Periodic waves in integrable equations: modulation instability and rogue waves

Jinbing Chen[1], Dmitry E. Pelinovsky[2], Jeremy Upsal[3], Robert White[2]

[1] School of Mathematics, Southeast University, Nanjing, Jiangsu, P.R. China
[2] Department of Mathematics, McMaster University, Hamilton, Ontario, Canada
[3] Department of Applied Mathematics, University of Washington, Seattle, WA 98195, USA

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1. Three examples of dynamics of periodic travelling waves
2. Characterization of periodic waves in the derivative NLS
3. Rogue waves on the periodic wave background in the NLS
4. Fluxon condensates in the sine–Gordon equation

Fluxon condensates in the semi-classical limit

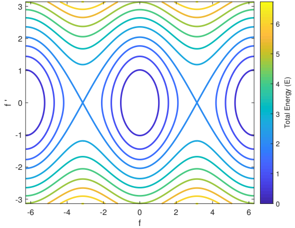
The sine–Gordon equation in the semi-classical limit is

$$\epsilon^2 u_{TT} - \epsilon^2 u_{XX} + \sin(u) = 0,$$

with small ϵ. Fluxon condensate arises in the evolution from the initial condition:

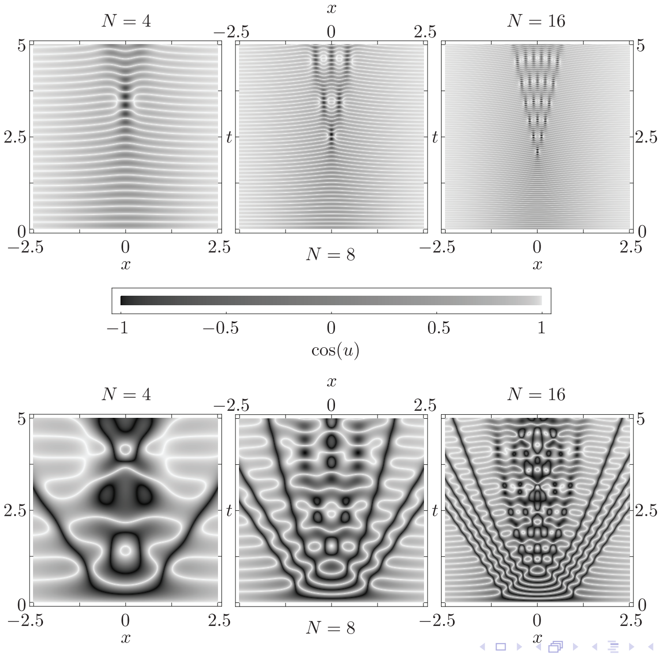
$$u(X,0) = 0, \quad \epsilon u_T(X,0) = G(X),$$

where G is fixed with either $\|G\|_{L^\infty} < 2$ (librational waves) or $\|G\|_{L^\infty} > 2$ (rotational waves). ϵ is selected at $\{\epsilon_N\}_{N\in\mathbb{N}}$ so that the solution is purely N-soliton potential and $\epsilon_N \to 0$ as $N \to \infty$.



Left: Orbits of $f'' + \sin(f) = 0$.

R.J. Buckingham–P.D. Miller (2012, 2013); B.Y. Lu–P.D. Miller (2020)


$N = 4$
$N = 8$
$N = 16$
x
t
-2.5
0
2.5
5
-1
-0.5
0.5
1
$\cos(u)$

Rogue waves on periodic background

The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

admits the exact solution

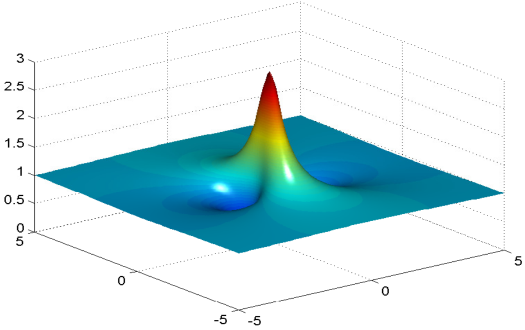
$$\psi(x,t) = \left[1 - \frac{4(1+2it)}{1+4x^2+4t^2}\right] e^{it}.$$

It was discovered by H. Peregrine (1983) and was labeled as *the rogue wave*.

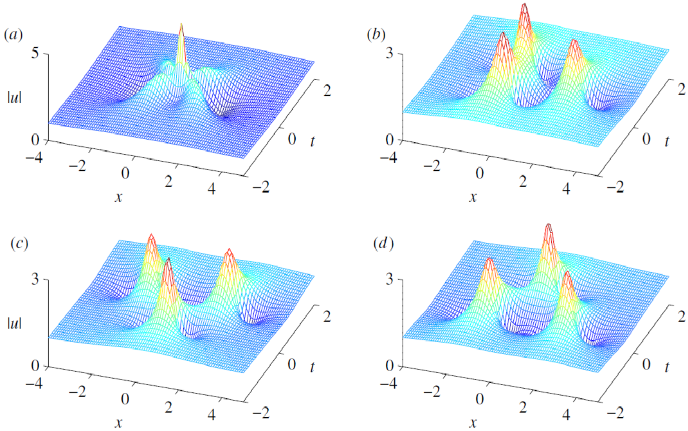
Properties of the rogue wave:

- It is localized in space and time on the background of $\psi_0(t) = e^{it}$
- It comes from nowhere and disappears without any trace.
- It is significantly magnified at the center: $M_0 := |\psi(0,0)| = 3$.

The surface plot of $|\psi(x,t)|$ for the rogue wave in NLS equation:

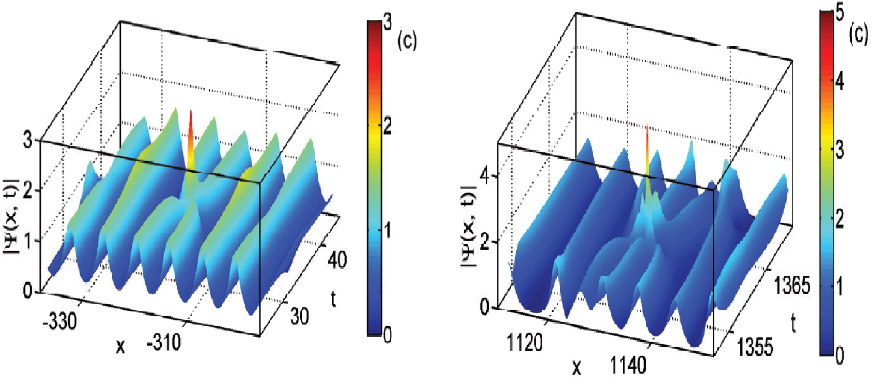


The rogue wave solution is related to the lump solution of the KP-I hierarchy.
D.Pelinovsky (1997);
P.Dubbard-V.B.Matveev (2013)



The ”second-order” rogue wave as in **Y.Ohta-J.Yang (2012)**

Peregrine's rogue wave has long believed to play the major role in more complicated dynamics of periodic waves in the NLS equation.



The result of numerical simulations in **D. Agafontsev–V.E. Zakharov (2016)**

Modulational instability of periodic waves

The derivative nonlinear Schrödinger (DNLS) equation

$$i\psi_t + \psi_{xx} + i(|\psi|^2\psi)_x = 0$$

admits the periodic traveling and standing wave solution

$$\psi(x,t) = e^{4ibt}u(x+2ct)$$

with two parameters b and c.

Linear stability of such solutions is defined by linearized evolution equation

$$iw_t - 4bw + 2icw_x + w_{xx} + i[2|u|^2 w_x + u^2 \bar{w}_x + 2(u\bar{u}_x + \bar{u}u_x)w + 2uu_x \bar{w}] = 0$$

for the perturbation w in $\psi(x,t) = e^{4ibt}\left[u(x+2ct) + w(x+2ct,t)\right]$.

Separating variables by $w(x,t) = w_1(x)e^{t\Lambda}$ and $\bar{w}(x,t) = w_2(x)e^{t\Lambda}$ results in the spectral problem for the eigenvector $(w_1, w_2)^T$ and eigenvalue Λ.

