# Nonlinear Schrödinger equation on a periodic graph

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# Summary

# Introduction: periodic potentials

Let us consider again the nonlinear Schrödinger (Gross–Pitaevskii) equation

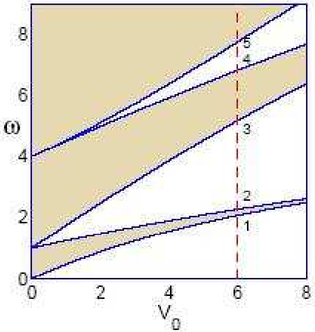
$$iu_t = -u_{xx} + V(x)u \pm |u|^2 u,$$

with a periodic potential, e.g. $V(x) = V_0 \sin^2(x)$.

Stationary solutions $u(x,t) = \phi(x)e^{-i\omega t}$ with $\omega \in \mathbb{R}$ satisfy a stationary Schrödinger equation with a periodic potential

$$\omega\phi = -\phi_{xx} + V(x)\phi \pm |\phi|^2\phi$$

Spectrum of $L = -\partial_x^2 + V(x)$ for $V(x) = V_0 \sin^2(x)$ and $N = 1$:



## Floquet–Bloch spectrum

The spectral problem with a bounded $2\pi$-periodic potential $V$,

$$\omega W = -\partial_x^2 W + V(x)W, \quad x \in \mathbb{R},$$

has a purely continuous spectrum, which can be found by using Bloch waves

$$W(x) = e^{i\ell x} f(\ell, x), \quad \ell,\ x \in \mathbb{R},$$

where $f(\ell, \cdot)$ is a $2\pi$-periodic function for every $\ell \in \mathbb{R}$. Since these functions satisfy the continuation conditions

$$f(\ell, x) = f(\ell, x + 2\pi), \quad f(\ell, x) = f(\ell + 1, x)e^{ix}, \quad \ell,\ x \in \mathbb{R},$$

we can restrict the definition of $f(\ell, x)$ to $x \in \mathbb{T}_{2\pi} = \mathbb{R}/(2\pi\mathbb{Z})$ and $\ell \in \mathbb{T}_1 = \mathbb{R}/\mathbb{Z}$.

For a fixed $\ell \in \mathbb{T}_1$, the Bloch waves are found from the periodic spectral problem,

$$-(\partial_x + i\ell)^2 f + V(x)f = \omega(\ell)f, \quad x \in \mathbb{T}_{2\pi}.$$

There exists a Schauder basis $\{f^{(m)}(\ell, \cdot)\}_{m\in\mathbb{N}}$ in $L^2(0, 2\pi)$
for an increasing sequence of eigenvalues $\{\omega^{(m)}(\ell)\}_{m\in\mathbb{N}}$.

# Homogenization of the NLS equation

The NLS equation with a bounded periodic potential $V$,

$$iu_t = -u_{xx} + V(x)u \pm |u|^2 u,$$

can be reduced to a homogeneous NLS equation

$$i\partial_T A = -\frac{1}{2}\partial_\ell^2 \omega^{(m_0)}(\ell_0)\partial_X^2 A \pm \nu |A|^2 A, \quad \nu = \frac{\|f^{(m_0)}(\ell_0,\cdot)\|_{L^4_{\rm per}}^4}{\|f^{(m_0)}(\ell_0,\cdot)\|_{L^2_{\rm per}}^2}$$

## Theorem (Schneider–Uecker, 2006; Dohnal, 2008; Ilan–Weinstein, 2010)

*Fix $m_0 \in \mathbb{N}$, $\ell_0 \in \mathbb{T}_1$, and assume $\omega^{(m)}(\ell_0) \neq \omega^{(m_0)}(\ell_0)$ for every $m \neq m_0$. Then, for every $C_0 > 0$ and $T_0 > 0$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all solutions $A \in C(\mathbb{R}, H^3(\mathbb{R}))$ of the homogeneous NLS equation with*

$$\sup_{T\in[0,T_0]} \|A(T,\cdot)\|_{H^3} \leq C_0$$

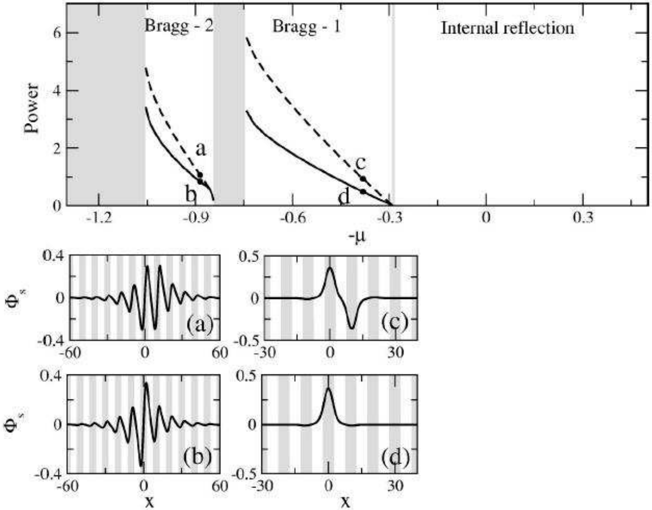
*and for all $\varepsilon \in (0,\varepsilon_0)$, there are solutions $u \in C([0,T_0/\varepsilon^2], L^\infty(\mathbb{R}))$ of the periodic NLS equation satisfying the bound*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} \left| u(t,x) - \varepsilon A(\varepsilon^2 t, \varepsilon(x - c_{\rm gr}t)) f^{(m_0)}(\ell_0,x) e^{i\ell_0 x} e^{-i\omega^{(m_0)}(\ell_0)t} \right| \leq C\varepsilon^{3/2}.$$

