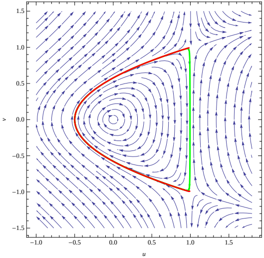
Stability of periodic waves in the reduced Ostrovsky equation

Dmitry Pelinovsky

Department of Mathematics, McMaster University, Canada



Joint work with Anna Geyer
(Delft University of Technology, Netherlands)

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The generalized *reduced Ostrovsky equation*

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where u is a real-valued function of (x,t) and $p \in \mathbb{N}$.

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- For $p = 1$, the equation arises as $\beta \to 0$ from the Ostrovsky equation

  $$(u_t + uu_x + \beta u_{xxx})_x = \gamma u$$

  derived in the context of long gravity waves in a rotating fluid, as a generalization of the KdV equation ($\gamma = 0$). [Ostrovsky, 1978]

- For $p = 2$, the equation arises from the modified equation

  $$(u_t + u^2 u_x + \beta u_{xxx})_x = \gamma u$$

  derived from Euler's equations in the context of internal waves [Grimshaw et al., 1998].

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- ▷ Solutions break in finite time for sufficiently large initial data. [Liu & P. & Sakovich 2009, 2010 for $p = 1$, $p = 2$.]

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- Solutions break in finite time for sufficiently large initial data.
  [Liu & P. & Sakovich 2009, 2010 for $p = 1$, $p = 2$.]
- Global solutions exist for sufficiently small initial data.
  [Stefanov et. al., 2010 for $p \geq 4$,
  P & Sakovich 2010 for $p = 2$,
  Grimshaw & P. 2014 for $p = 1$]

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  [Vakhnenko & Parkes, 1998], [Kraenkel & Leblond & Manna 2014]
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- For $p = 2$: the equation is different from the short-pulse equation derived from Maxwell's equations. [Schäfer & Wayne, 2004]

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- Part I: *Stability* of smooth periodic waves for arbitrary $p \in \mathbb{N}$.
- Part II: *Instability* of the limiting peaked periodic wave for $p = 1$.

Traveling wave solutions

We are interested in existence and stability of traveling wave solutions of the form

$$u(x,t) = U(x - ct),$$

where $z = x - ct$ is the travelling wave coordinate and $c > 0$ is the wave speed. The wave profile U is $2T$-periodic.

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The wave profile U satisfies the boundary-value problem

$$\frac{d}{dz}\left((c - U^p)\frac{dU}{dz}\right) + U(z) = 0, \qquad \left. \begin{array}{l} U(-T) = U(T), \\ U'(-T) = U'(T), \end{array} \right\} \quad \text{(ODE)}$$

where $\int_{-T}^{T} U(t)dt = 0$, i.e. the periodic waves have zero mean.

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Using $u(t,x) = U(z) + v(z)e^{\lambda t}$, where $z = x - ct$, the spectral stability problem for a perturbation of the wave profile U is given by

$$\boxed{\partial_z L v = \lambda v}$$

with the self-adjoint linear operator

$$L = P_0 \left(\partial_z^{-2} + c - U^p \right) P_0 : \ \dot{L}^2_{\rm per}(-T,T) \to \dot{L}^2_{\rm per}(-T,T).$$

Here $\dot{L}^2_{\rm per}$ denote the space of $L^2_{\rm per}$ functions with zero mean and $P_0 : L^2_{\rm per} \mapsto \dot{L}^2_{\rm per}$ is the projection operator that sets mean to zero.

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Definition

The travelling wave is *spectrally stable* with respect to co-periodic perturbations if the spectral problem $\partial_z L v = \lambda v$ with $v \in \dot{H}^1_{\rm per}(-T,T)$ has no eigenvalues $\lambda \notin i\mathbb{R}$.

Stability - course of action

- Construct a Lyapunov-type functional:

$$F[u] := H[u] + cQ[u],$$

where

$$\text{(energy)} \quad H[u] = -\frac{1}{2}\|\partial_x^{-1} u\|^2_{L^2_{\text{per}}} - \frac{1}{(p+1)(p+2)} \int_{-T}^{T} u^{p+2} dx$$

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- We will show that

a traveling wave U is a constrained minimizer of the energy $H[u]$ with fixed momentum $Q[u]$.

