Turing Bifurcation on Deterministic and Random Graphs

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Plan of the seminar

1. Pinning phenomena for traveling waves in lattices based on
   [H.J. Hupkes, D.P, B. Sandstede, Proceedings of AMS 139 (2011) 3537]

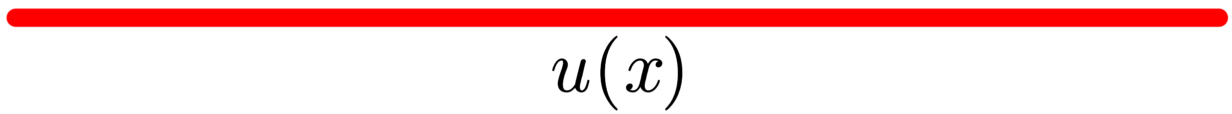
2. Turing Bifurcation in graphons based on
   [G. Medvedev, D.P, Journal of Nonlinear Science 34 (2024) 88]

3. Pinning of optical solitons in Lugiato–Lefever model with two pumping forces based on [L. Bengel, D.P., W. Reichel, SIAM J. Math. Anal. 56 (2024) 3679]

Section 1. Broken translational symmetry for Nagumo model

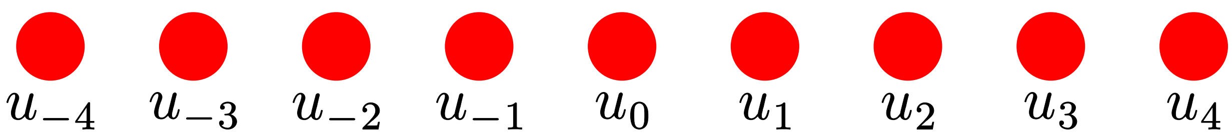
The PDE with the translational symmetry $u(t,x) \to u(t,x+x_0)$, $\forall x_0 \in \mathbb{R}$:

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + f\big(u(t,x)\big), \qquad x \in \mathbb{R}.$$



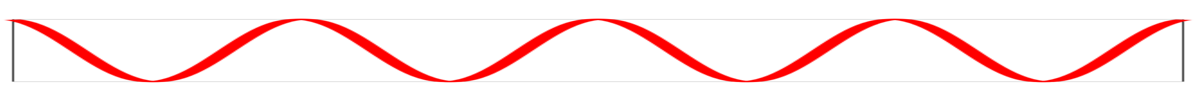
Lattice differential equations are ODEs on a spatial lattice with the step size h:

$$\frac{d}{dt} u_j(t) = h^{-2}\big(u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)\big) + f\big(u_j(t)\big), \qquad j \in \mathbb{Z}.$$



Spatial inhomogeneity can be modeled with spatially dependent potentials

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + f\big(u(t,x)\big) + V(x)u(t,x), \qquad x \in \mathbb{R}.$$



Example: TW in PDEs

Consider the continuous Nagumo equation,

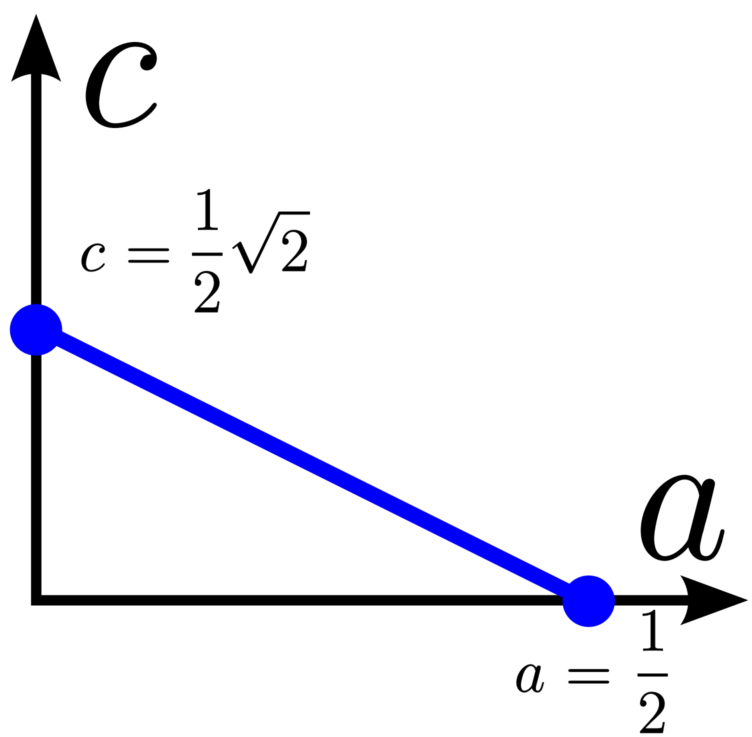
$$\partial_t u = \partial_x^2 u + u(a-u)(u-1), \quad a \in (0,1).$$

Travelling wave $u(x,t) = \phi(x+ct)$ satisfies:

$$c\phi'(\xi) \quad = \quad \phi''(\xi) + \phi(\xi)\big(a - \phi(\xi)\big)\big(\phi(\xi) - 1\big).$$

There exists a heteroclinic connection between stable equilibrium states 0 and 1:

$$\begin{aligned} \phi(\xi) &= \tfrac{1}{2} + \tfrac{1}{2}\tanh\big(\tfrac{1}{4}\sqrt{2}\,\xi\big), \\ c(a) &= \tfrac{1}{\sqrt{2}}(1-2a). \end{aligned}$$



Example: TW in LDEs

Consider the discrete Nagumo equation,

$$\frac{d}{dt}U_j(t) = \frac{1}{h^2}\left[U_{j+1}(t) + U_{j-1}(t) - 2U_j(t)\right] + U_j(t)(a - U_j(t))(U_j(t) - 1),\ j \in \mathbb{Z}.$$

Travelling front solutions $U_j(t) = \phi(j + ct)$ must satisfy:

$$c\phi'(\xi) = \frac{1}{h^2}\left[\phi(\xi + h) + \phi(\xi - h) - 2\phi(\xi)\right] + \phi(\xi)(a - \phi(\xi))(\phi(\xi) - 1),$$

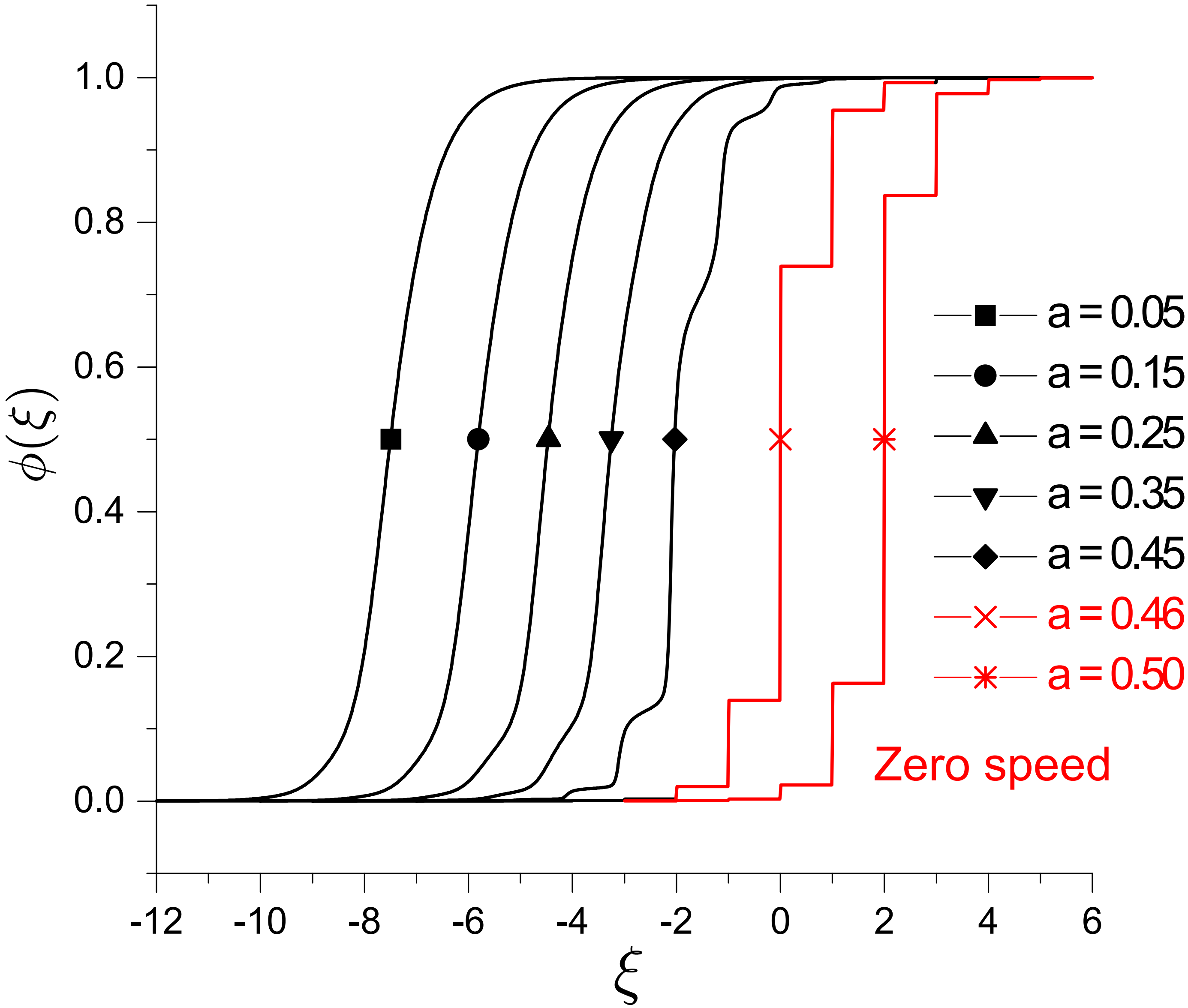
subject to

$$\lim_{\xi\to-\infty}\phi(\xi) = 0, \quad \lim_{\xi\to+\infty}\phi(\xi) = 1.$$

- When $c \neq 0$, this is a differential advance-delay equation with $\phi \in C^\infty(\mathbb{R})$.
- When $c = 0$, this is an advance-delay equation with generally $\phi \notin C^0(\mathbb{R})$.
- The limit $c \to 0$ is a singular perturbation theory.