## Application of the NLS equation to existence of nonlinear bound states

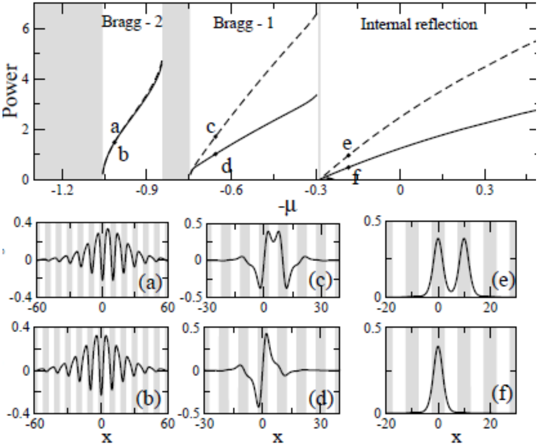
In the defocusing case, the nonlinear bound states bifurcate if $\partial_\ell^2 \omega^{(m_0)}(\ell_0) < 0$. In the focusing case, the nonlinear bound states bifurcate if $\partial_\ell^2 \omega^{(m_0)}(\ell_0) > 0$.

For $V(x) = V_0 \sin^2(x)$ and the defocusing case, the bifurcation diagram is

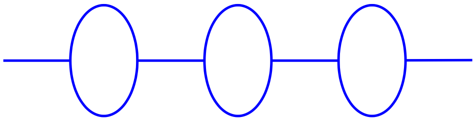


## Application of the NLS equation to existence of nonlinear bound states

For $V(x) = V_0 \sin^2(x)$ and the focusing case, the bifurcation diagram is



# Periodic Graph



Let the periodic graph $\Gamma$ consist of the circles of the normalized length $2\pi$ and the horizontal links of the length $L$. Writing the periodic graph as

$$\Gamma = \oplus_{n\in\mathbb{Z}}\Gamma_n, \qquad \text{with} \qquad \Gamma_n = \Gamma_{n,0} \oplus \Gamma_{n,+} \oplus \Gamma_{n,-},$$

we parameterize $\Gamma_{n,0} := [nP, nP + L]$ and $\Gamma_{n,\pm} := [nP + L, (n+1)P]$, where $P = L + \pi$ is the graph period.

The NLS equation on the periodic graph $\Gamma$,

$$i\partial_t u + \partial_x^2 u + |u|^2 u = 0, \quad t \in \mathbb{R}, \quad x \in \Gamma, \tag{1}$$

subject to the Kirchhoff boundary conditions at the vertices.

# Motivations

- Understand differences between analysis of bounded periodic potentials and of singularities related to the periodic graph.
- Study homogenizations of the NLS equation on the periodic graph.
- Construct nonlinear bound states and the ground state on the periodic graph.

S. Gilg, D.P., and G. Schneider, "Validity of the NLS approximation for periodic quantum graphs" (2016)
D.P. and G. Schneider, arXiv: 1603.05463

# Linear spectral problem

The spectral problem with a bounded $2\pi$-periodic potential $V$,

$$\lambda w = -\partial_x^2 w, \quad x \in \Gamma,$$

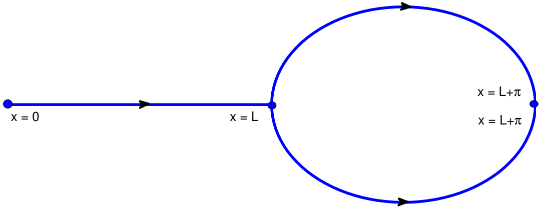
subject to the Kirchhoff boundary conditions for $n \in \mathbb{Z}$,

$$\begin{cases} w_{n,0}(nP+L) = w_{n,+}(nP+L) = w_{n,-}(nP+L), \\ w_{n+1,0}((n+1)P) = w_{n,+}((n+1)P) = w_{n,-}((n+1)P), \end{cases}$$

and

$$\begin{cases} \partial_x w_{n,0}(nP+L) = \partial_x w_{n,+}(nP+L) + \partial_x w_{n,-}(nP+L), \\ \partial_x w_{n+1,0}((n+1)P) = \partial_x w_{n,+}((n+1)P) + \partial_x w_{n,-}((n+1)P). \end{cases}$$

E. Korotyaev and I. Lobanov, Ann. Henri Poincare **8** (2007), 1151
P. Kuchment and O. Post, Commun Math. Phys. **275** (2007), 805



# Decomposition of the spectrum on $\Gamma$

### Lemma

*The linear operator $-\partial_x^2 : \mathcal{D}(\Gamma) \to L^2(\Gamma)$ is self-adjoint. Its spectrum $\sigma(-\partial_x^2)$ is positive and consists of two parts.*

Integrating by parts with Kirchhoff boundary conditions, we have

$$\lambda \|w\|^2_{L^2(\Gamma)} = \|\partial_x w\|^2_{L^2(\Gamma)} \geq 0.$$

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$$\lambda \|w\|^2_{L^2(\Gamma)} = \|\partial_x w\|^2_{L^2(\Gamma)} \geq 0.$$

The first part of $\sigma(-\partial_x^2)$ corresponds to the eigenfunctions of the form

$$\left\{ \begin{array}{ll} w_{n,0}(x) = 0, & x \in [nP, nP+L], \\ w_{n,+}(x) = -w_{n,-}(x), & x \in [nP+L, (n+1)P], \end{array} \right. \qquad n \in \mathbb{Z}.$$

Clearly, $\lambda = m^2$, $m \in \mathbb{N}$ is an eigenvalue of infinite multiplicity with the eigenfunction $w_{n,\pm}(x) = \pm\delta_{n,k} \sin[m(x - 2\pi n)]$, $k \in \mathbb{Z}$.

