consistent with local well-posedness for $u_0 \in H^1 \cap W^{1,\infty}$ and wave breaking in a finite time: $u_x(t,x) \to -\infty$ at some $x \in \mathbb{R}$.
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2. By means of characteristics, it follows that if $v_0 \in C^1(\mathbb{R})$, then $v(t,\cdot) \notin C^1(\mathbb{R})$ for $t > 0$ because of the single peak at $x = \xi(t)$.

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3. Since $v_0(0) + v_0'(0) < 0$ for instability, the unstable solution actually breaks in a finite time [L. Brandolese (2014)].

4. The same instability can be detected in the linearized equation

$$\frac{d}{dt}(V_0 + V_0') = V_0 + V_0',$$

from which $V_0(t) + V_0'(t) = e^t \left[V_0(0) + V_0'(0)\right]$.

# Linearized instability

Consider the linearized equation at the single peakon:

$$\begin{cases} v_t = (1-\varphi)v_x + \varphi w, \\ v|_{t=0} = v_0, \end{cases}$$

where $w(t,x) = \int_0^x v(t,y)dy$.

**Theorem (F. Natali–D.P. (2019))**

*For every $v_0 \in H^1$, there exists a unique global solution $v \in C(\mathbb{R}, H^1)$ satisfying*

$$\begin{aligned} \|v(t,\cdot)\|^2_{H^1(0,\infty)} &= \|v_0\|^2_{H^1(0,\infty)} \\ &+2(e^t-1)\int_0^\infty \varphi(s)\left([v_0(s)]^2 + \frac{1}{2}[v_0'(s)]^2\right)ds \end{aligned}$$

Linear instability in $H^1$ contradicts orbital stability of peakons in $H^1$!

# Summary

1. Global solutions and wave breaking in the Ostrovsky equation

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- *Peaked* wave is spectrally and linearly *unstable*.

2. Global solutions and breaking in the Camassa–Holm equation

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**Thank you! Questions ???**