Stability - course of action

- The constraint of fixed momentum $Q[u] := \frac{1}{2}\|u\|^2_{L^2_{\rm per}} = q$ is equivalent to restricting the self-adjoint linear operator L to the subspace

$$U^\perp = \left\{ v \in \dot{L}^2_{\rm per}(-T,T) : \quad \langle U, v \rangle_{L^2_{\rm per}} = 0 \right\}$$

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Indeed,

$$\begin{aligned} 0 = Q[U+v] - Q[U] &= \frac{1}{2}\int_{-T}^{T} (U+v)^2 dz - \frac{1}{2}\int_{-T}^{T} U^2 dz \\ &= \int_{-T}^{T} U\, v\, dz + O(v^2) \\ &= \langle U, v\rangle. \end{aligned}$$

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states that [Haragus & Kapitula, 08]

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- **Result:** the smooth periodic wave U is stable. [Geyer & P., LMP '17]

Existence of periodic traveling waves

Let $c > 0$ and $p \in \mathbb{N}$. A function U is a smooth periodic solution of

$$\frac{d}{dz}\left((c - U^p)\frac{dU}{dz}\right) + U = 0 \qquad \text{(ODE)}$$

iff $(u, v) = (U, U')$ is a periodic orbit γ_E of the planar system

$$\begin{cases} u' = v, \\ v' = \dfrac{-u + pu^{p-1}v^2}{c - u^p}, \end{cases}$$

which has the first integral

$$E(u, v) = \frac{1}{2}(c - u^p)^2 v^2 + \frac{c}{2}u^2 - \frac{1}{p+2}u^{p+2}.$$

Note that $c - U(z)^p > 0$ for every z if U is smooth.

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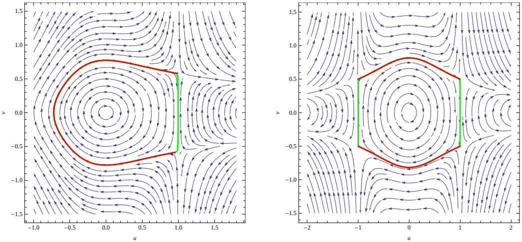
if and only if $(u, v) = (U, U')$ is a periodic orbit γ_E of the planar system with first integral $E(u, v) = \frac{1}{2}(c - u^p)^2 v^2 + \frac{c}{2}u^2 - \frac{1}{p+2}u^{p+2}$.



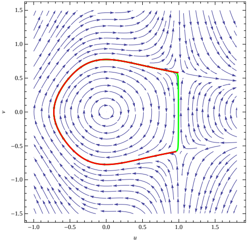
There exists a smooth family of periodic solutions $U \in \dot{H}^{\infty}_{\text{per}}$ of (ODE) parametrized by the energy $E \in (0, E_c)$.

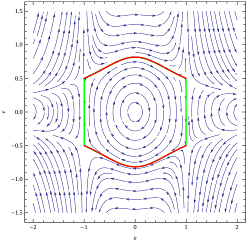
Monotonicity of energy-to-period map

For every $c > 0$ and $p \in \mathbb{N}$ the *period function*

$$T : (0, E_c) \longrightarrow \mathbb{R}^+, \quad E \longmapsto T(E) = \frac{1}{2} \int_{\gamma_E} \frac{du}{v},$$

is strictly monotonically decreasing: $T'(E) < 0$





Classical monotonicity criteria do not apply. [Chicone, Schaaf, 1980's]

Our proof is inspired by [Mañosas & Villadelprat, 2009].

Monotonicity of energy-to-period map $T(E) = \frac{1}{2} \int_{\gamma_E} \frac{du}{v}$

Recall the first integral

$$E(u,v) = B(u)v^2 + A(u), \quad B(u) := \frac{1}{2}(c - u^p)^2, \quad A(u) := \frac{c}{2}u^2 - \frac{1}{p+2}u^{p+2}.$$

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$$2E\,T(E) = \int_{\gamma_E} B(u) v du + \int_{\gamma_E} A(u) \frac{du}{v}.$$

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and choosing $g = \frac{2B}{A'}A$ we find

$$0 = \int_{\gamma_E} G(u) v \mathrm{d}u - \int_{\gamma_E} A \frac{\mathrm{d}u}{v}.$$

[Grau, Mañosas & Villadelprat, '11]

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$$T'(E) = -\frac{p}{4(2+p)E}\int_{\gamma_E} \frac{u^p}{(c-u^p)}\frac{du}{v} < 0.$$

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The period function is strictly monotone!