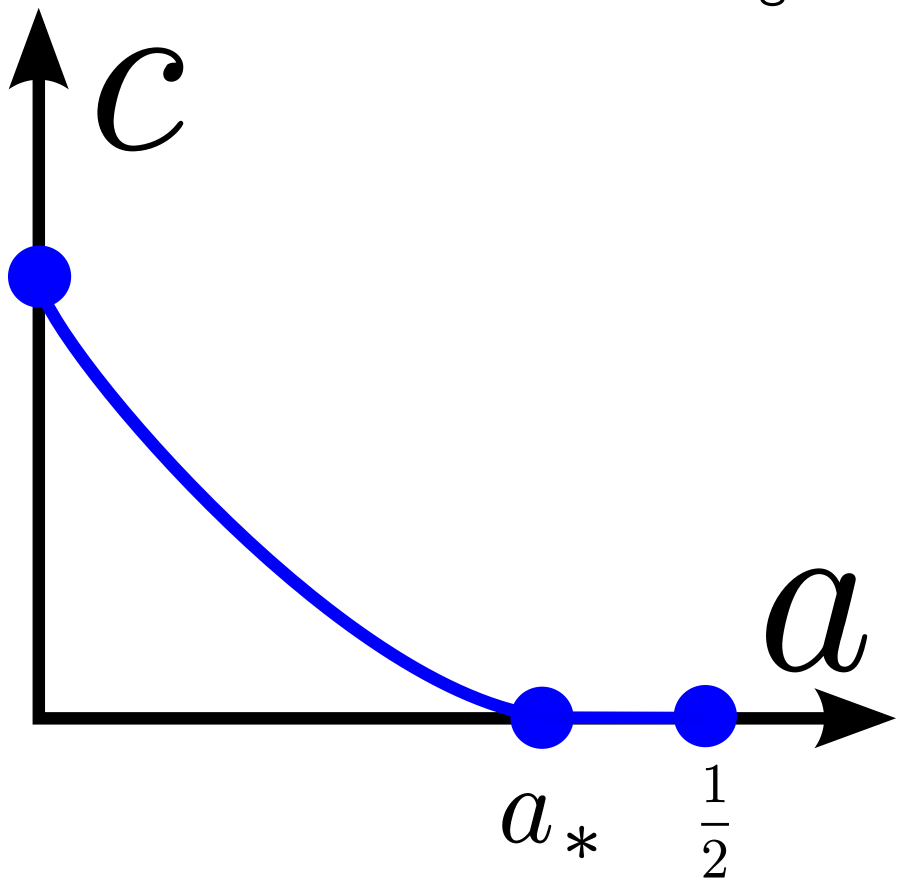
Numerical results for heteroclinic connections

Travelling waves for the discrete Nagumo LDE connecting $0 \to 1$.



Propagation failure

Plot of the wave speed c versus a for the discrete Nagumo LDE:



If $a_* < \frac{1}{2}$, we say that TW suffers from propagation failure or pinning of stationary solutions to lattice sites.

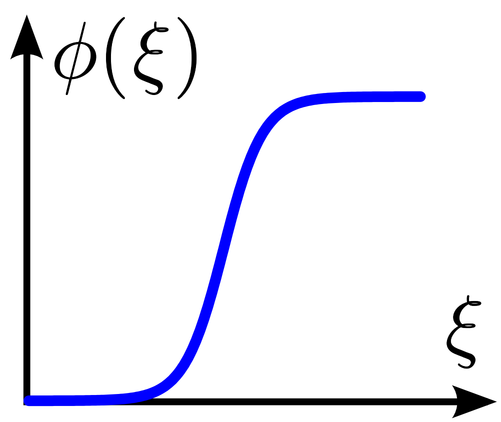
Example with no propagation failure

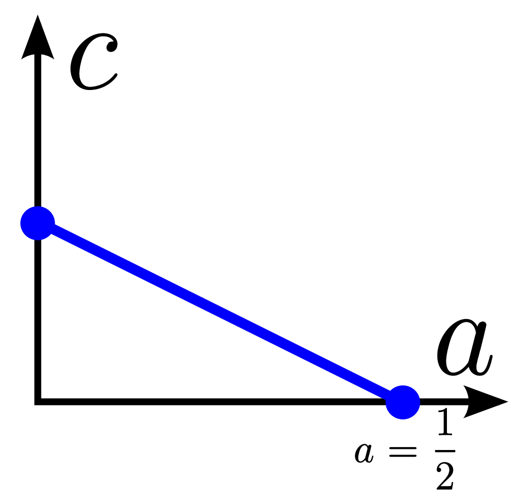
Consider the discrete Nagumo equation with a modified cubic nonlinearity:

$$\frac{d}{dt}U_j = \frac{1}{h^2}[U_{j-1} + U_{j+1} - 2U_j] + \frac{1}{2}U_j(2a - U_{j+1} - U_{j-1})(U_j - 1).$$

Explicit solutions available:

$$U_j(t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\operatorname{arcsinh}(\frac{1}{4}\sqrt{2}\,h)(j + ct)\right), \qquad c(a) = \frac{(1-2a)}{4\operatorname{arcsinh}(\frac{1}{4}\sqrt{2}\,h)}.$$





No propagation failure; smooth wave profile.

Why does pinning occur and how it can be avoided?

Consider LDEs for $c = 0$ with variables $p_j = \phi(j)$ and $r_j = \phi(j+1)$:

$$\begin{aligned} p_{j+1} &= r_j \\ r_{j+1} &= -p_j + 2r_j - h^2 r_j (a - r_j)(r_j - 1). \end{aligned}$$

Two fixed point $(0, 0)$ and $(1, 1)$ are saddles. Two heteroclinic orbits exist for $a = \frac{1}{2}$ (symmetric case) at the two points of discrete symmetry:

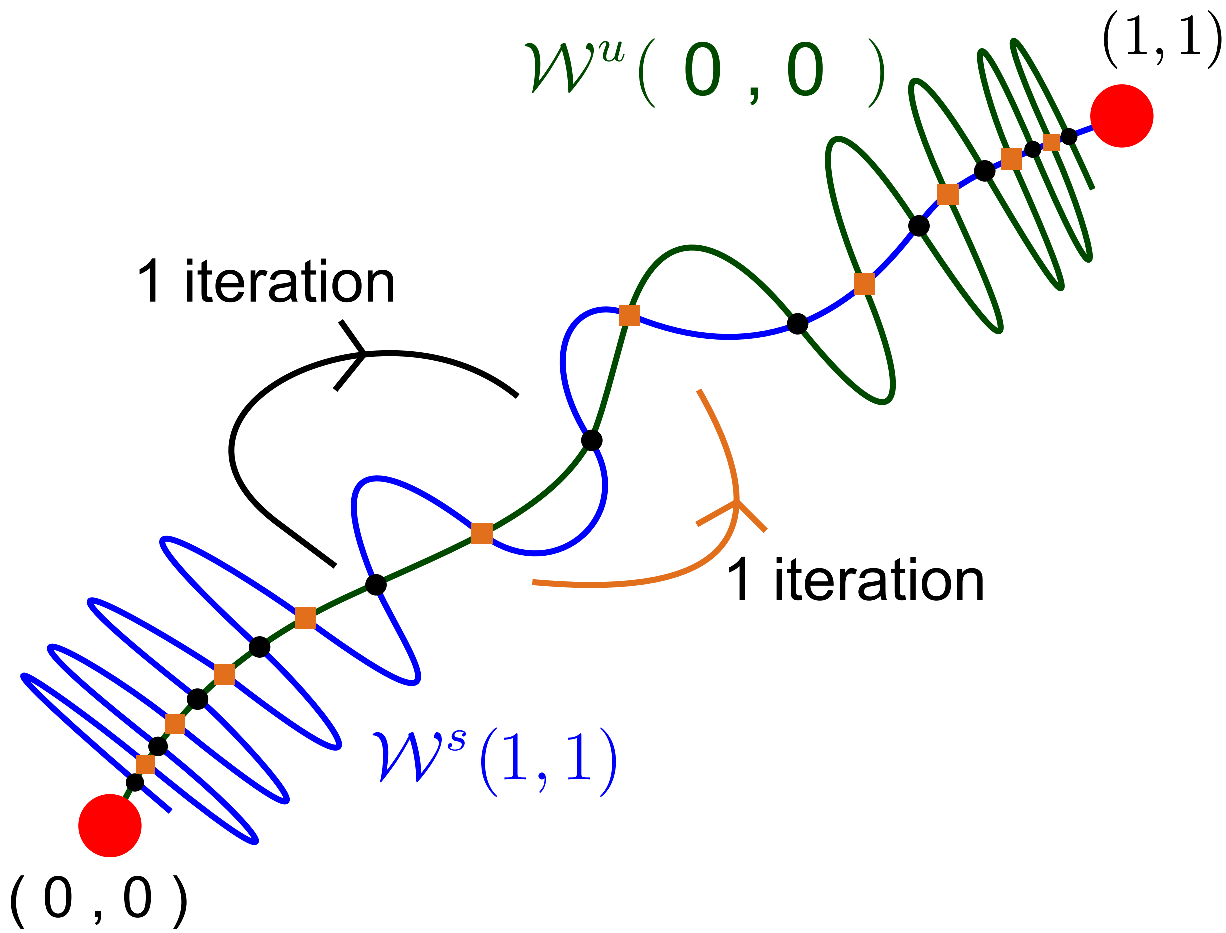
$$p^{(s)}_{-j} = 1 - p^{(s)}_j, \quad p^{(b)}_{-j+1} = 1 - p^{(b)}_j,$$

called site-symmetric and bond-symmetric fronts.