Stability spectrum is the union of all Λ for which $(w_1, w_2)^T$ is bounded (Floquet spectrum related to periodic u).

Definition

The periodic wave u is spectrally unstable if there exists Λ with $\mathrm{Re}(\Lambda) > 0$ such that $(w_1, w_2) \in L^\infty(\mathbb{R})$. The periodic wave is modulationally unstable if there exists an unstable band of Floquet spectrum with $\mathrm{Re}(\Lambda) > 0$ that intersects $\Lambda = 0$.

Stability spectrum Λ can be characterized from the linear Lax system representing the DNLS equation for $\phi(x,t) = e^{2bt\sigma_3}\varphi(x+2ct,t)$:

$$\varphi_x = U(u,\lambda)\varphi, \quad \varphi_t + 2ib\sigma_3\varphi + 2c\varphi_x = V(u,\lambda)\varphi,$$

where

$$U = \begin{pmatrix} -i\lambda^2 & \lambda u \\ -\lambda\bar{u} & i\lambda^2 \end{pmatrix}, \quad V = \begin{pmatrix} -2i\lambda^4 + i\lambda^2|u|^2 & 2\lambda^3 u + \lambda(iu_x - |u|^2 u) \\ -2\lambda^3\bar{u} + \lambda(i\bar{u}_x + |u|^2\bar{u}) & 2i\lambda^4 - i\lambda^2|u|^2 \end{pmatrix}$$

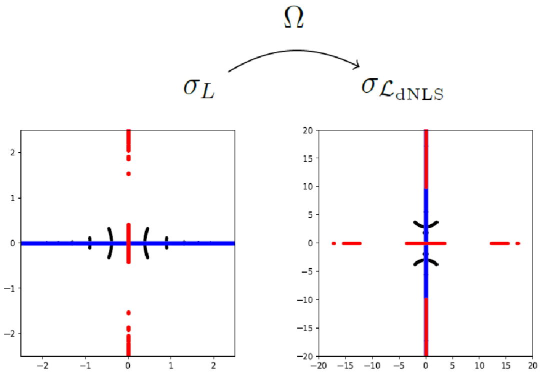
If $\lambda \in \mathbb{C}$ belongs to Lax spectrum (Floquet spectrum related to periodic u), then eigenvector $\varphi(x,t) = \chi(x)e^{t\Omega(\lambda)}$ for some specific $\Omega(\lambda)$ determines solution of the spectral stability problem for $(w_1, w_2)^T$ and Λ in

$$w_1 = \partial_x \chi_1^2, \quad w_2 = \partial_x \chi_2^2, \quad \Lambda = 2\Omega.$$

Squared eigenfunctions were found in **X.G. Chen-J.Yang (2002)**.
Recent study was done in **J. Upsal-B. Deconinck (2020)**
after similar studies of NLS in **B. Deconinck–B.L. Segal (2017)**.

Main result from **J. Upsal-B. Deconinck (2020)**:

If $\Lambda \in i\mathbb{R}$ for a given $\lambda \in \mathbb{R} \cup i\mathbb{R}$, then $\lambda \in \mathbb{R} \cup i\mathbb{R}$ belongs to the Lax spectrum.



Spectral stability of periodic waves in DNLS subject to perturbations of the same period was studied in **S.Hakkaev-A.Stefanov-M.Stanislavova (2020)**.

Our methods and results

- We explore construction of periodic waves of integrable equations by using complex-valued Hamiltonian systems arising in the nonlinearization of the Lax equations **(Cao–Geng, 1990)** Also **Z. Qiao; R. Zhou; J. Chen**.
- This allows to characterize the periodic waves in terms of eigenvalues of the Lax equations associated with the periodic eigenfunctions for $\Lambda = 0$.
- We give precise information on the location of Lax and stability spectra, with assistance of a numerical package based on Hill's method.
- We obtain solutions describing localized structures on the background of periodic waves (either rogue waves or propagating algebraic solitons), with assistance of the Darboux transformations.

A particularly interesting outcome is the explicit relation between the existence of modulational instability and the existence of a rogue wave on the background of periodic travelling waves.

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A particularly interesting outcome is the explicit relation between the existence of modulational instability and the existence of a rogue wave on the background of periodic travelling waves.

Periodic travelling and standing waves

The derivative nonlinear Schrödinger (DNLS) equation

$$i\psi_t + \psi_{xx} + i(|\psi|^2\psi)_x = 0$$

admits the periodic traveling and standing wave solution

$$\psi(x,t) = e^{4ibt} u(x + 2ct)$$

with two parameters b and c. The envelope $u = u(x)$ satisfies

$$u'' + 2|u|^2 u + 2icu' - 4bu = 0,$$

From here, solutions are usually constructed by separation of variables $u(x) = R(x)e^{i\Theta(x)}$ with two additional integrations:

$$\frac{d\Theta}{dx} = -\frac{a}{R^2} - \frac{3}{4}R^2 - c$$

and

$$\left(\frac{dR}{dx}\right)^2 + \frac{a^2}{R^2} + \frac{1}{16}R^6 + \frac{c}{2}R^4 + R^2\left(c^2 - 4b - \frac{a}{2}\right) + 2ac - 4d = 0,$$

where a and d are constants of integration.

Consider the first-order equation:

$$\boxed{\left(\frac{dR}{dx}\right)^2 + \frac{a^2}{R^2} + \frac{1}{16}R^6 + \frac{c}{2}R^4 + R^2\left(c^2 - 4b - \frac{a}{2}\right) + 2ac - 4d = 0.}$$

For $a \neq 0$, phase singularity is unfolded for $\rho := \frac{1}{2}R^2$ and the solutions are found from the quadrature

$$\left(\frac{d\rho}{dx}\right)^2 + Q(\rho) = 0,$$

where $Q(\rho) = \rho^4 + 4c\rho^3 + 2(2c^2 - a - 8b)\rho^2 + 4(ac - 2d)\rho + a^2$.

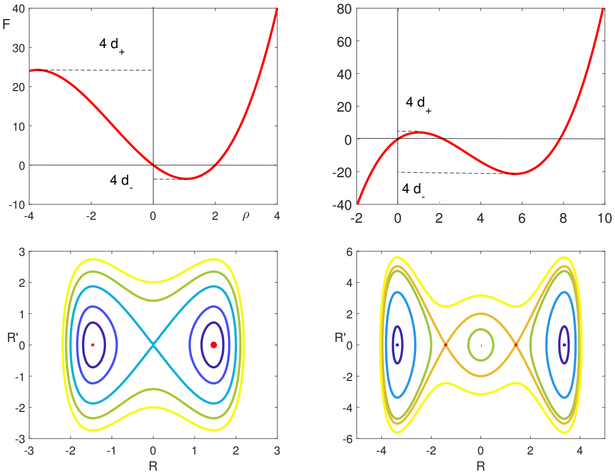
For $a = 0$, there is no phase singularity and solutions are obtained from Newton's dynamics:

$$\left(\frac{dR}{dx}\right)^2 + F(R) = 4d,$$

where $F(R) = \frac{1}{16}R^6 + \frac{c}{2}R^4 + (c^2 - 4b)R^2$.

Families of periodic waves for $a = 0$

Left: $c^2 < 4b$. Right: $c^2 > 4b$, $c < 0$, and $b > 0$. Here $\rho = \frac{1}{2}R^2 > 0$.



More properties from the integrability structure

A solution to the derivative nonlinear Schrödinger (DNLS) equation is the compatibility condition of the Lax system discovered by **D.Kaup–A.Newell (1978)**, where the first equation is now called the Kaup–Newell problem:

$$\varphi_x = \begin{pmatrix} -i\lambda^2 & \lambda u \\ -\lambda \bar{u} & i\lambda^2 \end{pmatrix} \varphi.$$

Definition

If $u(x) = R(x)e^{i\Theta(x)}$ with L-periodic R and Θ', then λ is called an eigenvalue w.r.t. periodic boundary conditions if $\varphi = (p,q)^T$ is given by $p(x) = P(x)e^{i\Theta(x)/2}$ and $q(x) = Q(x)e^{-i\Theta(x)/2}$ with L-periodic P and Q.