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This is true if the following two conditions hold:
[Vakhitov-Kolokolov, 1975], [Grillakis–Shatah–Strauss, 1987]

- L has exactly one negative eigenvalue,
  a simple zero eigenvalue with eigenvector $\partial_z U$,
  and the rest of its spectrum is positive and bounded away from 0
- $\langle L^{-1}U, U\rangle = -\frac{d}{dc}\|U\|^2_{L^2_{\rm per}(-T,T)} < 0$, where the period T is fixed.

We show that these conditions hold using the fact that the energy-to-period map $T(E)$ is strictly monotone.

Spectral properties of the operator L

Recall the self-adjoint linear operator

$$L = P_0 \left(\partial_z^{-2} + c - U^p \right) P_0 : \; \dot{L}^2_{\rm per}(-T,T) \to \dot{L}^2_{\rm per}(-T,T).$$

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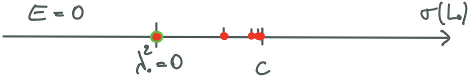
When $E \to 0$, then $U \to 0$, $T(E) \to T(0) = \sqrt{c}\pi$, and

$$L \to L_0 = P_0 \left(\partial_z^{-2} + c \right) P_0.$$

$\sigma(L_0) = \{c(1 - n^{-2}), n \in \mathbb{Z} \setminus \{0\}\}$ all eigenvalues are double.



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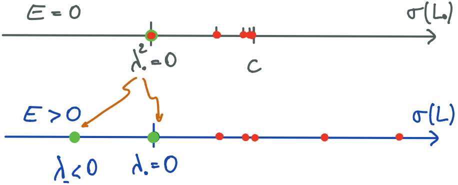

When $E > 0$ the double zero eigenvalue splits into a simple negative eigenvalue and a simple zero eigenvalue of L.

Spectral properties of the operator L

Consider the eigenvalue problem

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Zero eigenvalue $\lambda_0 = 0$:

- $\partial_z U$ is an eigenvector for λ_0: $L\partial_z U = 0$
- U_E is also a solution of the spectral equation for $\lambda_0 = 0$:

$$\partial_E(\text{ODE}) \iff U_E + \partial_z^2[(c - U^p)U_E] = 0$$

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- $\partial_z U$ is an eigenvector for λ_0: $L\partial_z U = 0$
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$$\partial_E(\text{ODE}) \iff U_E + \partial_z^2[(c - U^p)U_E] = 0$$

Differentiating the BC $U(\pm T(E);E) = 0$ w.r.t. E yields

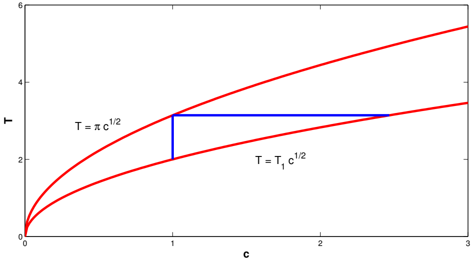
$$\partial_E U(-T(E);E) - T'(E)\underbrace{\partial_z U(-T(E);E)}_{\neq 0} = \partial_E U(T(E);E) + T'(E)\underbrace{\partial_z U(T(E);E)}_{\neq 0}.$$

Since $T'(E) \neq 0$ the solution U_E is not $2T(E)$-periodic!

$\rightsquigarrow$ the zero eigenvalue is simple, i.e. $\text{Ker}(L) = \text{span}\{U_z\}$.

Spectral properties of the operator L

Sign condition $-\frac{d}{dc}\|U\|^2_{L^2_{\rm per}(-T,T)} < 0$, where the period T is fixed. Here the monotonicity $T'(E) < 0$ also plays a role.



For fixed c, the map $E \mapsto T$ is monotonically decreasing for $E \in (0, E_c)$ with $T(0) = \pi c^{1/2}$.

For fixed T, the map $c \mapsto E$ is monotonically increasing for $c \in (c_0, c_*)$ with $c_0 = T^2/\pi^2$.

Summary - Part I

- We consider smooth periodic traveling waves $u(x,t) = U(x-ct)$ of the generalized reduced Ostrovsky equation

$$(u_t + u^p u_x)_x = u.$$

- The spectral stability problem is given by

$$\boxed{\partial_z L v = \lambda v}$$

- For every $p \in \mathbb{N}$ and every c for which smooth U exists, the operator $L|_{U^\perp}$ has a simple zero eigenvalue and a positive spectrum bounded away from zero.