There are generally no other heteroclinic orbits except those generated from the two families by the discrete group of symmetries.

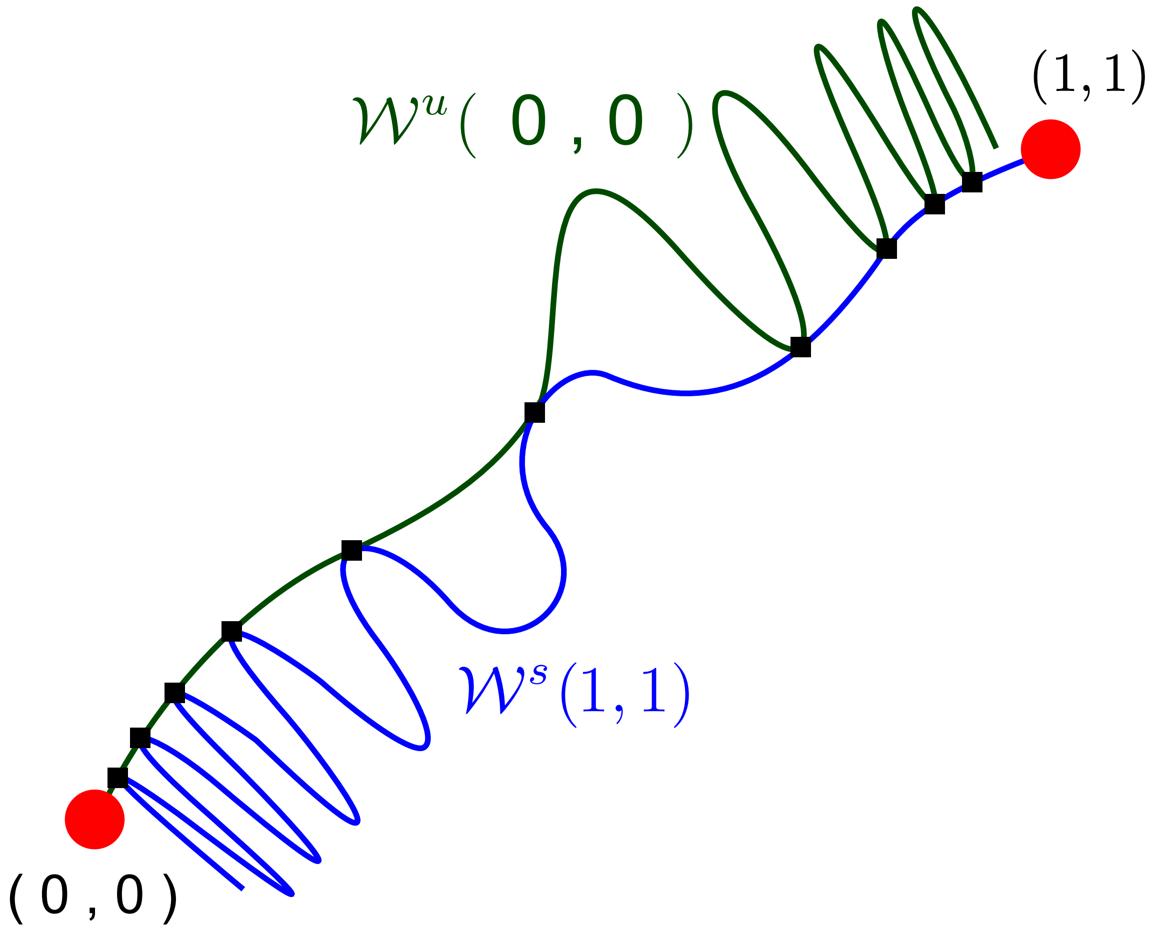
Why does pinning occur?

For $a = \frac{1}{2}$, site-symmetric (orange) and bond-symmetric (black) solutions:



Why does pinning occur?

For $a < \frac{1}{2}$, the distance between nodes decreases. At $a = a_* < \frac{1}{2}$, two branches of stationary front solutions coincide and annihilate via a saddle-node bifurcation.

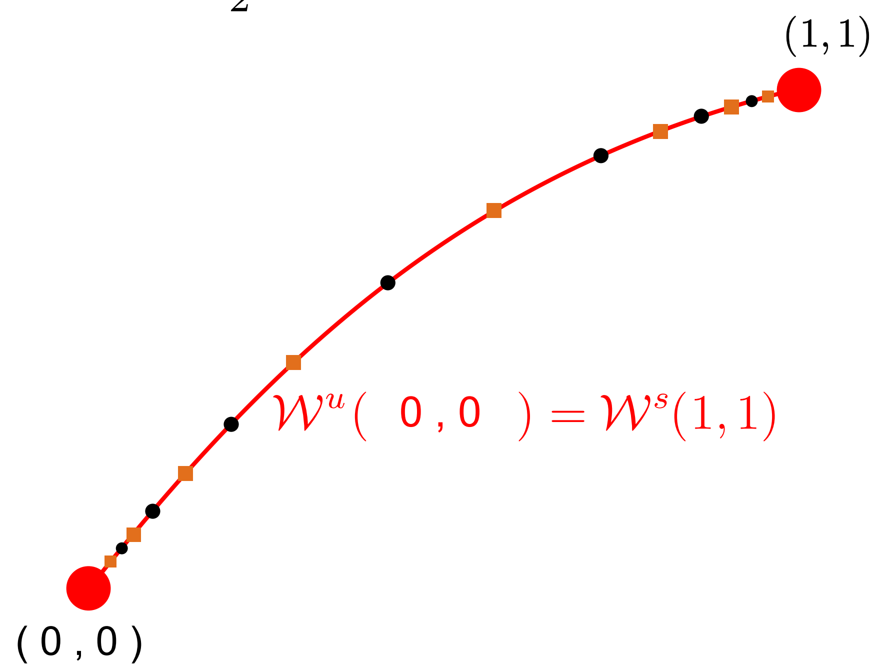


How can pinning be avoided?

For the system

$$\begin{aligned} p_{j+1} &= r_j \\ r_{j+1} &= -p_j + 2r_j - \tfrac{1}{2}h^2 r_j (2a - r_{j+1} - r_{j-1})(r_j - 1) \end{aligned}$$

with the smooth profile at $a = \frac{1}{2}$, we have



Site-symmetric and bond-symmetric solutions are connected by a continuous branch of "continuously translationally invariant" heteroclinic solutions.

The continuous branch is continued for $a \neq \frac{1}{2}$ as a TW solution with $c \neq 0$ under some technical assumptions [H.J. Hupkes, D.P, B. Sandstede (2011)]

Section 2. Turing bifurcations in graphons

Two distinct nodes of a graph i and j are connected with probability $a_{ij} \in [0,1]$,

$$\mathbb{P}(\tilde{a}_{ij} = 1) = a_{ij}, \quad \mathbb{P}(\tilde{a}_{ij} = 0) = 1 - a_{ij}, \tag{1}$$

with $\tilde{a}_{ji} = \tilde{a}_{ij}$ and $a_{ii} = 0$. The indirect graph is assumed to have $2N$ nodes.

The adjacency matrix is $\tilde{A}^N = (\tilde{a}_{ij})_{-N+1 \le i,j \le N}$ in the random case and $A^N = (a_{ij})_{-N+1 \le i,j \le N}$ in the deterministic case. The corresponding linear operators acting on $u = (u_j)_{-N+1 \le j \le N}$ are denoted by

$$\tilde{L}^N u := \frac{1}{2N}\left(\tilde{A}^N - \tilde{D}^N\right) u, \quad L^N u := \frac{1}{2N}\left(A^N - D^N\right) u,$$

where $\tilde{D}^N$ and D^N are diagonal matrices called degree matrices, e.g.

$$(\tilde{D}^N)_{ii} := \sum_{j=-N+1}^{N} \tilde{a}_{ij}, \quad (D^N)_{ii} := \sum_{j=-N+1}^{N} a_{ij}.$$

Small-world graph as an example of Cayley graphs

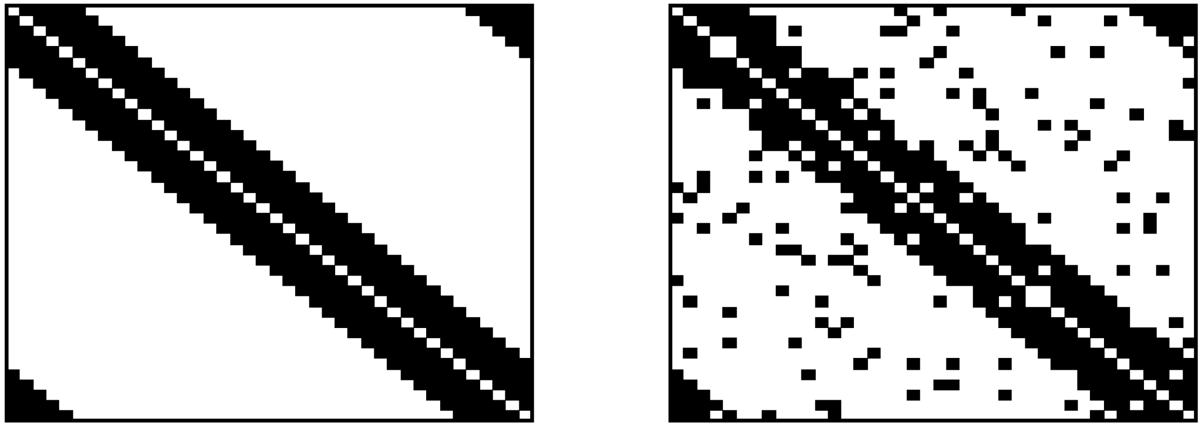


Figure 1: Adjacency matrix in the deterministic (left) and random (right) cases.