The second part of $\sigma(-\partial_x^2)$ corresponds to the eigenfunctions of the form

$$w_{n,+}(x) = w_{n,-}(x), \quad x \in [nP+L, (n+1)P], \quad n \in \mathbb{Z}.$$

# Construction of symmetric eigenfunctions

Let us parameterize the spectral parameter $\lambda = \omega^2$. Then, solutions of ODEs are found in terms of the boundary conditions:

$$\begin{cases} w_{n,0}(x) = a_n \cos(\omega(x - nP)) + b_n \sin(\omega(x - nP)), & x \in [nP, nP + L], \\ w_{n,\pm}(x) = c_n \cos(\omega(x - nP - L)) + d_n \sin(\omega(x - nP - L)), & x \in [nP + L, (n+1)P], \end{cases}$$

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Kirchhoff boundary conditions yield

$$\begin{cases} c_n = a_n \cos(\omega L) + b_n \sin(\omega L), \\ 2d_n = -a_n \sin(\omega L) + b_n \cos(\omega L), \end{cases}$$

and

$$\begin{cases} a_{n+1} = c_n \cos(\omega \pi) + d_n \sin(\omega \pi), \\ b_{n+1} = -2c_n \sin(\omega \pi) + 2d_n \cos(\omega \pi). \end{cases}$$

The monodromy matrix

$$M(\omega) := \begin{bmatrix} \cos(\omega\pi) & \sin(\omega\pi) \\ -2\sin(\omega\pi) & 2\cos(\omega\pi) \end{bmatrix} \begin{bmatrix} \cos(\omega L) & \sin(\omega L) \\ -\frac{1}{2}\sin(\omega L) & \frac{1}{2}\cos(\omega L) \end{bmatrix}$$

satisfies $\det(M) = 1$ and $\mathrm{tr}(M) = 2\cos(\omega\pi)\cos(\omega L) - \frac{5}{2}\sin(\omega\pi)\sin(\omega L)$.

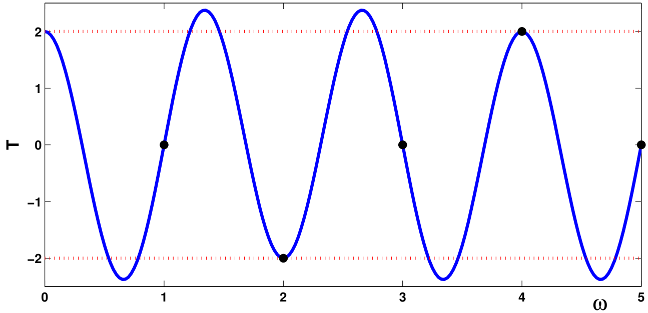
# The symmetric part of the spectrum

Trace of the monodromy matrix:

$$T(\omega) = 2\cos(\omega\pi)\cos(\omega L) - \frac{5}{2}\sin(\omega\pi)\sin(\omega L) \in [-2, 2].$$

Note that $T(m) = 2(-1)^m \cos(mL) \in [-2, 2]$ for every $m \in \mathbb{N}$.

*The spectrum $\sigma(-\partial_x^2)$ in $L^2(\Gamma)$ consists of eigenvalues $\{m^2\}_{m\in\mathbb{N}}$ of infinite multiplicity and a countable set of spectral bands $\{\sigma_k\}_{k\in\mathbb{N}}$. Moreover,*
$m^2 \in \cup_{k\in\mathbb{N}}\sigma_k$ for every $m \in \mathbb{N}$.



# Floquet–Bloch spectrum

For simplicity, take $L = \pi$ and define the Bloch waves

$$W(x) = e^{i\ell x} f(\ell, x), \quad \ell, \ x \in \mathbb{R},$$

where $f(\ell, \cdot) = (f_0, f_+, f_-)(\ell, \cdot)$ is a $2\pi$-periodic function for every $\ell \in \mathbb{R}$ satisfying the $\ell$-dependent Kirchhoff boundary conditions

$$\left\{ \begin{array}{l} f_0(\ell, \pi) = f_+(\ell, \pi) = f_-(\ell, \pi), \\ f_0(\ell, 0) = f_+(\ell, 2\pi) = f_-(\ell, 2\pi) \end{array} \right.$$

and

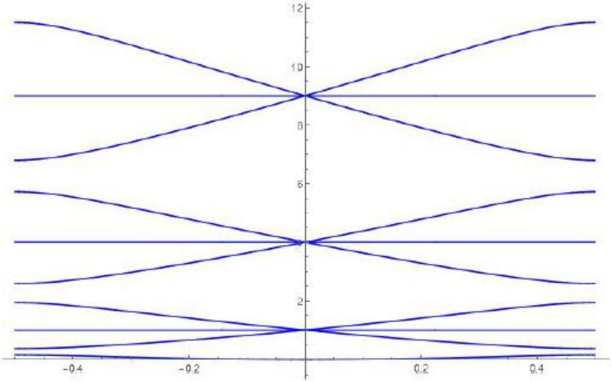
$$\left\{ \begin{array}{l} (\partial_x + i\ell) f_0(\ell, \pi) = (\partial_x + i\ell) f_+(\ell, \pi) + (\partial_x + i\ell) f_-(\ell, \pi), \\ (\partial_x + i\ell) f_0(\ell, 0) = (\partial_x + i\ell) f_+(\ell, 2\pi) + (\partial_x + i\ell) f_-(\ell, 2\pi). \end{array} \right.$$

Note that $e^{i\ell x}$ is defined for $x \in \mathbb{R}$ but is not defined for $x \in \Gamma$.