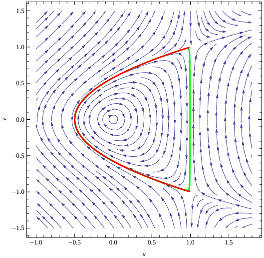
- Hamilton-Krein index theory implies

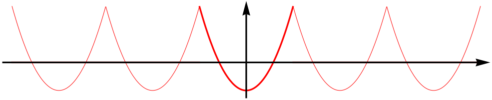
$$\#\text{ unstable EV of } \partial_z L \leq \#\text{negative EV of } L|_{U^\perp}$$

- **Result**: the smooth periodic traveling waves U are *spectrally stable*. [Geyer & P., LMP '17]

We now consider the *peaked* periodic traveling waves of the reduced Ostrovsky equation ($p = 1$)

$$(u_t + uu_x)_x = u.$$





Part II - Peaked periodic wave

Some results for periodic waves of other equations:

- KdV equation: smooth solutions are stable, no peaked solutions
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  [Carter, Kalisch et. al. 2014]
- Ostrovsky equation: all smooth solutions are stable, but the limiting *peaked solution is unstable*.
  [Geyer & P. 2018]

Peaked periodic wave

The 2π periodic traveling wave solutions $U(z)$ satisfy the BVP

$$\left\{ \begin{array}{l} [c - U(z)]\, U'(z) + (\partial_z^{-1} U)(z) = 0, \quad z \in (-\pi, \pi) \\ U(-\pi) = U(\pi), \end{array} \right.$$

where $z = x - ct$ and $\int_{-\pi}^{\pi} U(z) dz = 0$.

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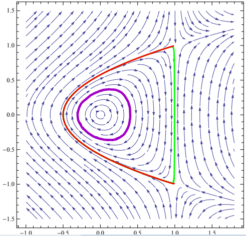
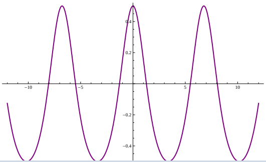
where $z = x - ct$ and $\int_{-\pi}^{\pi} U(z) dz = 0$.

Lemma (Existence of smooth periodic traveling waves)

There exists $c_* > 1$ such that for every $c \in (1, c_*)$, the BVP admits a unique smooth periodic wave U satisfying $U(z) < c$ for $z \in [-\pi, \pi]$.

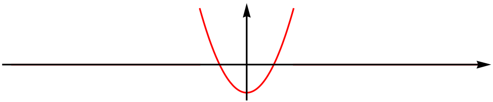




Peaked periodic wave

For $c = c_* := \pi^2/9$ there exists a solution with parabolic profile

$$U_*(z) := \frac{3z^2 - \pi^2}{18}, \quad z \in [-\pi, \pi],$$



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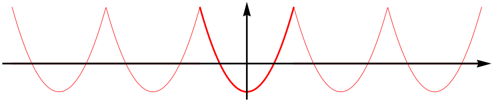

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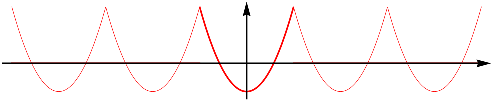

▷ The peaked periodic wave $U_* \in \dot{H}^s_{\rm per}(-\pi, \pi)$ for $s < 3/2$:

$$U_*(z) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{3n^2} \cos(nz),$$

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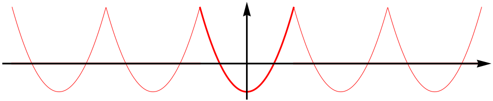

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▷ $U_*(z) < c_*$ for $z \in (-\pi,\pi)$, $U_*(\pm\pi) = c_*$, and $U_*'(\pm\pi) = \pm\pi/3$.

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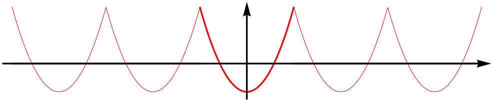

Lemma

The peaked periodic wave U_* is the unique *solution with a jump discontinuity in the derivative at $z = \pm\pi$.*

Spectral stability of the peaked periodic wave

Consider the linearized evolution for a co-periodic perturbation v to the travelling wave U:

$$\left\{ \begin{array}{l} v_t + \partial_z \left[(U_*(z) - c_*) v \right] = \partial_z^{-1} v, \quad t > 0, \\ v|_{t=0} = v_0, \end{array} \right.$$

or equivalently

$$v_t = \partial_z L v, \quad \text{where} \;\; L = P_0 \left(\partial_z^{-2} + c_* - U_* \right) P_0 : \quad \dot{L}^2_{\rm per} \to \dot{L}^2_{\rm per}.$$

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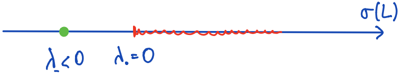
Lemma

The spectrum of the self-adjoint operator L is $\sigma(L) = \{\lambda_-\} \cup \left[0, \frac{\pi^2}{6}\right]$.