Three properties of eigenvalues and eigenvectors

1. Let $\lambda \in i\mathbb{R}$ be a simple eigenvalue with the periodic eigenvector $\varphi = (p, q)^T$. Then, there is $c \in \mathbb{C}$ with $|c| = 1$ such that $q = c\bar{p}$.
2. Let $\lambda \in \mathbb{C}$ be a simple eigenvalue with the periodic eigenvector $\varphi = (p, q)^T$. Then, $\bar{\lambda}$ is a simple eigenvalue with $\varphi = (\bar{q}, -\bar{p})^T$.
3. Let $\lambda \in \mathbb{R}$ be an eigenvalue with the periodic eigenvector $\varphi = (p, q)^T$. Then, it is at least double with two linearly independent eigenvectors.

Complex Hamiltonian system

Fix $\lambda = \lambda_1$ with $\varphi = (p_1, q_1)^T$ and $\lambda = \lambda_2$ with $\varphi = (p_2, q_2)^T$ s.t. $\lambda_1 \neq \lambda_2$. Consider the potential u of the Kaup–Newell problem given by either

$$\lambda_1 \in \mathbb{C}\backslash i\mathbb{R}, \quad \lambda_2 = \bar{\lambda}_1 : \qquad \left\{ \begin{array}{l} u = \lambda_1 p_1^2 + \bar{\lambda}_1 \bar{q}_1^2, \\ \bar{u} = \bar{\lambda}_1 \bar{p}_1^2 + \lambda_1 q_1^2 \end{array} \right.$$

or

$$\begin{array}{ll} \lambda_1 = i\beta_1, & \lambda_2 = i\beta_2 \\ q_1 = -i\bar{p}_1, & q_2 = -i\bar{p}_2 \end{array} : \qquad \left\{ \begin{array}{l} u = i\beta_1 p_1^2 + i\beta_2 p_2^2, \\ \bar{u} = -i\beta_1 \bar{p}_1^2 - i\beta_2 \bar{p}_2^2. \end{array} \right.$$

The Kaup–Newell problem becomes a complex Hamiltonian system generated by the Hamiltonian function

$$H = i\lambda_1^2 p_1 q_1 + i\lambda_2^2 p_2 q_2 - \frac{1}{2}(\lambda_1 p_1^2 + \lambda_2 p_2^2)(\lambda_1 q_1^2 + \lambda_2 q_2^2).$$

with additional conserved quantity

$$M = i(p_1 q_1 + p_2 q_2).$$

Both conserved quantities are real for the two cases above.

Travelling wave reduction

Differentiating the constraint between u and eigenfunctions:

$$\begin{aligned} & u = \lambda_1 p_1^2 + \lambda_2 p_2^2, \\ \Rightarrow \quad & \frac{du}{dx} + i|u|^2 u + 2iHu + 2i(\lambda_1^3 p_1^2 + \lambda_2^3 p_2^2) = 0, \\ \Rightarrow \quad & \frac{d^2 u}{dx^2} + i\frac{d}{dx}(|u|^2 u) + 2iH\frac{du}{dx} + 4(\lambda_1^5 p_1^2 + \lambda_2^5 p_2^2 + i\lambda_1^4 u p_1 q_1 + i\lambda_2^4 u p_2 q_2) = 0. \end{aligned}$$

The last equation yields the travelling wave reduction of DNLS:

$$\boxed{\frac{d^2 u}{dx^2} + i\frac{d}{dx}(|u|^2 u) + 2ic\frac{du}{dx} - 4bu = 0,}$$

where $b := \lambda_1^2 \lambda_2^2 (1 + M)$ and $c := \lambda_1^2 + \lambda_2^2 + H$.

Integrability of the complex Hamiltonian system

The complex Hamiltonian system on (p_1, q_1) and (p_2, q_2) is a compatibility condition of the Lax equation

$$\frac{d}{dx}\Psi = U(\lambda, u)\Psi - \Psi U(\lambda, u),$$

where $U(\lambda, u)$ is the same as in the Kaup–Newell system and

$$\Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & -\Psi_{11} \end{pmatrix}$$

with

$$\Psi_{11} = -i - \frac{\lambda_1^2 p_1 q_1}{\lambda^2 - \lambda_1^2} - \frac{\lambda_2^2 p_2 q_2}{\lambda^2 - \lambda_2^2} = \frac{-i[\lambda^4 - (c + \frac{1}{2}|u|^2)\lambda^2 + b]}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)},$$

$$\Psi_{12} = \lambda\left[\frac{\lambda_1 p_1^2}{\lambda^2 - \lambda_1^2} + \frac{\lambda_2 p_2^2}{\lambda^2 - \lambda_2^2}\right] = \frac{\lambda[\lambda^2 u + \frac{i}{2}(\frac{du}{dx} + i|u|^2 u) - cu}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)}.$$

$\det\Psi$ is constant and has simple poles at $(\pm\lambda_1, \pm\lambda_2)$:

$$\det\Psi = 1 - \frac{2H\lambda^2 - \lambda_1^2\lambda_2^2 M(M+2)}{(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_2^2)} = \frac{P(\lambda)}{(\lambda^2-\lambda_1^2)^2(\lambda^2-\lambda_2^2)^2}$$

with

$$P(\lambda) = \lambda^8 - 2c\lambda^6 + (a+2b+c^2)\lambda^4 + (d - c(a+2b))\lambda^2 + b^2.$$

Here the new constants a and d appear in the conserved quantities

$$\boxed{2i\left(\bar{u}\frac{du}{dx} - u\frac{d\bar{u}}{dx}\right) - 3|u|^4 - 4c|u|^2 = 4a}$$

and

$$\boxed{2\left|\frac{du}{dx}\right|^2 - |u|^6 - 2c|u|^4 - 4(a+2b)|u|^2 = 8d}$$

New parameters are related to parameters of the algebraic method:
$a := \lambda_1^2\lambda_2^2 M^2 - H^2$ and $d := \lambda_1^2\lambda_2^2 MH(M+2) - H^2(\lambda_1^2 + \lambda_2^2 + H)$.

Characterization of periodic waves

Thus, the periodic waves of the DNLS are related to the polynomial

$$\boxed{P(\lambda) = \lambda^8 - 2c\lambda^6 + (a + 2b + c^2)\lambda^4 + (d - c(a + 2b))\lambda^2 + b^2.}$$

Denote four pairs of roots of $P(\lambda)$ by $\{\pm\lambda_1, \pm\lambda_2, \pm\lambda_3, \pm\lambda_4\}$, where any two roots can be picked for the algebraic method.

Recall the periodic waves are given by the first-order equation for $\rho = \frac{1}{2}|u|^2$:

$$\boxed{\left(\frac{d\rho}{dx}\right)^2 + Q(\rho) = 0, \; Q(\rho) = \rho^4 + 4c\rho^3 + 2(2c^2 - a - 8b)\rho^2 + 4(ac - 2d)\rho + a^2}$$

Denote four roots of $Q(\rho)$ by $\{u_1, u_2, u_3, u_4\}$.

The remarkable property of periodic wave is the explicit relation:

$$\left\{ \begin{array}{rcl} u_1 & = & -\frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)^2, \\ u_2 & = & -\frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)^2, \end{array} \right. \qquad \left\{ \begin{array}{rcl} u_3 & = & -\frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)^2, \\ u_4 & = & -\frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^2. \end{array} \right.$$

A. Kamchatnov (1990)

First family of periodic waves

Four roots of $Q(\rho)$ are real: $u_4 \leq u_3 \leq u_2 \leq u_1$. Then,

$$\rho(x) = u_4 + \frac{(u_1 - u_4)(u_2 - u_4)}{(u_2 - u_4) + (u_1 - u_2)\mathrm{sn}^2(\mu x; k)},$$

with $2\mu = \sqrt{(u_1 - u_3)(u_2 - u_4)}$ and $2\mu k = \sqrt{(u_1 - u_2)(u_3 - u_4)}$.