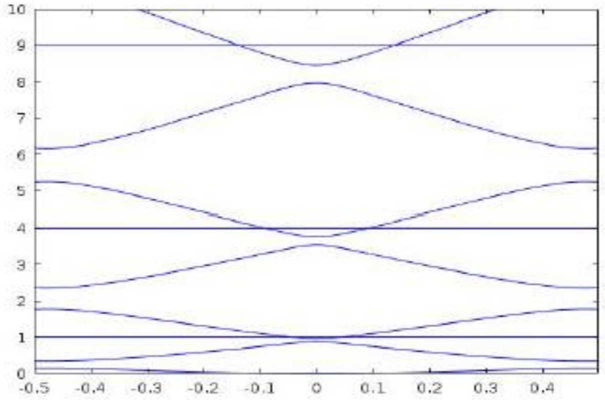
For a fixed $\ell \in \mathbb{T}_1$, the Bloch waves are found from the periodic spectral problem,

$$-(\partial_x + i\ell)^2 f = \omega(\ell) f, \quad x \in \mathbb{T}_{2\pi}.$$

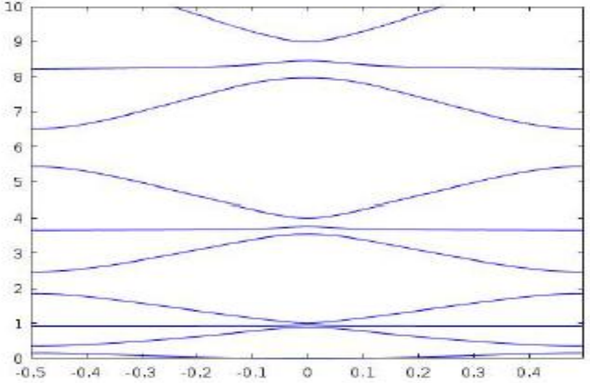
# Numerical approximation of spectral bands: $L = \pi$



# Numerical approximation of spectral bands: $L > \pi$



# Numerical approximation of spectral bands: semi-rings of different lengths



# The NLS equation on the periodic graph

Define piecewise functions for solutions of the NLS equation on the periodic graph $\Gamma$:

$$u_0(x) = \cup_{n\in\mathbb{Z}} \begin{cases} u_{n,0}(x), & x \in I_{n,0} = [2\pi n, 2\pi n + \pi], \\ 0, & \text{elsewhere}, \end{cases}$$

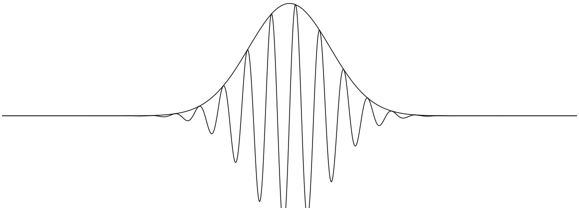
and

$$u_\pm(x) = \cup_{n\in\mathbb{Z}} \begin{cases} u_{n,\pm}(x), & x \in I_{n,\pm} = [2\pi n + \pi, 2\pi(n+1)], \\ 0, & \text{elsewhere}. \end{cases}$$

The NLS equation on the periodic graph $\Gamma$ can be written as the evolutionary problem for $U = (u_0,\ u_+, u_-)$:

$$i\partial_t U + \partial_x^2 U + |U|^2 U = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\},$$

subject to the Kirchhoff boundary conditions at the vertex points.



## Homogeneous NLS equation

The asymptotic solution in the form

$$U(t,x) = \varepsilon A(T,X) f^{(m_0)}(\ell_0, x) e^{i\ell_0 x} e^{-i\omega^{(m_0)}(\ell_0) t} + \text{higher-order terms},$$

with $T = \varepsilon^2 t$ and $X = \varepsilon(x - c_g t)$ satisfies the homogeneous NLS equation

$$i\partial_T A + \frac{1}{2}\partial_\ell^2 \omega^{(m_0)}(\ell_0)\partial_X^2 A + \nu |A|^2 A = 0, \quad \nu = \frac{\|f^{(m_0)}(\ell_0,\cdot)\|_{L^4_{\text{per}}}^4}{\|f^{(m_0)}(\ell_0,\cdot)\|_{L^2_{\text{per}}}^2}.$$

### Theorem (Gilg–Schneider-P, 2016)

*Fix $m_0 \in \mathbb{N}$, $\ell_0 \in \mathbb{T}_1$, and assume $\omega^{(m)}(\ell_0) \neq \omega^{(m_0)}(\ell_0)$ for every $m \neq m_0$. Then, for every $C_0 > 0$ and $T_0 > 0$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all solutions $A \in C(\mathbb{R}, H^3(\mathbb{R}))$ of the homogeneous NLS equation with*

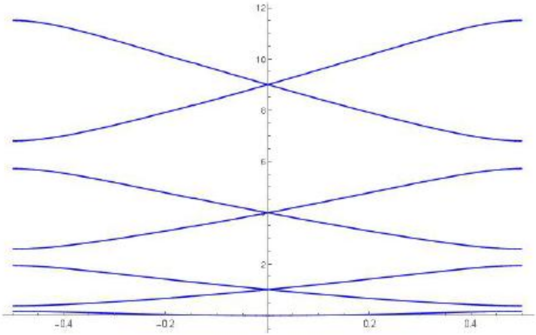
$$\sup_{T \in [0,T_0]} \|A(T,\cdot)\|_{H^3} \leq C_0$$

*and for all $\varepsilon \in (0, \varepsilon_0)$, there are solutions $U \in C([0, T_0/\varepsilon^2], L^\infty(\mathbb{R}))$ to the NLS equation on the periodic graph $\Gamma$ satisfying the bound*

$$\sup_{t \in [0,T_0/\varepsilon^2]} \sup_{x \in \mathbb{R}} \left| U(t,x) - \varepsilon A(T,X) f^{(m_0)}(\ell_0, x) e^{i\ell_0 x} e^{-i\omega^{(m_0)}(\ell_0) t} \right| \leq C\varepsilon^{3/2}.$$

# Extension to the Dirac equations

The symmetry constraints $u_{n,+}(t,x) = u_{n,-}(t,x)$ is invariant under the time evolution of the NLS equation on the periodic graph $\Gamma$. Under the constraints, the spectral bands feature Dirac points and no flat bands.