Fix $p \in [0,1]$, $r \in (0, \frac{1}{2})$, and define

$$S(x) = \begin{cases} 1-p, & |x| \le r, \\ p, & r < |x| \le \frac{1}{2}. \end{cases}$$

The small-world graph is generated by the discrete convolution with

$$a_{ij} := S_{i-j}.$$

This has a long history since Watts–Strogatz (1998) and has been used in Medvedev (2014) and Medvedev–Tang (2015).

Swift–Hohenberg equation

The main equation of motion is the Swift–Hohenberg equation on the graphon:

- Random: $\dot{u} = -\left(\tilde{L}^N - \kappa\right)^2 u + \gamma u - u^3$
- Deterministic: $\dot{u} = -\left(L^N - \kappa\right)^2 u + \gamma u - u^3$,

where κ and γ are parameters.

Both models converge as $N \to \infty$ to the limiting model given by the continuous Swift–Hohenberg equation (Medvedev, 2014):

$$\partial_t u = -\left(L - \kappa\right)^2 u + \gamma u - u^3, \quad x \in \mathbb{T},$$

where $\mathbb{T}$ is the unit torus and

$$(Lf)(x) = \int_{\mathbb{T}} S(x-y)\left[f(y) - f(x)\right] dy =: (Af)(x) - Df(x).$$

The Swift–Hohenberg is nonlocal and translationally invariant.

Turing bifurcations

The limiting Swift–Hohenberg equation,

$$\partial_t u = -\left(L-\kappa\right)^2 u + \gamma u - u^3, \quad x \in \mathbb{T},$$

is well-posed for $u \in C^1([0,\infty), L^\infty(\mathbb{T}))$ if $S \in L^1(\mathbb{T})$.

Eigenvalues of L in $L^2(\mathbb{T})$ are real if S is even on $\mathbb{T}$

$$\lambda_k = \int_{\mathbb{T}} S(x) e^{-2\pi i k x} dx - D, \quad k \in \mathbb{Z}.$$

Let $\kappa = \lambda_1 = \lambda_{-1}$ and γ be a small parameter. Then, the center manifold theory gives the existence of the Turing bifurcation (Iooss–Haragus, 2011).

Theorem 1. *There exists $\gamma_0 > 0$ and $C_0 > 0$ such that for every $\gamma \in (0, \gamma_0)$ there exists a non-trivial time-independent solution $u_\gamma(\cdot + \delta)$ of the continuous SHE, where u_γ is an even function satisfying*

$$\sup_{x \in \mathbb{T}} \left| u_\gamma(x) - \frac{2\sqrt{\gamma}}{\sqrt{3}} \cos(2\pi x) \right| \leq C_0 \sqrt{\gamma^3}, \tag{2}$$

and $\delta \in \mathbb{T}$ is an arbitrary translational parameter. The orbit $\{u_\gamma(\cdot + \delta)\}_{\delta \in \mathbb{T}}$ is asymptotically stable in the time evolution in $L^\infty(\mathbb{T})$.

Why to consider Turing bifurcations on graphons?

This has a long history in physics literature: Asllani et al. (2014), Hutt et al. (2022), Kouvaris et al. (2015), Nakao-Mikhailov (2010), Wolfrum (2012). See also tutorial of M.Porter–J.Gleeson (2016).

A mathematical study started from the paper of J. Bramburger–M. Holtzer (2023), where bifurcations on graphons were concluded from bifurcations in the continuous SHE.

In our work [Medvedev–P (2024)], we clarify more precisely how the translational symmetry is broken and how pinnning of periodic patterns is determined.

Our strategy is to "upgrade" normal forms consequently: from continuous SHE, to the discrete deterministic SHE, and to the discrete random SHE.

Turing bifurcations for discrete graphs

The continuous SHE on $\mathbb{T}$ admits the following symmetries:

- the spatial translation $x \mapsto x + h$, $\forall h \in \mathbb{R}$ due to periodicity,
- the spatial reflection $x \mapsto -x$ due to even S,
- the sign reflection $u \mapsto -u$ due to odd nonlinearity.

The discrete deterministic SHE on $\mathbb{Z}_N := \mathbb{Z}/(2N\mathbb{Z})$ admits

- the discrete spatial translation $j \mapsto j + m$, $\forall m \in \mathbb{Z}_N$ due to periodicity,
- the spatial reflection $j \mapsto -j$ due to even $\{S_j\}_{j \in \mathbb{Z}_N}$,
- the sign reflection $u \mapsto -u$ due to odd nonlinearity.

The discrete random SHE on $\{-N+1, \dots, N\}$ admits only

- the sign reflection $u \mapsto -u$ due to odd nonlinearity.

Normal form for the continuous SHE

Starting with

$$\partial_t u = -\left(L - \lambda_1\right)^2 u + \gamma u - u^3, \quad x \in \mathbb{T},$$

use Fourier series

$$u(t,x) = \sum_{k \in \mathbb{Z}} a_k(t) e^{2\pi i k x}$$

and use the center manifold parameterization

$$a_1 = A, \quad a_{-1} = \bar{A}, \quad a_m = \Psi_m(A, \bar{A}), \quad a_{-m} = \bar{\Psi}_m(A, \bar{A}), \quad m \in \{3, 5, \cdots\}.$$

Due to symmetries of the continuous SHE, the amplitude equations are transformed to the normal form (Iooss–Haragus, 2011):

$$\dot{A} = A P_1(|A|^2),$$

where P_1 is a C^∞ function in $|A|^2$ with γ-dependent real-valued coefficients such that $P_1(|A|^2) = \gamma - 3|A|^2 + \mathcal{O}(|A|^4)$. This is the Ginzburg–Landau equation

$$\dot{A} = (\gamma - 3|A|^2 + \mathcal{O}(|A|^4))A.$$

Turing pattern for the continuous SHE

The time-independent solution of the normal form is

$$A_{\gamma,\delta} := \frac{\sqrt{\gamma}}{\sqrt{3}} \left[1 + \mathcal{O}(\gamma)\right] e^{2\pi i \delta},$$

where $\delta \in \mathbb{T}$ is arbitrary parameter.

The orbit is asymptotically stable since all but one eigenvalues of the linearized operator are in the left half-plane.

The simple zero eigenvalue represents the translational symmetry. This creates difficulties in the persistence argument from the continuous to discrete cases, when the continuous translational symmetry is destroyed.

Center manifold for the discrete deterministic SHE

We have now

$$\dot{u}_j = -[(L^N - \lambda_1^N)^2 u]_j + \gamma u_j - u_j^3, \quad j \in \mathbb{Z}_N := \mathbb{Z}/(2N\mathbb{Z}),$$

where

$$[L^N u]_j = \frac{1}{2N} \sum_{l \in \mathbb{Z}_N} S_{j-l} u_l - \left(\frac{1}{2N} \sum_{l \in \mathbb{Z}_N} S_l \right) u_j, \quad j \in \mathbb{Z}_N.$$

We use discrete Fourier transform

$$u_j(t) = \sum_{k \in \mathbb{Z}_N} a_k(t) e^{\frac{i\pi k j}{N}}, \quad j \in \mathbb{Z}_N,$$

and the center manifold parameterization

$$a_1 = A, \quad a_{-1} = \bar{A}, \quad a_m = \Psi_m(A, \bar{A}), \quad a_{-m} = \bar{\Psi}_m(A, \bar{A}), \quad m \in \{3, 5, \cdots\}.$$

Normal form for the discrete deterministic SHE

However, the finite discrete lattice implies that

$$\left[Ae^{\frac{i\pi kj}{N}}\right]^{2N+1} = A^{2N+1}e^{\frac{i\pi kj}{N}} \quad \text{and} \quad \left[\bar{A}e^{\frac{-i\pi kj}{N}}\right]^{2N-1} = \bar{A}^{2N-1}e^{\frac{i\pi kj}{N}}.$$

Due to the discrete group of symmetries, the amplitude equations are transformed to the normal form (Chossat–Lauterbach, 2000):

$$\dot{A} = AQ_1(|A|^2, A^{2N}, \bar{A}^{2N}) + \bar{A}^{2N-1}R_1(|A|^2, A^{2N}, \bar{A}^{2N}),$$

where Q_1 and R_1 are C^∞ functions in $|A|^2$, A^{2N}, $\bar{A}^{2N}$ with γ-dependent real-valued coefficients such that, if $N \geq 3$,

$$Q_1(|A|^2, A^{2N}, \bar{A}^{2N}) = \gamma - 3|A|^2 + \mathcal{O}(|A|^4)$$

and

$$R_1(|A|^2, A^{2N}, \bar{A}^{2N}) = r_N + \mathcal{O}(|A|^2),$$

where $r_N \neq 0$ for every $N \geq 3$.