This family occurs only in two cases:
Two complex quadruplets when $u_4 \leq u_3 \leq 0 \leq u_2 \leq u_1$,

$$\lambda_1 = \bar{\lambda}_2 = \alpha_1 + i\beta_1, \qquad \lambda_3 = \bar{\lambda}_4 = \alpha_2 + i\beta_2.$$

Four pairs of purely imaginary eigenvalues when $0 \leq u_4 \leq u_3 \leq u_2 \leq u_1$,

$$\lambda_1 = i\beta_1, \qquad \lambda_2 = i\beta_2, \qquad \lambda_3 = i\beta_3, \qquad \lambda_4 = i\beta_4.$$

Second family of periodic waves

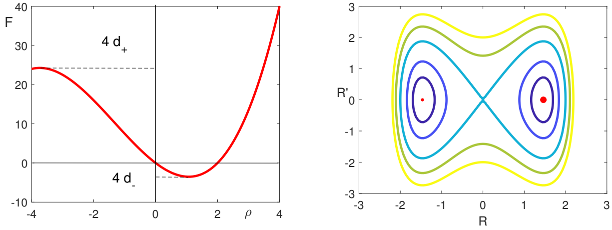
Two roots of $Q(\rho)$ are real $u_2 \leq u_1$ and two roots of $Q(\rho)$ are complex-conjugate $u_{3,4} = \gamma \pm i\eta$. Then,

$$\rho(x) = u_1 + \frac{(u_2 - u_1)(1 - \mathrm{cn}(\mu x; k))}{1 + \delta + (\delta - 1)\mathrm{cn}(\mu x; k)},$$

with δ, μ, and k are given in terms of u_1, u_2, γ, and η.

This family occurs only in one case:
One complex quadruplet and two pairs of purely imaginary eigenvalues when $0 \leq u_2 \leq u_1$.

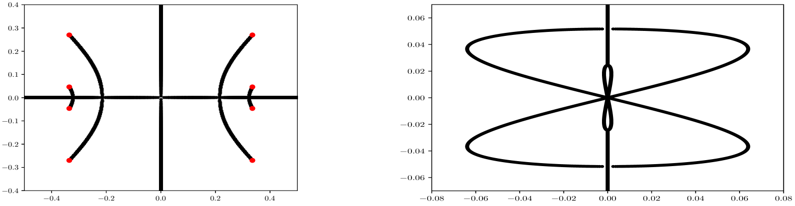
Periodic waves for $a = 0$ and $c^2 < 4b$



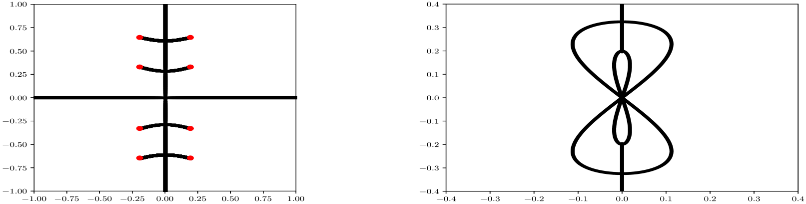
Three different cases:

- $d \in (d_-, 0)$: positive periodic solutions of the first family with two complex quadruplets;
- $d \in (0, d_+)$: sign-indefinite periodic solutions of the first family with two complex quadruplets;
- $d \in (d_+, \infty)$: sign-indefinite periodic solutions of the second family with one complex quadruplet and two pairs of purely imaginary eigenvalues.

Two complex quadruplets

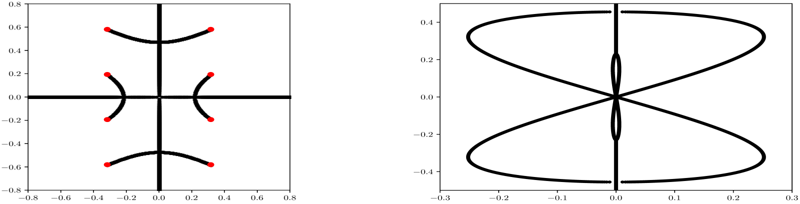


(a) $u_1 = 0.2,\ u_2 = 0.1,\ u_3 = 0,\ u_4 = -0.9.$

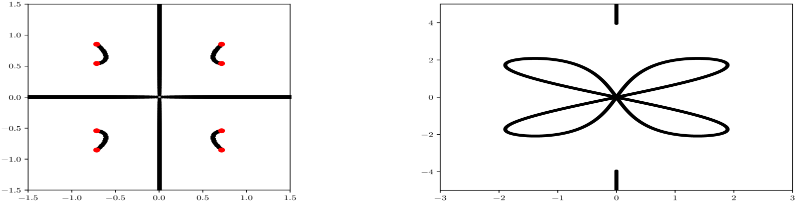


(b) $u_1 = 1.9,\ u_2 = 0.2,\ u_3 = 0,\ u_4 = -0.3.$

Two complex quadruplets

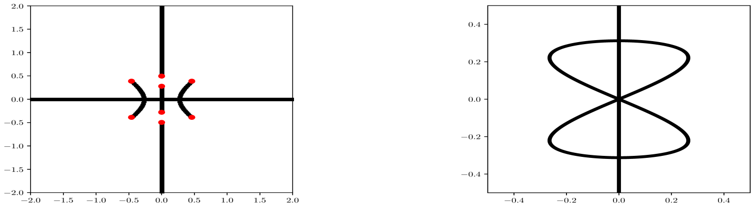


(a) $u_1 = 1.2$, $u_2 = 0.3$, $u_3 = 0$, $u_4 = -0.8$.

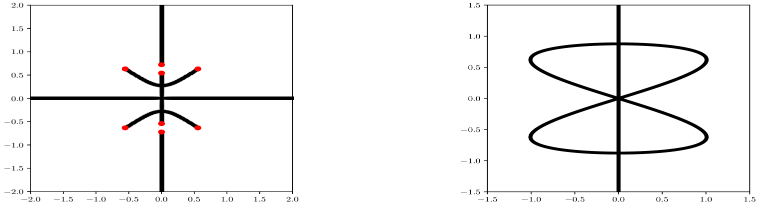


(b) $u_1 = 3.9$, $u_2 = 0.193012$, $u_3 = 0$, $u_4 = -4.090301$.

One complex quadruplet and two pairs of imaginary

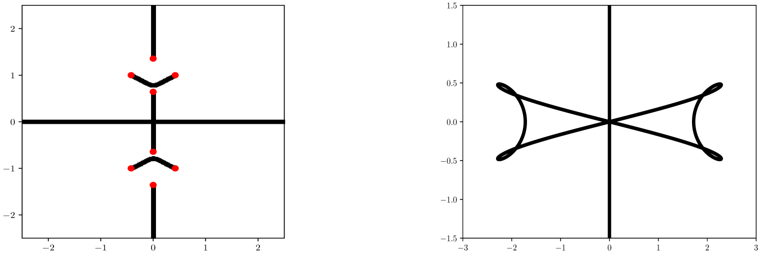


(a) $u_1 = 1.2$, $u_2 = 0$, $u_3 = -0.4 - 0.2i$, $u_4 = -0.4 + 0.2i$.



(b) $u_1 = 3.2$, $u_2 = 0$, $u_3 = -0.6 + 0.2i$, $u_4 = -0.6 - 0.2i$.

One complex quadruplet and two pairs of imaginary

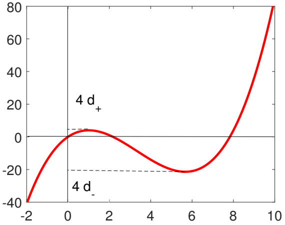


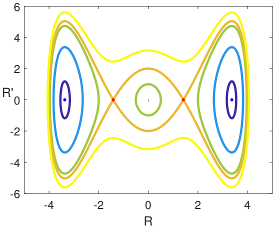
(a) $u_1 = 8$, $u_2 = 0$, $u_3 = -0.1 + 0.6i$, $u_4 = -0.1 - 0.6i$.