# Homogeneous Dirac equations

The asymptotic solution in the form

$$U(t,x) = \varepsilon A_+(T,X) f^+(0,x) e^{-i\omega^+(0)t} + \varepsilon A_-(T,X) f^-(0,x) e^{-i\omega^-(0)t} + \text{higher-order terms},$$

with $T = \varepsilon^2 t$ and $X = \varepsilon^2 x$ satisfies the homogeneous Dirac equations

$$\begin{cases} i\partial_T A_+ + i\partial_\ell \omega^+(0)\partial_X A_+ + \sum_{j_1,j_2,j_3\in\{+,-\}} \nu^+_{j_1 j_2 j_3} A_{j_1} A_{j_2} \overline{A_{j_3}} = 0, \\ i\partial_T A_- + i\partial_\ell \omega^-(0)\partial_X A_- + \sum_{j_1,j_2,j_3\in\{+,-\}} \nu^-_{j_1 j_2 j_3} A_{j_1} A_{j_2} \overline{A_{j_3}} = 0, \end{cases}$$

### Theorem (Gilg–Schneider-P, 2016)

*For every $C_0 > 0$ and $T_0 > 0$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all solutions $A_\pm \in C(\mathbb{R}, H^2(\mathbb{R}))$ of the Dirac equations with*

$$\sup_{T\in[0,T_0]} \|A_\pm(T,\cdot)\|_{H^2} \leq C_0$$

*and for all $\varepsilon \in (0,\varepsilon_0)$, there are solutions $U \in C([0,T_0/\varepsilon^2], L^\infty(\mathbb{R}))$ of the NLS equation on the periodic graph $\Gamma$ satisfying the bound*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} |U(t,x) - \varepsilon\Psi_{\text{dirac}}(t,x)| \leq C\varepsilon^{3/2}.$$

# Function spaces

The operator $L = -\partial_x^2$ is considered in the space

$$\mathcal{L}^2 = \{U = (u_0, u_+, u_-) \in (L^2(\mathbb{R}))^3 : \quad \text{supp}(u_{n,j}) = I_{n,j}, \quad n \in \mathbb{Z}, \;\; j \in \{0, +, -\}\}$$

with the domain of definition

$$\mathcal{H}^2 \quad := \quad \{U \in \mathcal{L}^2 : \;\; u_{n,j} \in H^2(I_{n,j}), \quad n \in \mathbb{Z}, \;\; j \in \{0, +, -\} \quad \text{Kirchhoff BCs}\}.$$

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- The space $\mathcal{H}^2$ is closed under pointwise multiplication.
- The skew symmetric operator $-iL$ defines a unitary semi-group $(e^{-iLt})_{t\in\mathbb{R}}$ in $\mathcal{L}^2$.
- There exists a positive constant $C_L$ such that

  $$\|e^{-iLt}U\|_{\mathcal{H}^2} \leq C_L \|U\|_{\mathcal{H}^2}$$

  for every $U \in \mathcal{H}^2$ and every $t \in \mathbb{R}$.
- There exists a unique local solution $U \in C([-T_0, T_0], \mathcal{H}^2)$ to the NLS equation on the periodic graph $\Gamma$.

# Bloch transform on the real line

For a function $f : \mathbb{R} \to \mathbb{C}$, Bloch transform is defined by

$$\widetilde{f}(\ell, x) = (\mathcal{T} f)(\ell, x) = \sum_{j \in \mathbb{Z}} e^{ijx} \widehat{f}(\ell + j),$$

where $\widehat{f}(\xi) = (\mathcal{F} f)(\xi)$, $\xi \in \mathbb{R}$ is the Fourier transform of $f$. The inverse transform is

$$f(x) = (\mathcal{T}^{-1} \widetilde{f})(x) = \int_{-1/2}^{1/2} e^{i\ell x} \widetilde{f}(\ell, x) d\ell.$$

By construction, $\widetilde{f}(\ell, x)$ is extended from $(\ell, x) \in \mathbb{T}_1 \times \mathbb{T}_{2\pi}$ to $(\ell, x) \in \mathbb{R} \times \mathbb{R}$ according to the continuation conditions:

$$\widetilde{f}(\ell, x) = \widetilde{f}(\ell, x + 2\pi) \quad \text{and} \quad \widetilde{f}(\ell, x) = \widetilde{f}(\ell + 1, x) e^{ix}.$$

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- $\mathcal{T}$ is an isomorphism between $H^s(\mathbb{R})$ and $L^2(\mathbb{T}_1, H^s(\mathbb{T}_{2\pi}))$.
- Multiplication in $x$ space corresponds to convolution in Bloch space.
- If $\chi : \mathbb{R} \to \mathbb{R}$ is $2\pi$ periodic, then

  $$\mathcal{T}(\chi u)(\ell, x) = \chi(x)(\mathcal{T} u)(\ell, x).$$

  In particular, if $\chi_j$ are periodic cut-off functions in $I_j$, $j \in \{0, +, -\}$, then $\mathcal{T}(u_j)(\ell, x) = \mathcal{T}(\chi_j u_j)(\ell, x) = \chi_j(x)(\mathcal{T} u_j)(\ell, x)$.