Turing pattern for the discrete deterministic SHE

After separation of variables with $A = \rho e^{i\theta}$, we get

$$\begin{cases} \dot{\rho} = \rho[\gamma - 3\rho^2 + \mathcal{O}(\rho^4)] + \cos(2N\theta)\rho^{2N-1}[r_N + \mathcal{O}(\rho^2)], \\ \dot{\theta} = -\sin(2N\theta)\rho^{2N-2}[r_N + \mathcal{O}(\rho^2)]. \end{cases}$$

There exist $4N$ time-independent solutions with $\theta_k = \frac{k\pi}{2N} \in [0, 2\pi)$ and $0 \leq k \leq 4N - 1$, for which

$$A_{\gamma,\delta_k} = \frac{\sqrt{\gamma}}{\sqrt{3}} \left[1 + \mathcal{O}(\gamma)\right] e^{\frac{i\pi k}{2N}}.$$

Stability is given by the linearized normal form:

$$\begin{cases} \dot{\rho}_1 = -2\gamma\rho_1 + \mathcal{O}(\gamma^2), \\ \dot{\theta}_1 = \mp \left(\frac{\gamma}{3}\right)^{N-1} \left[r_N + \mathcal{O}(\gamma)\right] \theta_1, \end{cases}$$

(2N) states are asymptotically stable and (2N) states are unstable with exactly one real positive eigenvalues.

All eigenvalues are nonzero but one eigenvalue is very small of the size $\mathcal{O}(\gamma^{1-N})$.

Main theorem for the discrete deterministic SHE

Theorem 2. *There exists* $\gamma_0 > 0$ *and* $C_0 > 0$ *such that for every* $\gamma \in (0, \gamma_0)$ *and every integer* $N \geq 3$ *there exist two non-trivial time-independent solutions* $u_\gamma^N, v_\gamma^N \in \mathbb{R}^{\mathbb{Z}_N}$ *of the discrete SHE, where* u_γ^N *is symmetric about* $j = 0$ *and satisfies*

$$\sup_{j \in \mathbb{Z}_N} \left| u_j - \frac{2}{\sqrt{3}} \sqrt{\gamma} \cos\left(\frac{\pi j}{N}\right) \right| \leq C_0 \sqrt{\gamma^3}.$$

and v_γ^G *is symmetric about the mid-point between* $j = 0$ *and* $j = 1$ *and satisfies*

$$\sup_{j \in \mathbb{Z}_N} \left| u_j - \frac{2}{\sqrt{3}} \sqrt{\gamma} \cos\left(\frac{\pi j}{N} - \frac{\pi}{2N}\right) \right| \leq C_0 \sqrt{\gamma^3}.$$

One of the two solutions is asymptotically stable in the time evolution of the discrete SHE in $C^1(\mathbb{R}, \mathbb{R}^{\mathbb{Z}_N})$ *and the other one is unstable. These solutions generate* $(2N)$ *asymptotically stable and* $(2N)$ *unstable solutions on* $\mathbb{Z}_N$ *via the discrete group of spatial translations.*

Center manifold for the discrete random SHE

We have now

$$\dot{u}_j = -[(\tilde{L}^N - \lambda_1^N)^2 u]_j + \gamma u_j - u_j^3, \quad j \in \mathbb{Z}_N := \mathbb{Z}/(2N\mathbb{Z}),$$

where

$$\tilde{L}^N = \frac{1}{2N}\left(\tilde{A}^N - \tilde{D}^N\right), \text{ with } (\tilde{D}^N)_{ii} := \sum_{j=-N+1}^{N} \tilde{a}_{ij}$$

We use discrete Fourier transform ($u \xrightarrow{\mathcal{F}} a$)

$$u_j(t) = \sum_{k \in \mathbb{Z}_N} a_k(t) e^{\frac{i\pi kj}{N}}, \quad j \in \mathbb{Z}_N,$$

and the center manifold parameterization

$$a_1 = A, \quad a_{-1} = \bar{A}, \quad a_m = \Psi_m(A, \bar{A}), \quad a_{-m} = \bar{\Psi}_m(A, \bar{A}), \quad m \in \{0, 2, 3, \cdots\}.$$

With probability of at least $1 - \mathcal{O}(25^{-N})$ (Guedon–Vershunin, 2016),

$$\max_{-N+1 \leq j,k \leq N} |\mathcal{F}^{-1}(\tilde{L}^N - L^N)\mathcal{F}| \leq CN^{-1/2}.$$

Normal form for the discrete random SHE

Due to proximity, we can introduce a new small parameter $\mu := CN^{-1/2}$. The amplitude equations are transformed to the normal form without any symmetries:

$$\dot{A} = F_1(A, \bar{A}),$$

where

$$\begin{aligned} F_1(A, \bar{A}) = (\gamma + \mu\alpha_1)A + \mu\alpha_2\bar{A} \\ + (-3 + \mu\beta_1)|A|^2 A + \mu\beta_2 A^3 + \mu\beta_3 |A|^2 \bar{A} + \mu\beta_4 \bar{A}^3 + \dots \\ + [r_N + \mathcal{O}(\mu)]\bar{A}^{2N-1} + \dots \end{aligned}$$

For fixed (small) $\gamma \in (0, \gamma_0)$, there is sufficiently small μ (sufficiently large N) such that

$$\gamma + \mu\alpha_1 > 0, \quad -3 + \mu\beta_1 < 0, \quad r_N + \mathcal{O}(\mu) \neq 0.$$

Turing pattern for the discrete random SHE

After separation of variables with $A = \rho e^{i\theta}$, we get

$$\begin{cases} \dot{\rho} = [\gamma + \mu\alpha_1 + \mu\alpha_2 \cos(2\theta)]\rho \\ \qquad\qquad +[-3 + \mu\beta_1 + \mu\beta_2 \cos(2\theta) + \mu\beta_3 \cos(2\theta) + \mu\beta_4 \cos(4\theta)]\rho^3 + \mathcal{O}(\rho^5), \\ \dot{\theta} = -\mu\alpha_2 \sin(2\theta) + \mu[\beta_2 \sin(2\theta) - \beta_3 \sin(2\theta) - \beta_4 \sin(4\theta)]\rho^2 + \dots \\ \qquad\qquad -[r_N + \mathcal{O}(\mu)]\rho^{2N-2} \sin(2N\theta) + \mathcal{O}(\rho^{2N}). \end{cases}$$

There exists only one positive root of the first equation for every $\theta \in [0, 2\pi)$:

$$\left| \rho - \frac{\sqrt{\gamma}}{\sqrt{3}} \right| \leq C(\gamma + \mu\gamma^{-1})\sqrt{\gamma},$$

For this root, the second equation becomes

$$\begin{aligned} &-\mu\alpha_2 \sin(2\theta) + \mu[\beta_2 \sin(2\theta) - \beta_3 \sin(2\theta) - \beta_4 \sin(4\theta)]\rho^2 + \dots \\ &- [r_N + \mathcal{O}(\mu)]\rho^{2N-2} \sin(2N\theta) + \mathcal{O}(\rho^{2N}) = 0, \end{aligned}$$

with at least 4 roots and at most $4N$ roots on $[0, 2\pi)$ $(r_N \neq 0)$.

Main theorem for the discrete random SHE

Theorem 3. *Fix $\gamma \in (0, \gamma_0)$. There exist $N_0 \geq 3$ such that with probability of $1 - \mathcal{O}(5^{-N})$, for every $N \geq N_0$ there exist at least 4 and at most $4N$ values of δ such that the discrete SHE admits time-independent solutions $u \in \mathbb{R}^{\mathbb{Z}_N}$ satisfying*

$$\sup_{j \in \mathbb{Z}_N} \left| u_j - \frac{2}{\sqrt{3}} \sqrt{\gamma} \cos\left(\frac{\pi j}{N} - \delta \right) \right| \leq C_0 \sqrt{\gamma^3},$$

where the constant $C_0 > 0$ is independent of $N \geq N_0$ and $\gamma \in (0, \gamma_0)$.

- In the generic situation when all roots of the phase equation are simple, half of time-independent solutions are asymptotically stable and the other half is unstable.

- Due to the sign reflection symmetry $u \mapsto -u$, half of solutions are related to the other half by the sign reflection.