Missing for analysis:

- No Lax spectrum between imaginary eigenvalues;
- No four reconnection across the outer branches

Periodic waves for $a = 0$, $c^2 > 4b$, $c < 0$, and $b > 0$

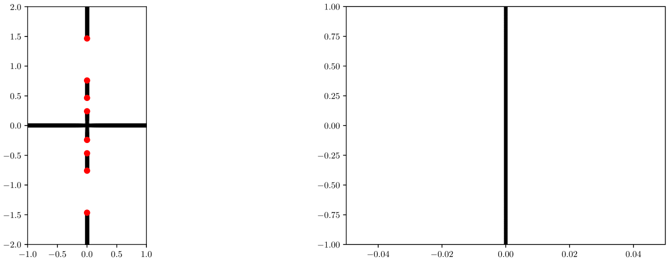




Three different cases:

- $d \in (d_-, 0)$: positive periodic solutions of the first family with two complex quadruplets;
- $d \in (0, d_+)$: positive and sign-indefinite periodic solutions of the first family with four pairs of purely imaginary eigenvalues;
- $d \in (d_+, \infty)$: sign-indefinite periodic solutions of the second family with one complex quadruplet and two pairs of purely imaginary eigenvalues.

Four pairs of imaginary eigenvalues



Remarkable properties:

- Stability is observed only for $c^2 > 4b$, $c < 0$, and $b > 0$ for periodic waves $\psi(x,t) = e^{4ibt}u(x+2ct)$
- Two different families of periodic waves (positive and sign-indefinite) share the same Lax spectrum and the same stability.

Commercial break: job posting at McMaster University

Tier 1 CRC in Mathematical Analysis and Applications

The Department of Mathematics and Statistics at McMaster University invites applications for a faculty position at the rank of Associate or full Professor to hold a proposed Tier 1 Canada Research Chair in Mathematical Analysis and Applications. This position is intended to be a tenure-track appointment, although a tenured appointment is possible. The expected start date for this position is July 1, 2021.

Deadline for application is December 1, 2020.

Further details on https://www.mathjobs.org/jobs/application/16437

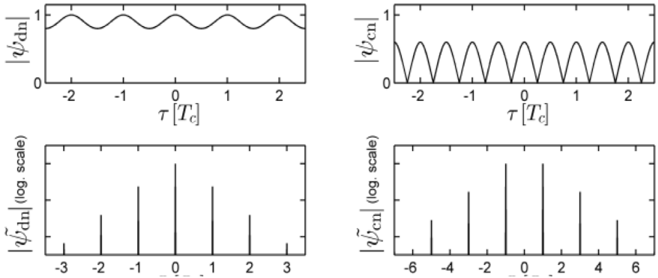
Periodic wave background

The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

also admits the periodic traveling and standing wave solutions, e.g. the dnoidal and cnoidal waves:

$$\psi_{\rm dn}(x,t) = {\rm dn}(x;k)e^{i(1-k^2/2)t}, \quad \psi_{\rm cn}(x,t) = k{\rm cn}(x;k)e^{i(k^2-1/2)t}.$$



Rogue waves on the periodic wave background

Can we obtain a rogue wave on the background ψ_0 such that

$$\inf_{x_0,t_0,\alpha_0\in\mathbb{R}} \sup_{x\in\mathbb{R}} \left|\psi(x,t) - \psi_0(x-x_0,t-t_0)e^{i\alpha_0}\right| \to 0 \quad \text{as} \quad t\to\pm\infty \quad ???$$

This rogue wave *appears from nowhere and disappears without trace*.

Let ψ be a solution of the NLS and $\phi = (p,q)^T$ be solution of the Lax system:

$$\phi_x = \begin{pmatrix} \lambda & \psi \\ -\bar{\psi} & -\lambda \end{pmatrix} \phi, \qquad \phi_t = i \begin{pmatrix} \lambda^2 + \frac{1}{2}|\psi|^2 & \frac{1}{2}\psi_x + \lambda\psi \\ \frac{1}{2}\bar{\psi}_x - \lambda\bar{\psi} & -\lambda^2 - \frac{1}{2}|\psi|^2 \end{pmatrix} \phi.$$

Let $\varphi = (p_1, q_1)$ be a nonzero solution of the Lax system for $\lambda = \lambda_1 \in \mathbb{C}$. The following one-fold Darboux transformation (DT):

$$\hat{\psi} = \psi + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1\bar{q}_1}{|p_1|^2 + |q_1|^2},$$

provides another solution $\hat{\psi}$ of the same NLS equation.

Lax and stability spectra for the dn-periodic wave

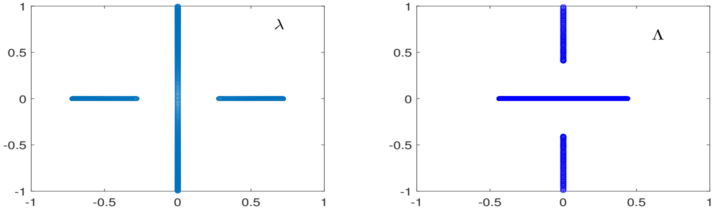
Similarly to the DNLS case, the periodic waves are related to the polynomial,

$$P(\lambda) = \lambda^4 - \frac{1}{2}(u_1^2 + u_2^2)\lambda^2 + \frac{1}{16}(u_1^2 - u_2^2)^2$$

with two pairs of roots $\{\pm\lambda_1, \pm\lambda_2\}$:

$$\lambda_1 = \frac{u_1 + u_2}{2}, \quad \lambda_2 = \frac{u_1 - u_2}{2}.$$

The dn-periodic wave has $u_1 = 1$, $u_2 = \sqrt{1-k^2}$:



Which value of λ to use for Darboux transformation to get rogue wave?

Let $\phi = (p_1, q_1)^T$ be the periodic eigenvector for the eigenvalue λ_1 with $\Lambda = 0$ (a root in the algebraic method). The second, linearly independent solution $\phi = (\hat{p}_1, \hat{q}_1)$ can be defined in several ways, e.g.

$$\hat{p}_1 = p_1\theta_1 - \frac{\bar{q}_1}{|p_1|^2 + |q_1|^2}, \quad \hat{q}_1 = q_1\theta_1 + \frac{\bar{p}_1}{|p_1|^2 + |q_1|^2},$$

such that $p_1\hat{q}_1 - \hat{p}_1 q_1 = 1$ (the Wronskian is normalized to 1).

The scalar function $\theta_1(x,t)$ satisfies

$$\frac{\partial \theta_1}{\partial x} = -\frac{4(\lambda_1 + \bar{\lambda}_1)\bar{p}_1\bar{q}_1}{(|p_1|^2 + |q_1|^2)^2}$$

and

$$\frac{\partial \theta_1}{\partial t} = -\frac{4i(\lambda_1^2 - \bar{\lambda}_1^2)\bar{p}_1\bar{q}_1}{(|p_1|^2 + |q_1|^2)^2} + \frac{2i(\lambda_1 + \bar{\lambda}_1)(\psi\bar{p}_1^2 + \bar{\psi}\bar{q}_1^2)}{(|p_1|^2 + |q_1|^2)^2}.$$

It is generally a linear growing function of (x,t) as $|x| + |t| \to \infty$.

Rogue wave on the dn-periodic background

Here we have $\psi(x,t) = \mathrm{dn}(x;k)e^{i(1-k^2/2)t}$, $|p_1|^2 + |q_1|^2 = \mathrm{dn}(x;k)$, and

$$\theta_1(x,t) = 2x + 2i(1 \pm \sqrt{1-k^2})t \pm 2\sqrt{1-k^2}\int_0^x \frac{dy}{\mathrm{dn}^2(y;k)},$$

such that $|\theta_1(x,t)| \to \infty$ as $|x| + |t| \to \infty$.