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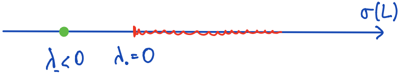

The spectral stability problem can not be solved by applying standard energy methods due to the lack of coercivity.

Linear stability of the peaked periodic wave

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Goal: show that the peaked periodic wave is *linearly unstable*.

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Definition

The travelling wave U is *linearly stable* if
for every $v_0 \in \dot{H}^1_{\rm per}$ satisfying $\langle U, v_0 \rangle_{L^2} = 0$,
there exists a unique global solution $v \in C(\mathbb{R}, \dot{H}^1_{\rm per})$ to (linO) s.t.

$$\|v(t)\|_{H^1_{\rm per}} \leq C \|v_0\|_{H^1_{\rm per}}, \quad t > 0.$$

Otherwise, it is said to be linearly unstable.

Linear instability of the peaked periodic wave

▷ **Step 1**: The *truncated problem*

$$\left\{ \begin{array}{l} v_t + \frac{1}{6}\partial_z \left[(z^2 - \pi^2) v \right] = 0, \quad t > 0, \\ v|_{t=0} = v_0 \in \dot{H}^1_{\rm per}. \end{array} \right. \qquad \text{(truncO)}$$

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Method of characteristics. The family of char. curves $z = Z(s,t)$ can be solved explicitly and the solution of $V(s,t) := v(Z(s,t),t)$ is

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This yields the linear instability result for the truncated problem:

Lemma

For every $v_0 \in \dot{H}^1_{\text{per}}$ $\exists!$ global solution $v \in C(\mathbb{R}, \dot{H}^1_{\text{per}})$ to (truncO).
If v_0 is odd, then the global solution satisfies

$$\frac{1}{2}\|v_0\|_{L^2} e^{\pi t/6} \leq \|v(t)\|_{L^2} \leq \|v_0\|_{L^2} e^{\pi t/6}, \quad t > 0.$$

Linear instability of the peaked periodic wave

▷ **Step 2**: The *full evolution problem*

$$\begin{cases} v_t + \frac{1}{6}\partial_z\left[(z^2-\pi^2)v\right] = \partial_z^{-1} v, \quad t>0, \\ v|_{t=0} = v_0 \in \dot{H}^1_{\rm per}. \end{cases} \tag{linO}$$

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Lemma

For every $v_0 \in \dot{H}^1_{\rm per}$ $\exists!$ *global solution* $v \in C(\mathbb{R}, \dot{H}^1_{\rm per})$ *to* (linO).
If v_0 *is odd, then the solution satisfies*

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Conclusion: The reduced Ostrovsky equation is *linearly unstable*.

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$$v_t = Av + F(v)$$

A is a linear operator generating a C^0-semigroup in Banach space X and F is strongly continuous in X

If A has positive spectrum $\{\mathcal{R}\lambda > 0\}$,
then $v = 0$ is nonlinearly unstable. [Shatah & Strauss '00]

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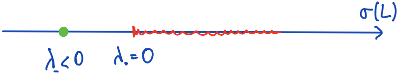
- Here: $A = \partial_z L$ but



  so we do not know whether the spectral assumption is satisfied.

- We need a different approach!

Nonlinear instability

Consider an orbit $\{U_*(z-a),\ \ a \in [-\pi, \pi]\}$ of the peaked wave U_*.

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Definition

The travelling wave U is said to be *orbitally stable* if for every $\epsilon > 0$, there exists $\delta > 0$ such that

for every $u_0 \in \dot{H}^1_{\rm per}$ satisfying $\|u_0 - U\|_{H^1_{\rm per}} < \delta$,

there exists a unique global solution $u \in C(\mathbb{R}, \dot{H}^1_{\rm per})$ to

$$\left\{ \begin{array}{l} u_t + u u_x = \partial_x^{-1} u, \quad t > 0, \\ u|_{t=0} = u_0, \end{array} \right. \tag{redO}$$

such that for every $t > 0$,

$$\inf_{a \in [-\pi,\pi]} \|u(t,\cdot) - U(\cdot - a)\|_{H^1_{\rm per}} < \epsilon.$$

Otherwise, the periodic wave U is said to be orbitally unstable.