Numerical illustrations of Turing patterns

Numerical solutions of the small-world graph with $N = 400$. Initial guess:

$$u_j = \frac{2}{\sqrt{3}}\sqrt{\gamma}\cos\left(\frac{\pi j}{N} - \delta\right).$$

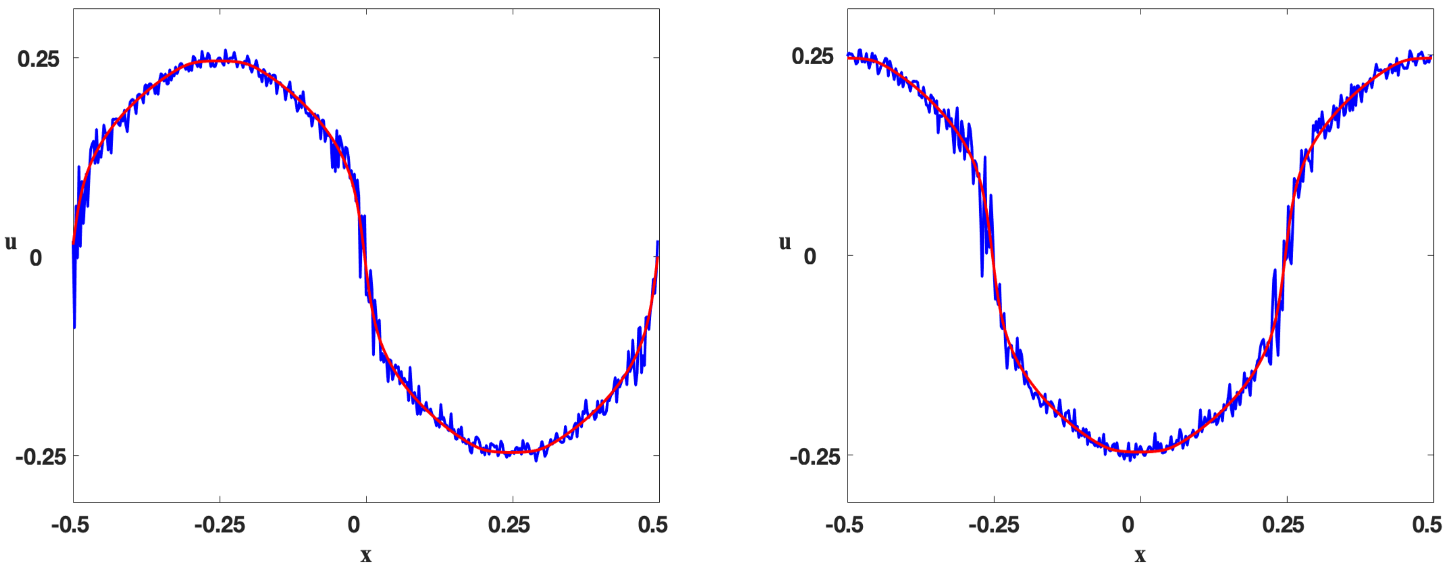
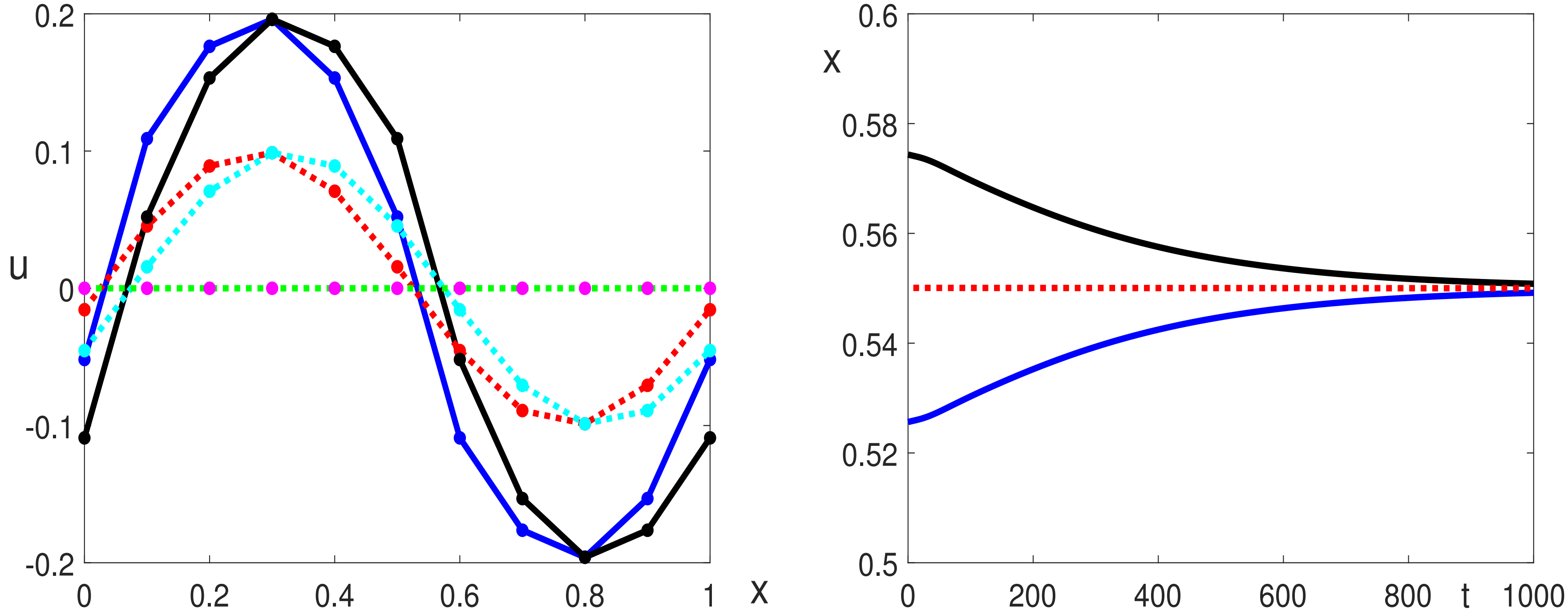


Figure 2: Red for deterministic SHE and blue for its random counterpart.

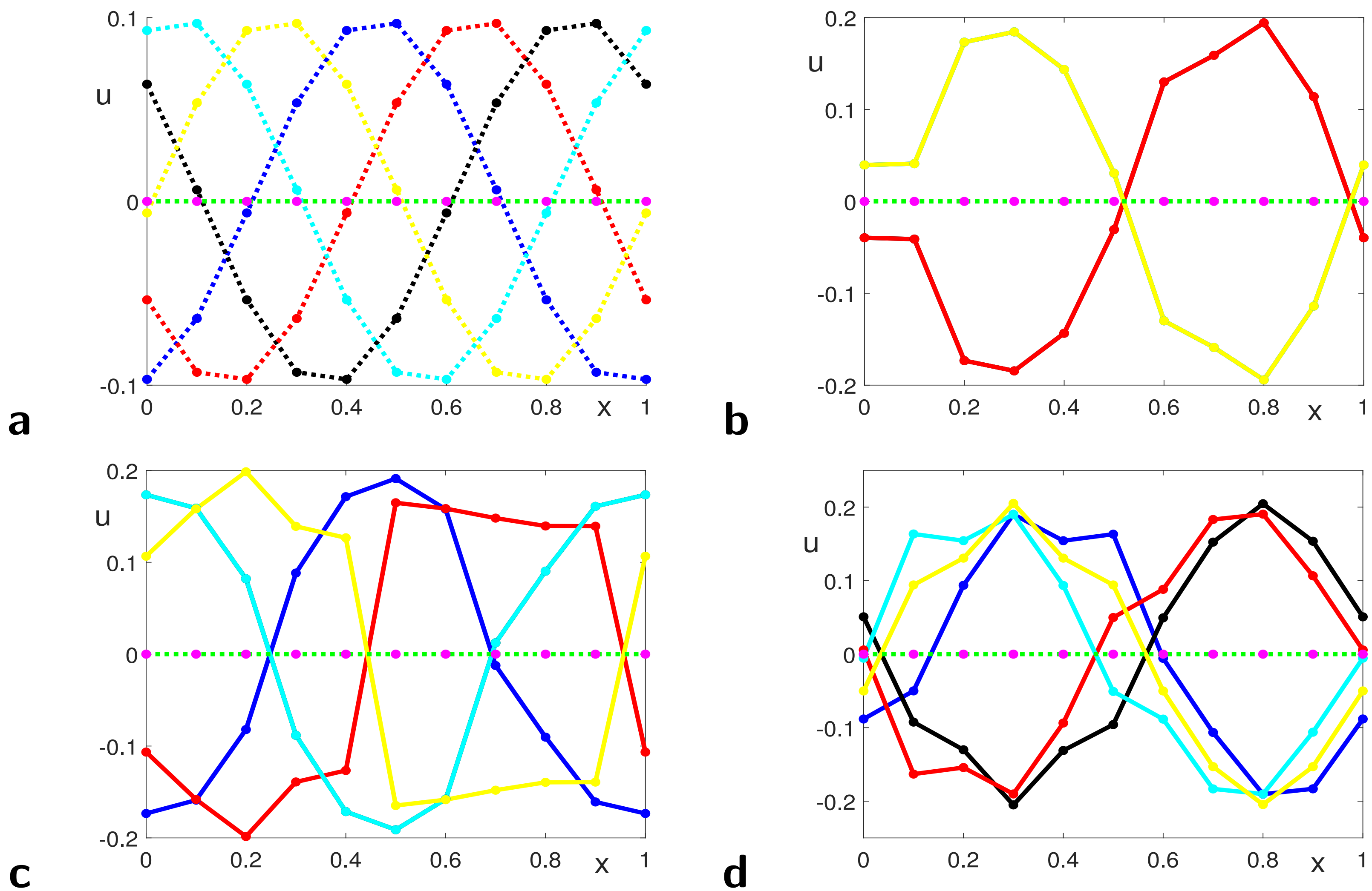
Numerical illustrations of the drift

Numerical solutions of the deterministic SHE with $N = 5$. Two initial guesses with different δ (red, cyan) and their nearly-final states (blue,black). The right panel shows the drift due to broken continuous translational symmetry.



Numerical illustrations of a random number of Turing patterns

Numerical solutions of the random SHE with $N = 5$. Five different initial guesses with different δ (a) and their nearly-final states in three different realizations (b,c,d). Only stable steady states are obtained in pairs (two, four, six).



Conclusion

- Broken translational symmetry transforms a continuous family of solutions to a finite number of solutions pinned to some exceptional spatial points.