Rogue wave on the background ψ is generated by the DT:

$$\hat{\psi} = \psi + \frac{2(\lambda_1 + \bar{\lambda}_1)\hat{p}_1 \bar{\hat{q}}_1}{|\hat{p}_1|^2 + |\hat{q}_1|^2},$$

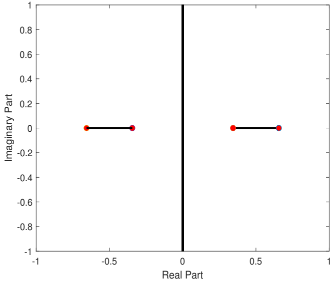
where

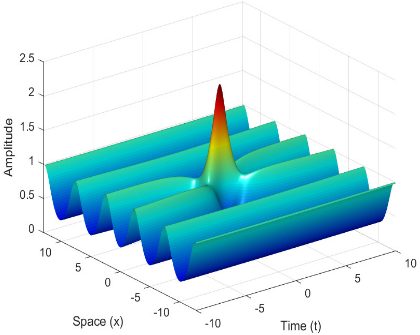
$$\hat{p}_1 = p_1\theta_1 - \frac{2\bar{q}_1}{|p_1|^2 + |q_1|^2}, \quad \hat{q}_1 = q_1\theta_1 + \frac{2\bar{p}_1}{|p_1|^2 + |q_1|^2}.$$

As $t \to \pm\infty$,

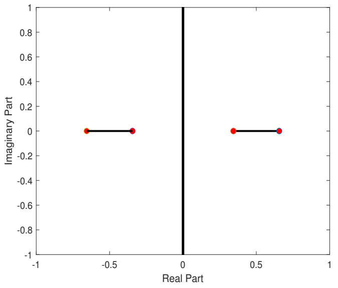
$$\hat{\psi}(x,t)|_{|\theta_1|\to\infty} = \psi + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1\bar{q}_1}{|p_1|^2 + |q_1|^2} = \mathrm{dn}(x + K(k);k)e^{i(1-k^2/2)t}.$$

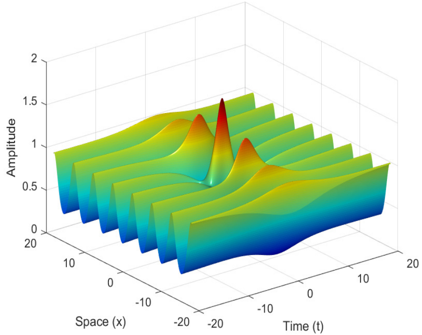
The rogue wave for the larger eigenvalue $\lambda_1 = \frac{1}{2}(u_1 + u_2)$ has the larger magnification $M(k) = 2 + \sqrt{1-k^2}$, $k \in [0,1]$.





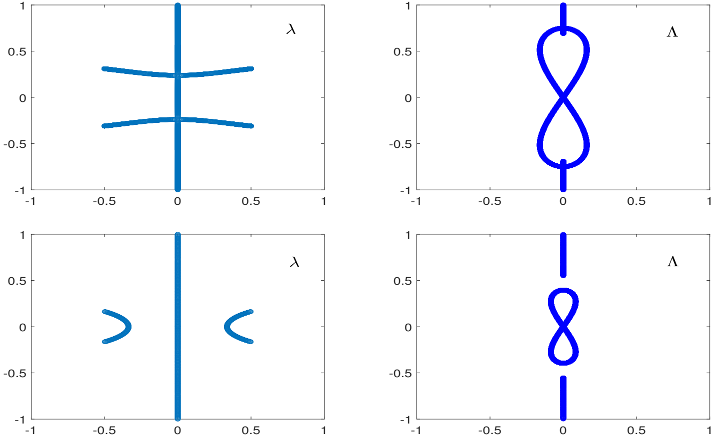
The rogue wave for the smaller eigenvalue $\lambda_2 = \frac{1}{2}(u_1 - u_2)$ has the smaller magnification $M(k) = 2 - \sqrt{1-k^2}$, $k \in [0, 1]$.





Lax and stability spectra for the cn-periodic wave

Here polynomial $P(\lambda) = \lambda^4 - \frac{1}{2}(u_1^2 + u_2^2)\lambda^2 + \frac{1}{16}(u_1^2 - u_2^2)^2$ has a quadruplet of roots $\{\pm\lambda_1, \pm\bar{\lambda}_1\}$ with $\lambda_1 = \frac{1}{2}(u_1 + u_2)$, where $u_1 = k$, $u_2 = i\sqrt{1-k^2}$.



Rogue wave on the cn-periodic background

Here we have $\psi(x,t) = k\mathrm{cn}(x;k)e^{i(k^2-1/2)t}$, $|p_1|^2 + |q_1|^2 = \mathrm{dn}(x;k)$, and

$$\theta_1(x,t) = 2k^2 \int_0^x \frac{\mathrm{cn}^2(y;k)dy}{\mathrm{dn}^2(y;k)} \mp 2ik\sqrt{1-k^2}\int_0^x \frac{dy}{\mathrm{dn}^2(y;k)} + 2ikt$$

such that $|\theta_1(x,t)| \to \infty$ as $|x| + |t| \to \infty$.

Rogue wave on the background u is generated by the DT:

$$\hat{\psi} = \psi + \frac{2(\lambda_1 + \bar{\lambda}_1)\hat{p}_1\bar{\hat{q}}_1}{|\hat{p}_1|^2 + |\hat{q}_1|^2},$$

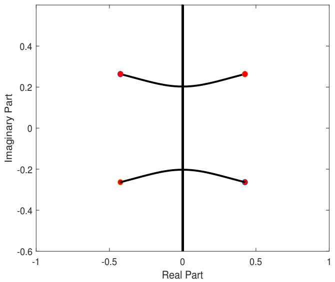
where

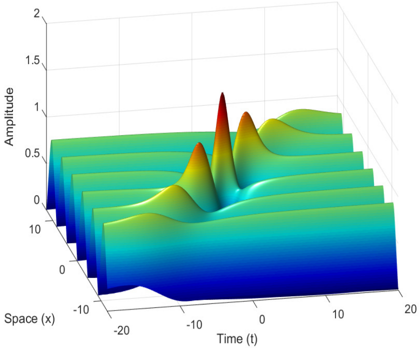
$$\hat{p}_1 = p_1\theta_1 - \frac{2\bar{q}_1}{|p_1|^2 + |q_1|^2}, \quad \hat{q}_1 = q_1\theta_1 + \frac{2\bar{p}_1}{|p_1|^2 + |q_1|^2}.$$

As $t \to \pm\infty$,

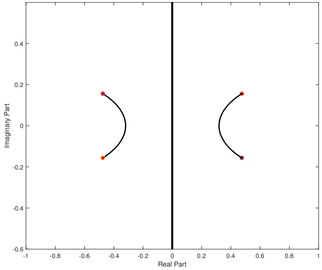
$$\hat{\psi}(x,t)|_{|\theta_1|\to\infty} = \psi + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1\bar{q}_1}{|p_1|^2 + |q_1|^2} = k\mathrm{cn}(x + K(k);k)e^{i(k^2-1/2)t}.$$

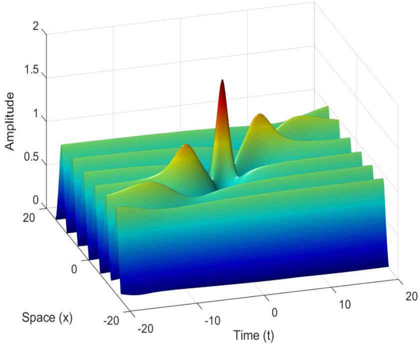
The rogue wave has exactly the double magnification factor.



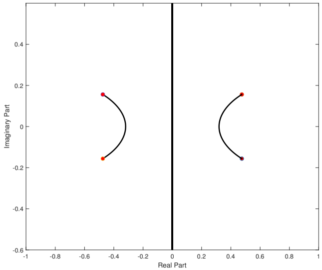


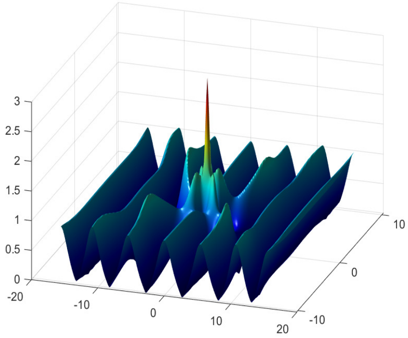
The rogue wave has exactly the double magnification factor.