Nonlinear instability

- We consider *decomposition of the solution* $u \in \dot{H}^1_{\rm per}$

$$u(t,x) = U_*(x - ct - a(t)) + v(t, x - ct - a(t)),$$

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$$\langle \partial_x U_*, v \rangle_{L^2} = 0.$$

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Such a decomposition always exists and is unique by an application of the inverse function theorem.

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  where $z = x - ct - a(t)$.

- Using the *orthogonality condition* we obtain an evolution equation for the translation parameter a:

  $$\left\{ \begin{array}{l} a'(t) = -\frac{\langle \partial_z U, \partial_z L v \rangle_{L^2} - \langle \partial_z U, v \partial_z v \rangle_{L^2}}{\|\partial_z U\|^2_{L^2} + \langle \partial_z U, \partial_z v \rangle_{L^2}}, \quad t > 0, \\ a(0) = 0. \end{array} \right. \tag{CPa}$$

Nonlinear instability

Theorem (Orbital instability)

There exists $\epsilon > 0$ *such that for every small* $\delta > 0$*,*
there exists $v_0 \in \dot{H}^s_{\text{per}}$ *satisfying*

$$\|v_0\|_{H^s_{\text{per}}} \leq \delta$$

s.t. the unique solution $v \in C([0,T], \dot{H}^s_{\text{per}})$ to (CPv)–(CPa) satisfies

$$\|v(t_1)\|_{L^2} \geq \epsilon$$

for some $t_1 \in (0,T)$ *with* $T = \mathcal{O}(\delta^{-1})$, $a \in C([0,T], \mathbb{R})$ *and* $s > 3/2$ *.*

Nonlinear instability – Proof

- Write (CPv)

$$\begin{cases} v_t + \frac{1}{6}\partial_z\left[(z^2-\pi^2)v\right] + v\partial_z v = \partial_z^{-1} v + a'(t)(\partial_z U_* + \partial_z v), \\ v|_{t=0} = v_0, \end{cases}$$

as the inhomogeneous evolution equation

$$v_t = Av + F(v)$$

where $A := A_0 + \partial_z^{-1}$ generates the C^0-semigroup in $\dot{L}^2_{\rm per}$ and $F(v) : \dot{L}^2_{\rm per} \to \dot{L}^2_{\rm per}$ is continuous.

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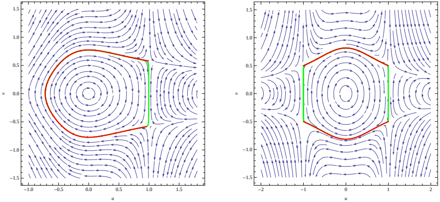
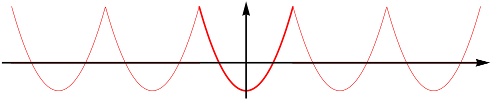
- This yields orbital instability of U_*.

- Periodic traveling waves of the reduced Ostrovsky equation

$$(u_t + u^p u_x)_x = u.$$



- The *smooth* periodic waves are spectrally *stable* for any $p \in \mathbb{N}$. [Geyer & P., LMP 2017]
- The *peaked* periodic wave is linearly and nonlinearly *unstable* for $p = 1$. [Geyer & P., SIMA 2018]



Further questions

- Periodic traveling waves of the reduced Ostrovsky equation

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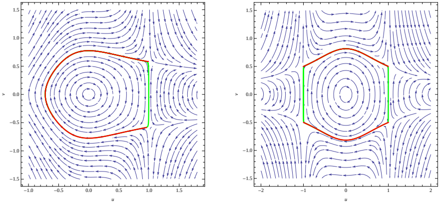

- Are the *smooth* periodic waves *transversally stable?*
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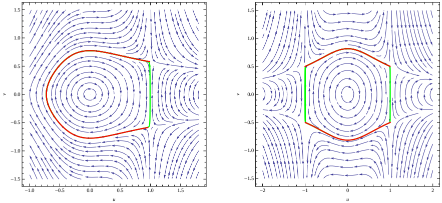

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Thank you for your attention!