- This phenomenon is generic in lattice differential equations and in PDEs with spatially dependent potentials.

- Existence and stability of pinned states can be studied with dynamical system methods (normal forms, Lyapunov–Schmidt reductions, perturbation theory).

Section 3. Pinning of optical solitons in Lugiato–Lefever model

Consider the NLS equation in a small bounded potential:

$$i\partial_t \psi = -\partial_x^2 \psi + \varepsilon V(x)\psi \pm |\psi|^2 \psi,$$

where $V \in W^{2,\infty}(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\varepsilon \ll 1$ is small.

- Bright solitons (focusing case): stable pinning at the minimum of V and unstable pinning at the maximum of V (Kapitula, 2001).

- Black solitons (defocusing case): unstable pinning at any extremal point of V (Kevrekidis–P, 2008)

- Domain walls (coupled NLS case): stable pinning at the maximum of V and unstable pinning at the minimum of V (Dror-Malomed-Zeng 2011, Alama–Bronsard–Contreras–P 2015)

Lugiato-Lefever equation with two pumping forces

The main model for electromagnetic field inside a ring-shaped cavity:

$$\mathrm{i}\partial_t \psi = -d\partial_x^2 \psi + (\zeta - \mathrm{i}\mu)\psi - |\psi|^2\psi + \mathrm{i}f_0 + \mathrm{i}f_1 \mathrm{e}^{\mathrm{i}(k_1 x - \nu_1 t)}, \qquad (x,t) \in \mathbb{T} \times \mathbb{R},$$

where d is dispersion, ζ is frequency detuning, μ is damping, f_0 is amplitude of the main force, and $f_1 \mathrm{e}^{\mathrm{i}(k_1 x - \nu_1 t)}$ is the second force.

In the limit $f_1 \ll f_0$, the main model can be reduced to the LL equation with a small bounded potential:

$$\mathrm{i}\partial_t u = -d\partial_x^2 u + \mathrm{i}\epsilon V(x)\partial_x u + (\zeta - \mathrm{i}\mu)u - |u|^2 u + \mathrm{i}f_0, \qquad (x,t) \in \mathbb{T} \times \mathbb{R}$$

Huanfa Peng (IPQ, KIT).
(Bengel–P–Reichel, 2024).

Persistence

Let $u_0 \in H^2_{per}(\mathbb{T},\mathbb{C})$ be a non-constant solution of the stationary equation

$$-du'' + (\zeta - \mathrm{i}\mu)u - |u|^2 u + \mathrm{i}f_0 = 0, \qquad x \in \mathbb{T}.$$

Assume that it is non-degenerate in the sense that the kernel of the linearized operator

$$\mathcal{L}\varphi := -d\varphi'' + (\zeta - \mathrm{i}\mu - 2|u_0|^2)\varphi - u_0^2\bar{\varphi}, \quad \varphi \in H^2_{per}(\mathbb{T},\mathbb{C})$$

consists only of $\mathrm{Span}\{u'\}$.

Theorem 4. *Let $V \in C^1_{per}(\mathbb{T},\mathbb{R})$ and $u_0 \in H^2_{per}(\mathbb{T},\mathbb{C})$ be non-constant and non-degenerate. If σ_0 is a simple zero of the function*

$$\sigma \mapsto V_{\text{eff}}(\sigma) := Re \int_{-\pi}^{\pi} \mathrm{i}V(x+\sigma)u_0'\bar{\phi}_0^*\, dx$$

then there exists a continuous curve $(-\epsilon^*, \epsilon^*) \ni \epsilon \to u(\epsilon) \in H^2_{per}(\mathbb{T},\mathbb{C})$ consisting of stationary solutions with $\|u(\epsilon) - u_0(\cdot - \sigma_0)\|_{H^2} \le C\epsilon$, $C > 0$.

In the limit where u_0 is highly localized around 0 compared to the potential V (e.g. the limit $d \to 0$), we have $V_{\text{eff}}(\sigma) \approx V(\sigma)$.

Spectral stability

Linearization with

$$u(x) + v(x,t) = u_1(x) + \mathrm{i}u_2(x) + v_1(x,t) + \mathrm{i}v_2(x,t)$$

yields the linearized system for $\boldsymbol{v} = (v_1, v_2)$ such that

$$\partial_t \boldsymbol{v} = \widetilde{L}_{u,\epsilon} \boldsymbol{v}$$

with

$$\widetilde{L}_{u,\epsilon} = JA_u - I(\mu - \epsilon V(x)\partial_x)$$

with

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A_u := \begin{pmatrix} -d\partial_x^2 + \zeta - (3u_1^2 + u_2^2) & -2u_1u_2 \\ -2u_1u_2 & -d\partial_x^2 + \zeta - (u_1^2 + 3u_2^2) \end{pmatrix}.$$

$\varepsilon = 0$: spectral and asymptotic stability by Delcey–Haragus (2018), Hakkaev–Stanislavova–Stefanov (2019), Stanislavova–Stefanov (2019), Haragus–Johnson–Perkins (2021), Haragus–Johnson–Perkins–de Rijk (2021).

Spectral stability

Stability analysis suggests exchange of stabilities, e.g. the transcritical bifurcation.

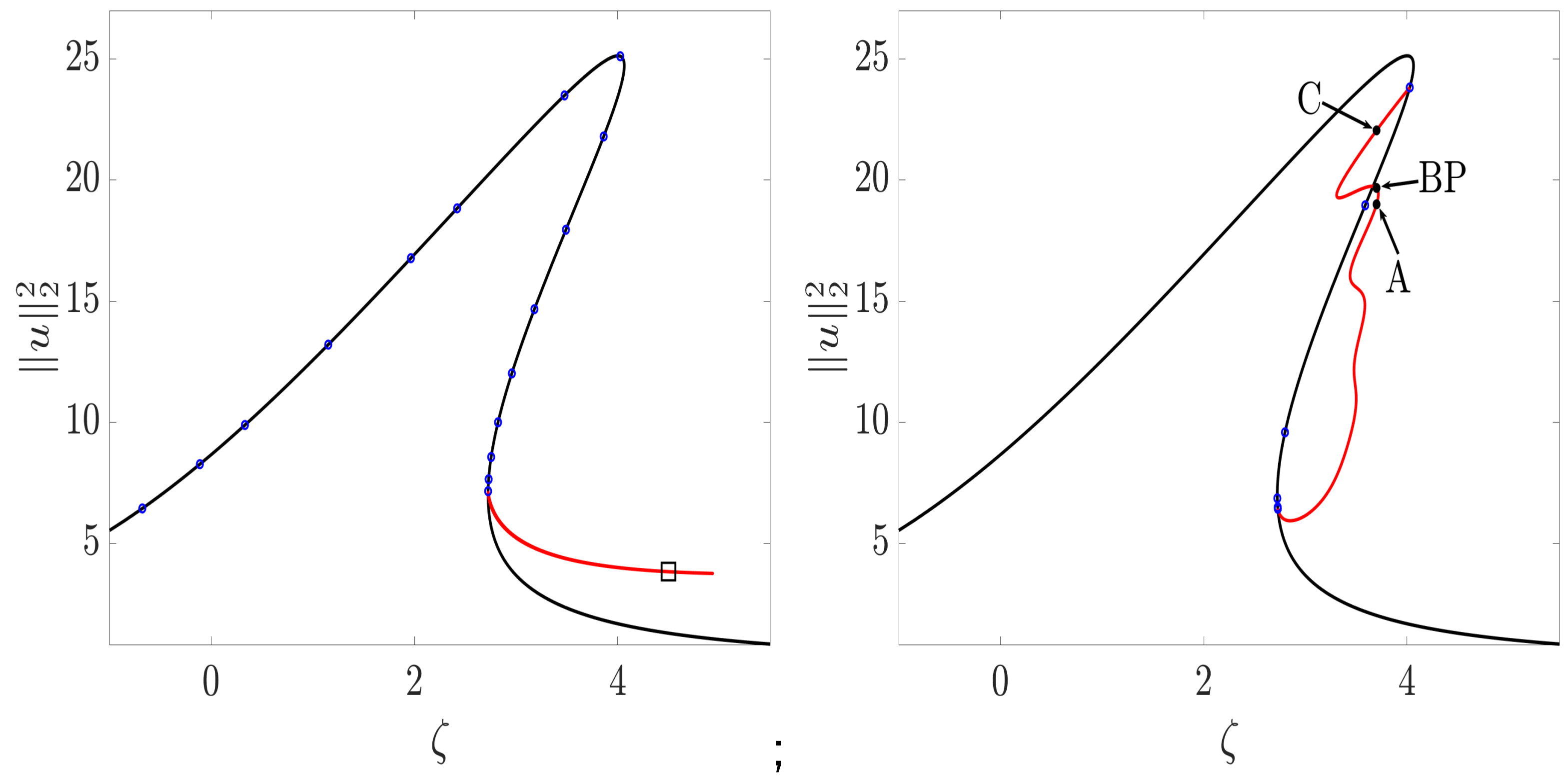
Theorem 5. *Assume that* $u_0 \in H^2_{per}(\mathbb{T},\mathbb{C})$ *is orbitally asymptotically stable with*

$$\sigma(\widetilde{L}_{u_0,0}) \subset \{z \in \mathbb{C} : \ \ Re z \leq -\xi\} \cup \{0\}.$$

and that σ_0 *is a simple zero of* V_{eff}*, that is,* $V'_{\text{eff}}(\sigma_0) \neq 0$*. Then there exists* $\epsilon_0 > 0$ *such that on the solution branch* $(-\epsilon_0, \epsilon_0) \ni \epsilon \to u(\epsilon) \in H^2_{per}(\mathbb{T},\mathbb{C})$ *with* $u(0) = u_{\sigma_0}$ *the solutions* $u(\epsilon)$ *are spectrally stable for* $V'_{\text{eff}}(\sigma_0) \cdot \epsilon > 0$ *and spectrally unstable for* $V'_{\text{eff}}(\sigma_0) \cdot \epsilon < 0$*.*

Moreover, spectrally stable solutions are also asymptotically stable [Bengel–P–Reichel (2024)].