# Function spaces for Bloch transforms

The operator $\tilde{L}(\ell) = -(\partial_x + i\ell)^2$ is self-adjoint in the space

$$L^2_\Gamma := \{ \widetilde{U} = (\widetilde{u}_0, \widetilde{u}_+, \widetilde{u}_-) \in (L^2(\mathbb{T}_{2\pi}))^3 : \quad \mathrm{supp}(\widetilde{u}_j) = I_{0,j}, \quad j \in \{0, +, -\}\}$$

with the domain of definition

$$H^2_\Gamma := \{\widetilde{U} \in L^2_\Gamma : \ \widetilde{u}_j \in H^2(I_{0,j}), \quad j \in \{0, +, -\}, \quad \text{Kirchhoff BCs}\}.$$

In Bloch space, we work with functions in $L^2(\mathbb{T}_1, L^2_\Gamma)$. Local well-posedness applies to smooth functions in $\widetilde{\mathcal{H}}^2 = L^2(\mathbb{T}_1, H^2_\Gamma)$.

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The operator $\tilde{L}(\ell) = -(\partial_x + i\ell)^2$ is self-adjoint in the space

$$L^2_\Gamma := \{ \, \widetilde{U} = (\widetilde{u}_0, \widetilde{u}_+, \widetilde{u}_-) \in (L^2(\mathbb{T}_{2\pi}))^3 : \quad \text{supp}(\widetilde{u}_j) = I_{0,j}, \quad j \in \{0,+,-\}\}$$

with the domain of definition

$$H^2_\Gamma := \{\widetilde{U} \in L^2_\Gamma : \ \ \widetilde{u}_j \in H^2(I_{0,j}), \quad j \in \{0,+,-\}, \quad \text{Kirchhoff BCs}\}.$$

In Bloch space, we work with functions in $L^2(\mathbb{T}_1, L^2_\Gamma)$. Local well-posedness applies to smooth functions in $\widetilde{\mathcal{H}}^2 = L^2(\mathbb{T}_1, H^2_\Gamma)$.

**Key Lemma:** *The Bloch transform $\mathcal{T}$ is an isomorphism between $\mathcal{H}^2$ and $\widetilde{\mathcal{H}}^2$.*

- Extend a piecewise $H^2$ function $u_0$ to $u_{0,ext} \in H^2(\mathbb{R})$.
- By Bloch transform on the real line, $\mathcal{T}(u_{0,ext}) \in L^2(\mathbb{T}_1, H^2(\mathbb{T}_{2\pi}))$.
- Compact support persists as $\widetilde{u}_0 = \mathcal{T}(u_0) = \mathcal{T}(\chi_0 u_{0,ext}) = \chi_0 \mathcal{T}(u_{0,ext})$.
- From the properties of $\mathcal{T}(u_{0,ext})$, we obtain $\widetilde{u}_0 \in L^2(\mathbb{T}_1, H^2(I_{0,0}))$.

# Rest of the proof

- Bloch transform for the NLS equation on the periodic graph $\Gamma$.
- Decomposition of solutions in the Bloch space

$$\widetilde{U}(t,\ell,x) = \widetilde{V}(t,\ell)f^{(m_0)}(\ell,x) + \widetilde{U}^{\perp}(t,\ell,x)$$

- Approximation of the principal part of the solution

$$\widetilde{V}_{\mathrm{app}}(t,\ell) = \widetilde{A}\left(\varepsilon^2 t, \frac{\ell-\ell_0}{\varepsilon}\right) e^{-i\omega^{(m_0)}(\ell_0)t} e^{-i\partial_\ell \omega^{(m_0)}(\ell_0)(\ell-\ell_0)t}.$$

  As $\varepsilon \to 0$, $\widetilde{A}$ satisfies the homogeneous NLS equation in the Fourier space.
- A near-identity transformation for $\widetilde{U}^{\perp}(t,\ell,x)$ with a suitable chosen approximation $\widetilde{U}^{\perp}_{\mathrm{app}}(t,\ell,x)$.
- Estimates of residual terms in Bloch spaces.
- Estimates of the approximation between the Fourier space and Bloch space.
- Estimates of the error term in time evolution with Gronwall's inequality.

## Homogeneous NLS equation

The asymptotic solution in the form

$$U(t,x) = \varepsilon A(T,X) f^{(m_0)}(\ell_0,x) e^{i\ell_0 x} e^{-i\omega^{(m_0)}(\ell_0)t} + \text{higher-order terms},$$

with $T = \varepsilon^2 t$ and $X = \varepsilon(x - c_g t)$ satisfies the homogeneous NLS equation

$$i\partial_T A + \frac{1}{2}\partial_\ell^2 \omega^{(m_0)}(\ell_0)\partial_X^2 A + \nu |A|^2 A = 0, \quad \nu = \frac{\|f^{(m_0)}(\ell_0,\cdot)\|^4_{L^4_{\mathrm{per}}}}{\|f^{(m_0)}(\ell_0,\cdot)\|^2_{L^2_{\mathrm{per}}}}.$$