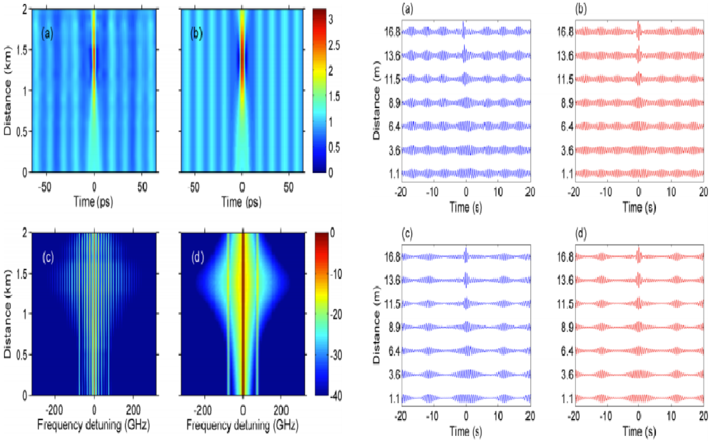
With the two-fold Darboux transformations, one can use both eigenvalues and construct a more symmetric rogue waves on the cn-periodic background. The rogue wave has exactly the triple magnification factor.





Experimental observations of rogue waves

The same rogue waves are observed in optics (left) and hydrodynamics (right). Thanks to G. Xu, B. Kibler (left) and A. Chabchoub (right).



Relation to modulation instability of the periodic wave

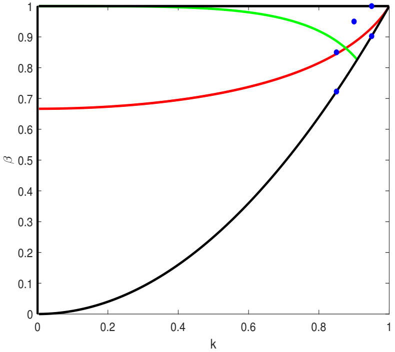
The NLS equation admits the periodic waves with nontrivial phase:

$$u(x) = R(x) e^{i\Theta(x)} e^{2ibt}$$

with

$$R(x) = \sqrt{\beta - k^2 \mathrm{sn}^2(x;k)}, \quad \Theta(x) = -2a \int_0^x \frac{dx}{R(x)^2},$$

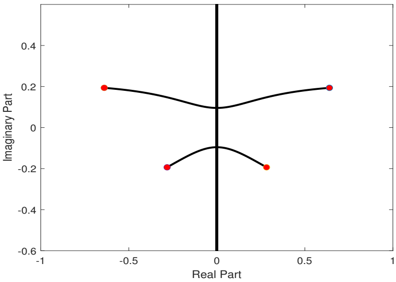
where β and k are two parameters. **B. Deconinck–B.L. Segal (2017)**.

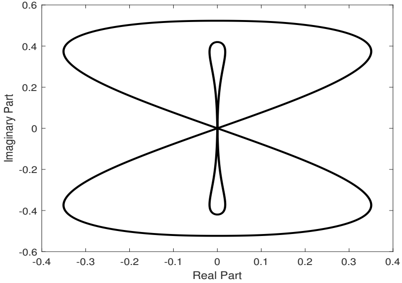


$\beta = 1$: dn-periodic waves.
$\beta = k^2$: cn-periodic waves.

Green curve:
separates two patterns of Lax spectrum.

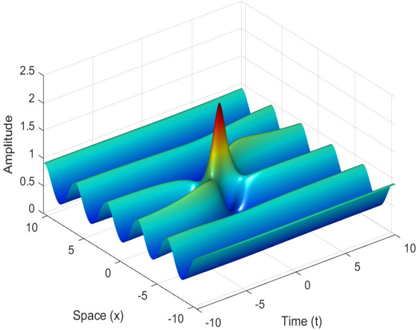
Red curve:
modulational instability of the second band vanishes.

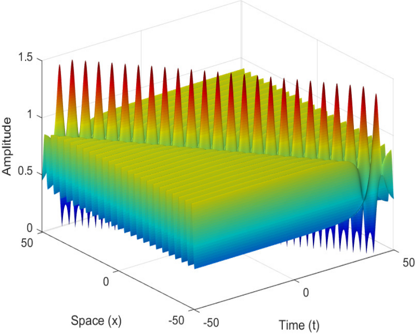

0.4
0.2
0
-0.2
-0.4
-0.6
-1
-0.5
0
0.5
1
Imaginary Part
Real Part


0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
Imaginary Part
Real Part

Here are two rogue waves obtained from the one-fold Darboux transformation associated with the eigenvalues $\lambda = \frac{1}{2}(\sqrt{\rho_1} \pm \sqrt{\rho_2}) \pm \frac{i}{2}\sqrt{-\rho_3}$. The rogue wave is defined by the growth of the function

$$\theta_1(x,t) = 2\int_0^x \frac{\rho(y) \pm \sqrt{\rho_1\rho_2} \mp i\sqrt{-\rho_3}(\sqrt{\rho_1} \pm \sqrt{\rho_2})}{\mathrm{dn}^2(y;k)} dy + 2i(\sqrt{\rho_1} \pm \sqrt{\rho_2})t$$



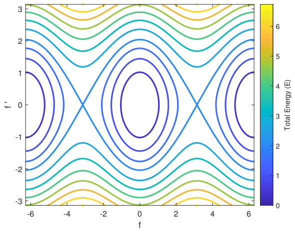


Travelling periodic waves in the sine–Gordon equation

The sine–Gordon equation is

$$u_{tt} - u_{xx} + \sin(u) = 0.$$

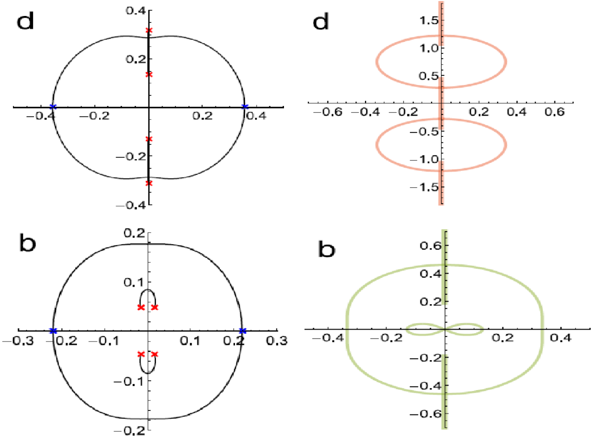
The travelling wave solutions $u(x,t) = f(x - ct)$ with $c > 1$ satisfy (after Lorentz transformation) the differential equation $f'' + \sin(f) = 0$.



Rotational solutions:
$f'(x) = \pm 2k^{-1}\mathrm{dn}(k^{-1}x, k)$.

Librational solutions:
$f'(x) = 2k\mathrm{cn}(x, k)$.

Spectral and modulational instability of periodic waves



Top:
Rotational waves
Bottom:
Librational waves

Left:
Lax spectrum
Right:
Stability spectrum

B. Deconinck–P. McGill–B.L. Segal (2017)
C. Jones, R. Marangell, P. Miller, R.G. Plaza (2013).

Real-valued Hamiltonian system

The Lax system of linear equations is written in characteristic form:

$$\frac{\partial}{\partial \xi}\begin{bmatrix} p \\ q \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \lambda & -u_\xi \\ u_\xi & -\lambda \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}$$

and

$$\frac{\partial}{\partial \eta}\begin{bmatrix} p \\ q \end{bmatrix} = \frac{1}{2\lambda}\begin{bmatrix} \cos(u) & \sin(u) \\ \sin(u) & -\cos(u) \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix},$$

where $\xi = \frac{1}{2}(x+t)$ and $\eta = \frac{1}{2}(x-t)$.

The real-valued Hamiltonian system is obtained with the constraint:

$$-u_\xi = p_1^2 + q_1^2,$$

where $(p_1, q_1)^T$ is an eigenvector for the eigenvalue λ_1, which is a root of the polynomial $P(\lambda)$, and $u = f(\xi - \eta)$ being the normalized periodic wave.