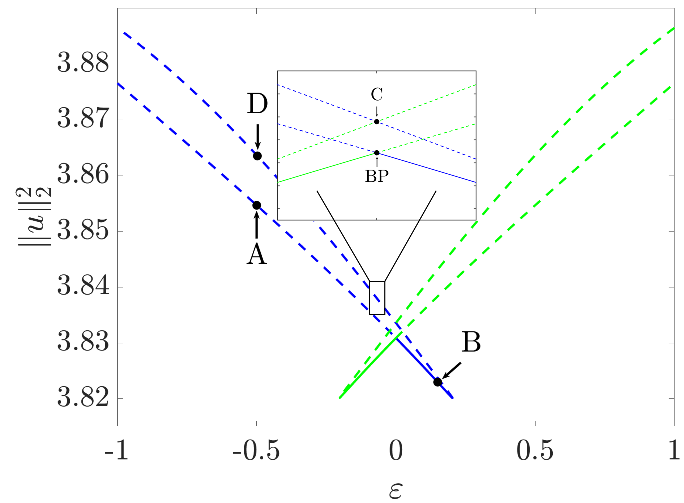
Numerically computed bifurcation diagram for $\epsilon = 0$

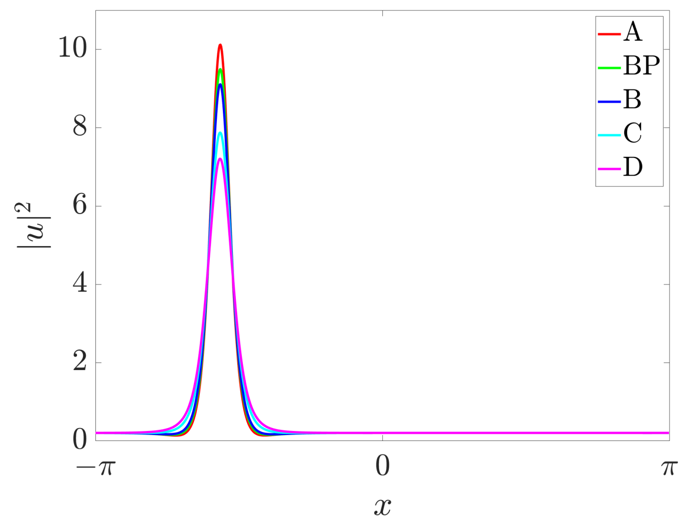


Left: $d > 0$. Right: $d < 0$.

Mandel–Reichel (2017)

Numerically computed bifurcation diagram for $\epsilon \neq 0$





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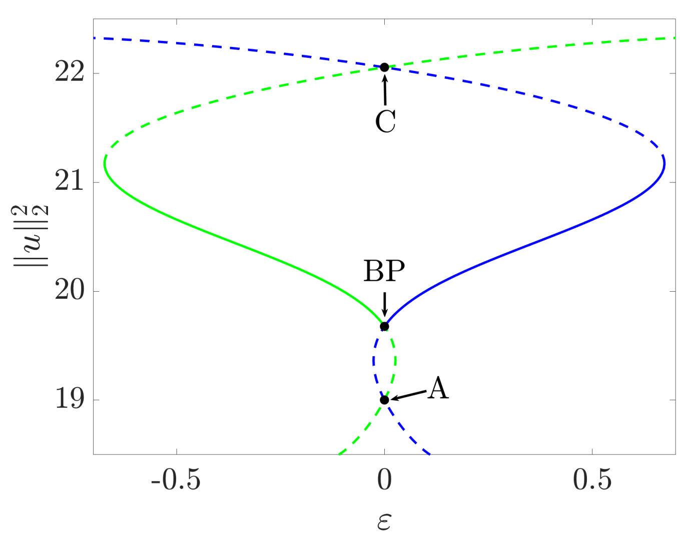
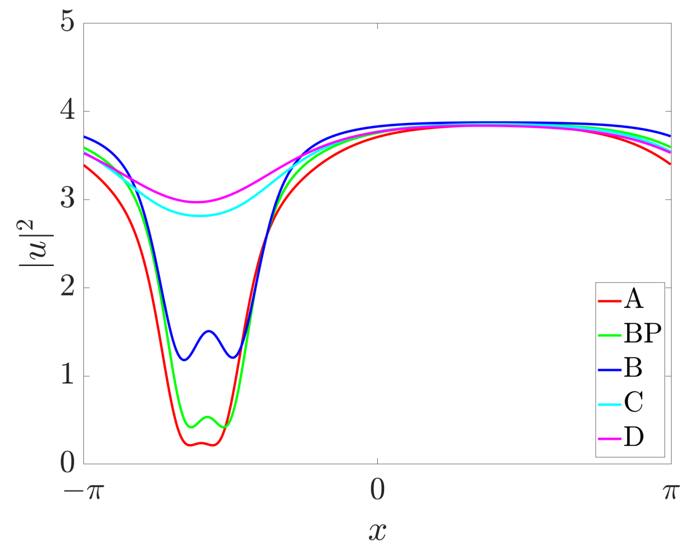
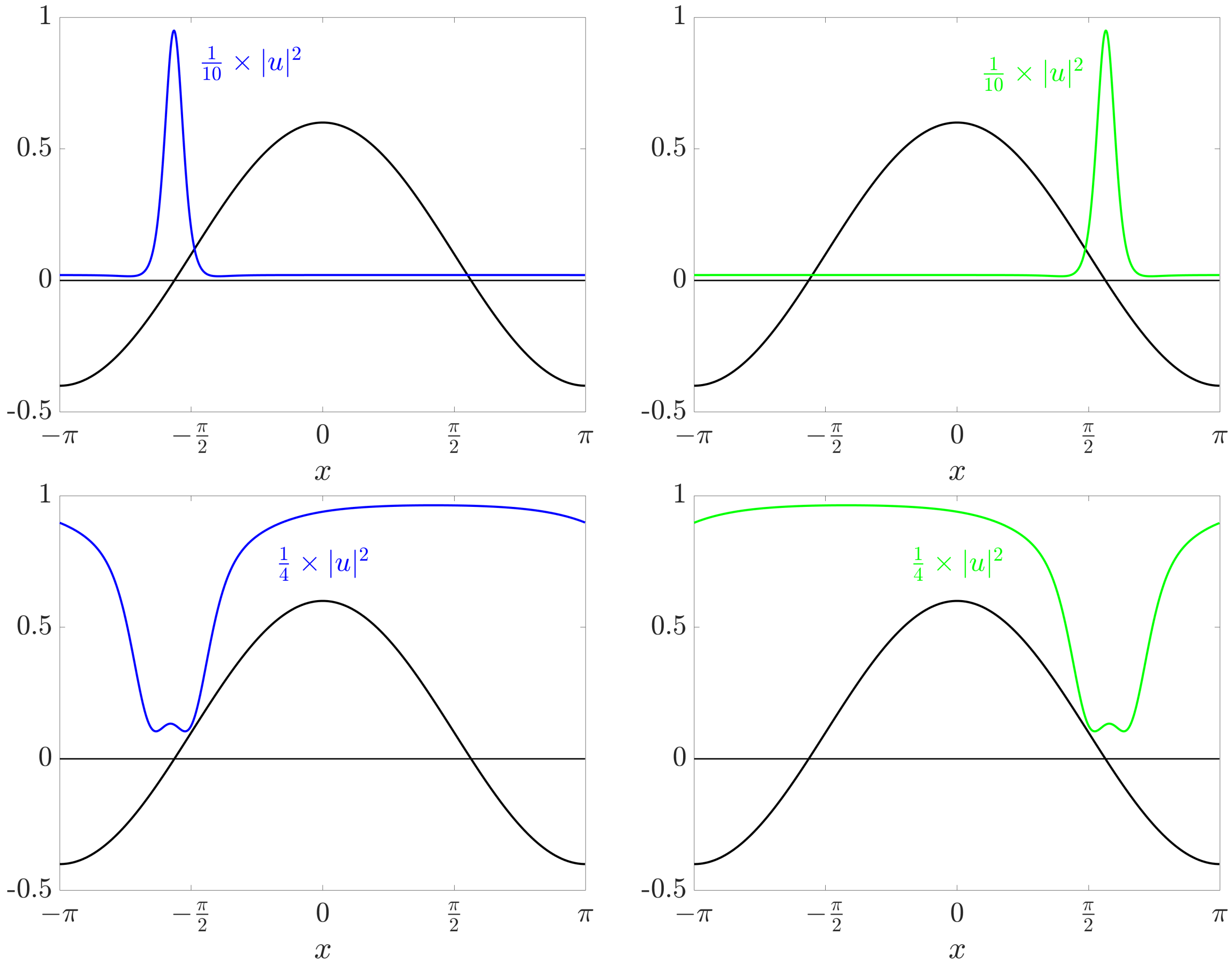




Up: $d > 0$. Down: $d < 0$.

Pinning in the external potential



Up: $d > 0$. Down: $d < 0$.