### Theorem (Gilg–Schneider-P, 2016)

*Fix $m_0 \in \mathbb{N}$, $\ell_0 \in \mathbb{T}_1$, and assume $\omega^{(m)}(\ell_0) \neq \omega^{(m_0)}(\ell_0)$ for every $m \neq m_0$. Then, for every $C_0 > 0$ and $T_0 > 0$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all solutions $A \in C(\mathbb{R}, H^3(\mathbb{R}))$ of the homogeneous NLS equation with*

$$\sup_{T \in [0,T_0]} \|A(T,\cdot)\|_{H^3} \leq C_0$$

*and for all $\varepsilon \in (0,\varepsilon_0)$, there are solutions $U \in C([0,T_0/\varepsilon^2], L^\infty(\mathbb{R}))$ to the NLS equation on the periodic graph $\Gamma$ satisfying the bound*

$$\sup_{t \in [0,T_0/\varepsilon^2]} \sup_{x \in \mathbb{R}} \left| U(t,x) - \varepsilon A(T,X) f^{(m_0)}(\ell_0,x) e^{i\ell_0 x} e^{-i\omega^{(m_0)}(\ell_0)t} \right| \leq C\varepsilon^{3/2}.$$

# Bifurcations of nonlinear bound states

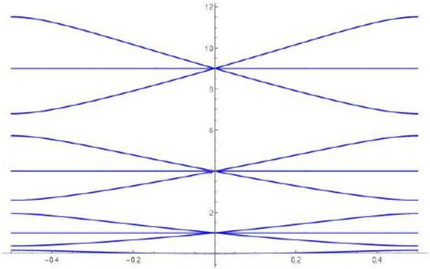
The stationary NLS equation on the periodic graph $\Gamma$:

$$-\partial_x^2 \phi - 2|\phi|^2 \phi = \Lambda \phi \qquad \Lambda \in \mathbb{R}, \quad \phi(x) : \Gamma \to \mathbb{R}.$$

The effective homogeneous NLS equation on the real line

$$-\frac{1}{2} \partial_\ell^2 \omega^{(m_0)}(\ell_0) \partial_X^2 A - \nu |A|^2 A = \Omega A, \quad A(X) : \mathbb{R} \to \mathbb{R}.$$

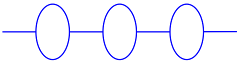
The stationary reduction is satisfied if $\partial_\ell \omega^{(m_0)}(\ell_0) = 0$.



# Nonlinear bound states on the periodic graph

Stable bound states bifurcate from the bottom of the linear spectrum at $\Lambda = 0$:

$$-\partial_x^2 \phi - 2|\phi|^2 \phi = \Lambda \phi \qquad \Lambda \in \mathbb{R}, \quad \phi(x) : \Gamma \to \mathbb{R}.$$



## Theorem

*There are positive constants $\Lambda_0$ and $C_0$ such that for every $\Lambda \in (-\Lambda_0, 0)$, there exist two bound states $\phi \in \mathcal{D}(\Gamma)$ (up to the discrete translational invariance) s.t. either*

$$\phi(x - L/2) = \phi(L/2 - x), \quad x \in \Gamma$$

*or*

$$\phi(x - L - \pi/2) = \phi(L + \pi/2 - x), \quad x \in \Gamma.$$

*Moreover, it is true for both bound states that*

(i) *$\phi$ is symmetric in upper and lower semicircles of $\Gamma$,*

(ii) *$\phi(x) > 0$ for every $x \in \Gamma$,*

(iii) *$\phi(x) \to 0$ as $|x| \to \infty$ exponentially fast.*

# Numerical approximations of the bound states with $L = \pi$

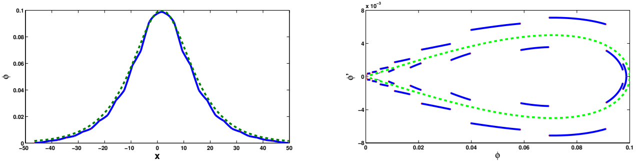



Figure : Profile of the numerically generated bound state on $(x, \phi)$ plane (left) and on $(\phi, \phi')$ plane (right). The red dots show the break points on the periodic graph $\Gamma$. The green dashed line shows the NLS soliton on the infinite line.

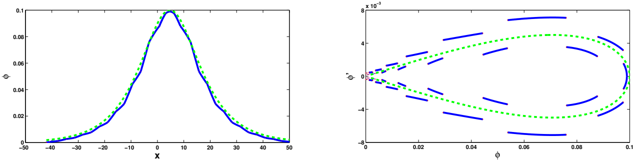



Figure : The same but for the other bound state.

# Discrete homogenization method

We set $\Lambda = -\epsilon^2$ and consider the limit $\epsilon \to 0$.

For every $(a, b) \in \mathbb{R}^2$ and every $\epsilon \in \mathbb{R}$, there is a unique solution $\psi(x; a, b, \epsilon) \in C^\infty(\mathbb{R})$ of the initial-value problem:

$$\begin{cases} \partial_x^2 \psi - \epsilon^2 \psi + 2|\psi|^2 \psi = 0, & x \in \mathbb{R}, \\ \psi(0) = a, \\ \partial_x \psi(0) = b, \end{cases}$$

For each $\Gamma_{n,0}$ and $\Gamma_{n,\pm}$, the solution can be defined in the implicit form:

$$\phi_{n,0}(x) = \psi(x - nP; a_n, b_n, \epsilon), \quad \phi_{n,\pm}(x) = \psi(x - nP - L; c_n, d_n, \epsilon).$$

Kirchhoff boundary conditions produces a two-dimensional map:

$$\begin{cases} a_{n+1} = \psi(\pi; c_n, d_n, \epsilon), \\ b_{n+1} = 2\partial_x \psi(\pi; c_n, d_n, \epsilon), \end{cases} \quad \begin{cases} c_n = \psi(L; a_n, b_n, \epsilon), \\ 2d_n = \partial_x \psi(L; a_n, b_n, \epsilon), \end{cases} \tag{2}$$

The nonlinear discrete map generalizes the linear transfer matrix method.