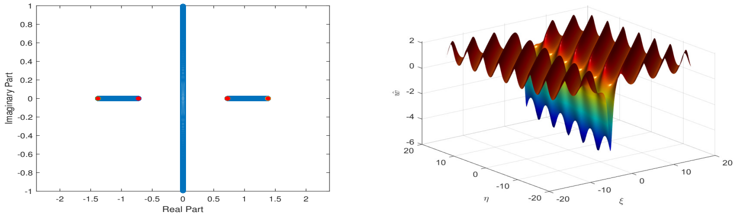
Rogue wave on the rotational background

One-fold transformation with the second solution

$$\hat{p}_1 = p_1\theta_R - \frac{q_1}{p_1^2+q_1^2}, \quad \hat{q}_1 = q_1\theta_R + \frac{p_1}{p_1^2+q_1^2},$$

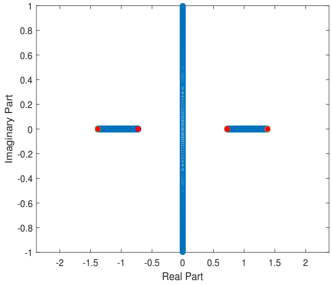
with $\theta_R(\xi,\eta) = C + \frac{1}{2}(\xi+\eta) - \frac{Hk^3}{2(1-k^2)}\int_0^{k^{-1}(\xi-\eta)} \mathrm{dn}^2(z+K(k);k)dz.$

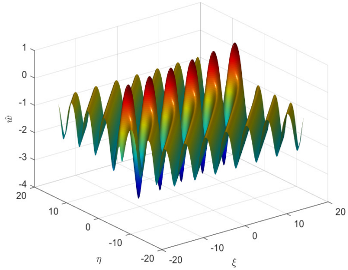
This rogue wave corresponds to the larger positive eigenvalue λ_1.



Another rogue wave on the rotational background

This rogue wave corresponds to the smaller positive eigenvalue λ_1.

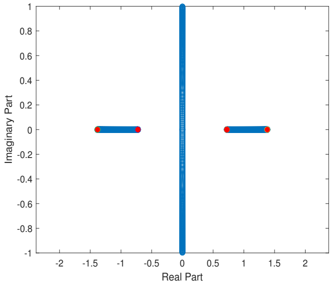


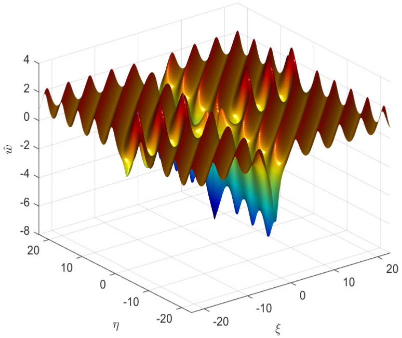


Both rogue waves on the rotational background

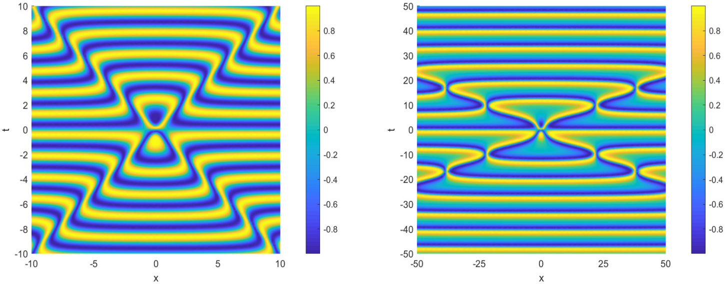
Using two eigenvalues in the two-fold Darboux transformation gives the kink-antikink solution with the speeds on (x,t) plane:

$$x = \pm \frac{E(k)}{\sqrt{1-k^2}K(k)} t,$$

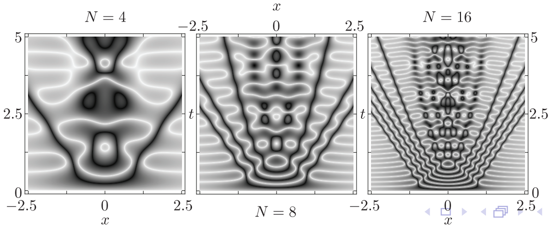




Computing $\sin(\hat{u}) = \hat{u}_{\xi\eta}$ by numerically differentiating $\hat{w} = -\hat{u}_{\xi}$ in η with a forward difference yields the surface plots of $\sin(\hat{u})$ in (x, t).



Compare with R.J. Buckingham–P.D. Miller (2013):

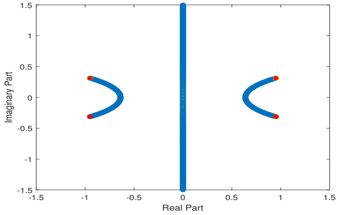


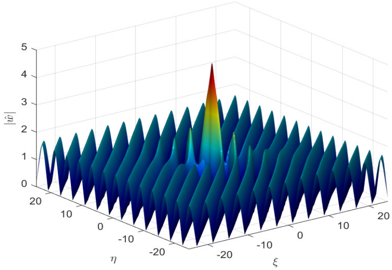
Rogue wave on the librational background

One-fold transformation with the second solution

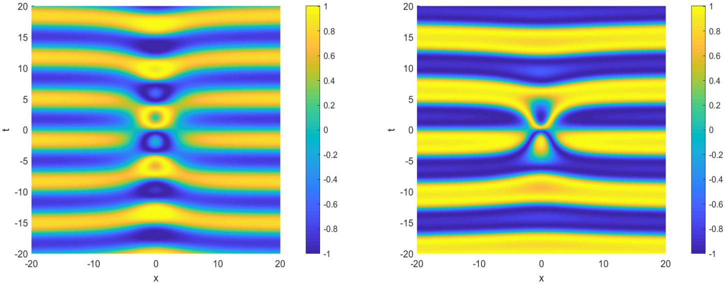
$$\hat{p}_1 = \frac{\theta_L - 1}{q_1}, \quad \hat{q}_1 = \frac{\theta_L + 1}{p_1},$$

with $\theta_L(\xi,\eta) = (4H - (f')^2)\left(C + \frac{\eta}{2\lambda_1} + \int_0^{\xi-\eta} \frac{2\lambda_1 (f')^2 dx}{(4H-(f')^2)^2}\right)$

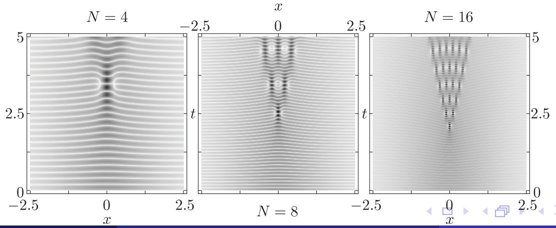




Computing $\sin(\hat{u}) = \hat{u}_{\xi\eta}$ by numerically differentiating $\hat{w} = -\hat{u}_{\xi}$ in η with a forward difference yields the surface plots of $\sin(\hat{u})$ in (x, t).



Compare with R.J. Buckingham–P.D. Miller (2013):



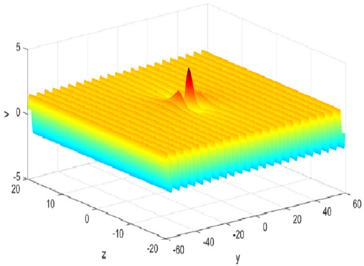
Rogue waves by other methods

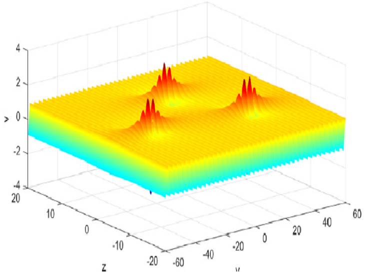
We are using the Darboux transformation (DT):

$$\hat{u}_\xi = u_\xi - \frac{4\lambda pq}{p^2 + q^2}.$$

R.Li–X.Geng (2020) used another DT in the form:

$$\hat{u} = u - 4\arctan\left[\tan(\arg(\lambda_1))\tanh(\mathrm{Im}(p_1/q_1))\right].$$





B.Y. Lu–P.D. Miller (2020) used DT in the Riemann–Hilbert problem.

Summary

- Periodic waves of integrable equations are constructed by using either real or complex Hamiltonian systems
- This allows us to characterize the periodic waves in terms of eigenvalues of the Lax equations associated with the periodic eigenfunctions
- We obtain the precise location of Lax and stability spectra, with assistance of the numerical package based on the Hill's method.
- We further obtain exact solutions describing localized structures on the background of periodic waves (either rogue waves or propagating algebraic solitons), with assistance of the Darboux transformations.
- Full localization of rogue waves is related to the modulational instability of the background periodic wave.

Thank you for listening!