# Approximate continuous solution

In the limit $\epsilon \to 0$, expand solution $\psi(x; \epsilon\alpha, \epsilon^2\beta, \epsilon)$ in the power series in $\epsilon$. The two-dimensional map is now available in the perturbative form:

$$\begin{cases} \alpha_{n+1} = \alpha_n + \epsilon(L + \pi/2)\beta_n + \frac{1}{2}\epsilon^2(L^2 + \pi L + \pi^2)(1 - 2\alpha_n^2)\alpha_n + \mathcal{O}(\epsilon^3), \\ \beta_{n+1} = \beta_n + \epsilon(L + 2\pi)(1 - 2\alpha_n^2)\alpha_n + \frac{1}{4}\epsilon^2(2L^2 + 4L\pi + \pi^2)(1 - 6\alpha_n^2)\beta_n + \mathcal{O}(\epsilon^3). \end{cases}$$

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Approximate continuous solution:

$$\alpha_n = A(X+X_0), \quad \beta_n = B(X+X_0), \quad X = \epsilon n, \quad n \in \mathbb{Z},$$

where $X_0$ is arbitrary and $A, B$ satisfy the continuous limit

$$\begin{cases} A'(X) = (L+\pi/2)B(X), \\ B'(X) = (L+2\pi)(1-2A^2)A(X), \end{cases}$$

with the continuous NLS solitons

$$A(X) = \text{sech}(\nu X), \quad B(X) = -\mu \tanh(\nu X)\text{sech}(\nu X), \quad X \in \mathbb{R},$$

## Justification of the approximate continuous solution

**Key Lemma:** *For a given $f \in \ell^2(\mathbb{Z})$ satisfying the reversibility symmetry $f_n = f_{1-n}$ for every $n \in \mathbb{Z}$, consider solutions of the linearized difference equation*

$$-\frac{\alpha_{n+1} - 2\alpha_n + \alpha_{n-1}}{\epsilon^2} + \nu^2(1 - 6A^2(\epsilon n))\alpha_n = f_n, \quad n \in \mathbb{Z}.$$

*For sufficiently small $\epsilon > 0$, there exists a unique solution $\alpha \in \ell^2(\mathbb{Z})$ satisfying the reversibility symmetry $\alpha_n = \alpha_{1-n}$ for every $n \in \mathbb{Z}$. Moreover there is a positive $\epsilon$-independent constant $C$ such that*

$$\epsilon^{-1} \|\sigma_+\alpha - \alpha\|_{\ell^2} \le C\|f\|_{\ell^2}, \quad \|\alpha\|_{\ell^2} \le C\|f\|_{\ell^2},$$

*where $\sigma_+$ is the shift operator defined by $(\sigma_+\alpha)_n := \alpha_{n+1}$, $n \in \mathbb{Z}$.*

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- Translational parameter $X_0$ can be chosen to satisfy the reversibility symmetry.
- Two reversibility symmetries give two nonlinear bound states.
- The symmetry $\phi_+ = \phi_-$ holds by construction.
- Positivity and exponential decay are not obtained from this method.

# Positivity and exponential decay

The perturbative two-dimensional map:

$$\begin{cases} \alpha_{n+1} = \alpha_n + \epsilon(L + \pi/2)\beta_n + \frac{1}{2}\epsilon^2(L^2 + \pi L + \pi^2)(1 - 2\alpha_n^2)\alpha_n + \mathcal{O}(\epsilon^3), \\ \beta_{n+1} = \beta_n + \epsilon(L + 2\pi)(1 - 2\alpha_n^2)\alpha_n + \frac{1}{4}\epsilon^2(2L^2 + 4L\pi + \pi^2)(1 - 6\alpha_n^2)\beta_n + \mathcal{O}(\epsilon^3). \end{cases}$$

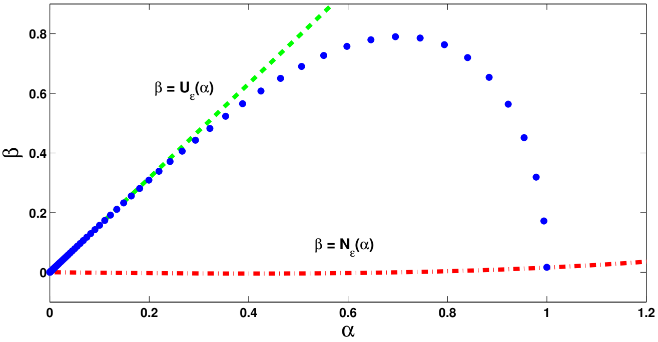
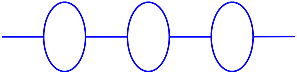



Figure : The plane $(\alpha, \beta)$, where the blue dots denote a sequence $\{\alpha_n, \beta_n\}_{n\in\mathbb{Z}}$, the green dashed line shows the unstable curve $\beta = \mathcal{U}_\epsilon(\alpha)$, and the red dash-dotted line shows the symmetry curve $\beta = \mathcal{N}_\epsilon(\alpha)$.

# Conclusion



For the periodic graph $\Gamma$, we have obtained the following results:

- We developed the Bloch transform on $\Gamma$ and justified homogenization of the NLS equation on $\Gamma$ with the homogeneous NLS or Dirac equations on the line.
- We approximated nonlinear bound states near the lowest spectral band by using NLS solitons.
- We used discrete maps and dynamical system methods to study linear spectrum of the periodic graph $\Gamma$ and the nonlinear bound states on $\Gamma$.
- Scattering and nonlinear dynamics on the periodic graph $\Gamma$ are still to be analyzed in some future.

**Thank you!**