

## Propagation Failure in the Discrete Nagumo Equation

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- Proceedings of the AMS, in press (2011)

# Lattice Differential Equations

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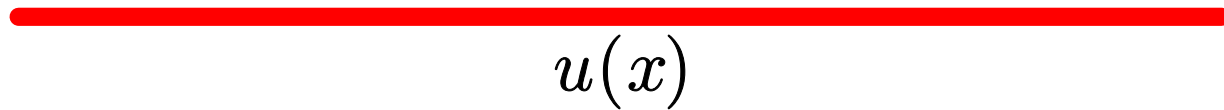
Lattice differential equations (LDEs) are ODEs indexed on a spatial lattice, e.g.

$$\frac{d}{dt}u_j(t) = \alpha(u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)) + f(u_j(t)), \quad j \in \mathbb{Z}.$$



If  $\alpha = h^{-2} \gg 1$ , LDE can be seen as a discretization with step size  $h$  of PDE

$$\partial_t u(t, x) = \partial_x^2 u(t, x) + f(u(t, x)), \quad x \in \mathbb{R}.$$

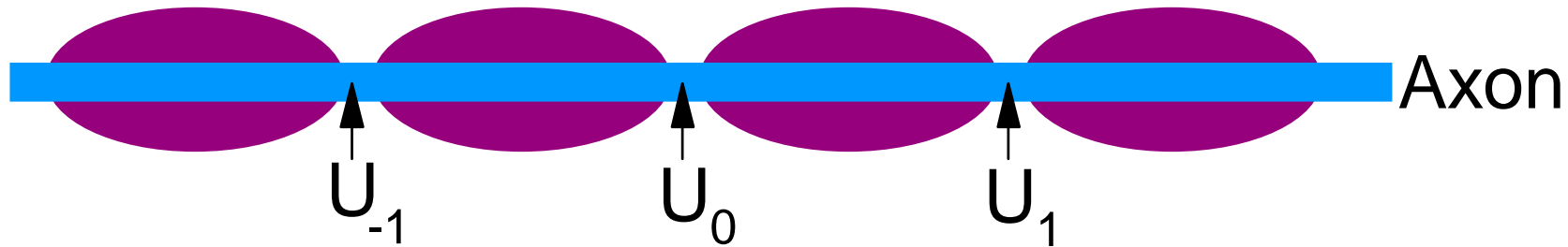


- Many physical models have a discrete spatial structure  $\rightarrow$  LDEs.
- Main theme: qualitative differences between PDEs and LDEs.

# Signal Propagation through Nerves

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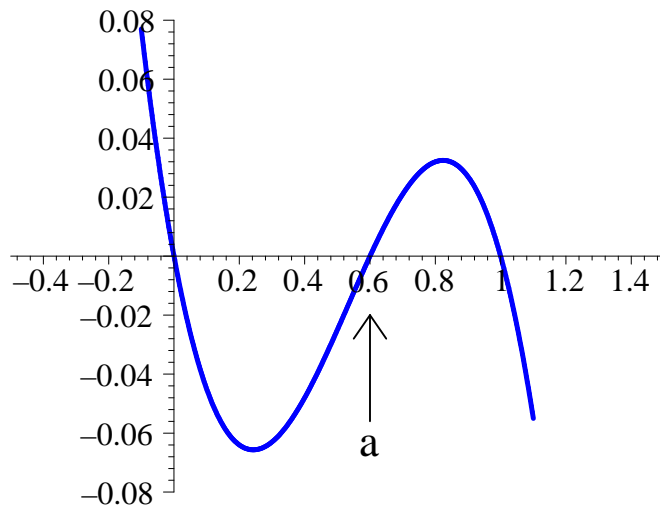
One is interested in the potential  $U_j$  at the node sites.



Signals appear to "hop" from one node to the next [Lillie, 1925].

Ignoring recovery, one arrives at the LDE, called the discrete Nagumo equation [Keener and Sneyd, 1998]

$$\frac{d}{dt}U_j(t) = U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t); a), \quad j \in \mathbb{Z}.$$



Bistable nonlinearity  $g$  given by

$$g(u; a) = u(a - u)(u - 1).$$

# Traveling front solutions

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In the continuum limit, the discrete Nagumo equation becomes the continuous Nagumo equation,

$$\partial_t u = \partial_x^2 u + u(a - u)(u - 1).$$

Travelling wave  $u(x, t) = \phi(x + ct)$  satisfies:

$$c\phi'(\xi) = \phi''(\xi) + \phi(\xi)(a - \phi(\xi))(\phi(\xi) - 1).$$

We are interested in the front solutions connecting stable equilibrium states 0 and 1 (heteroclinic orbits). These solutions satisfy the boundary conditions,

$$\lim_{\xi \rightarrow -\infty} \phi(\xi) = 0, \quad \lim_{\xi \rightarrow +\infty} \phi(\xi) = 1.$$

# Exact solutions

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Recall the travelling wave ODE

$$c\phi'(\xi) = \phi''(\xi) + \phi(\xi)(a - \phi(\xi))(\phi(\xi) - 1),$$

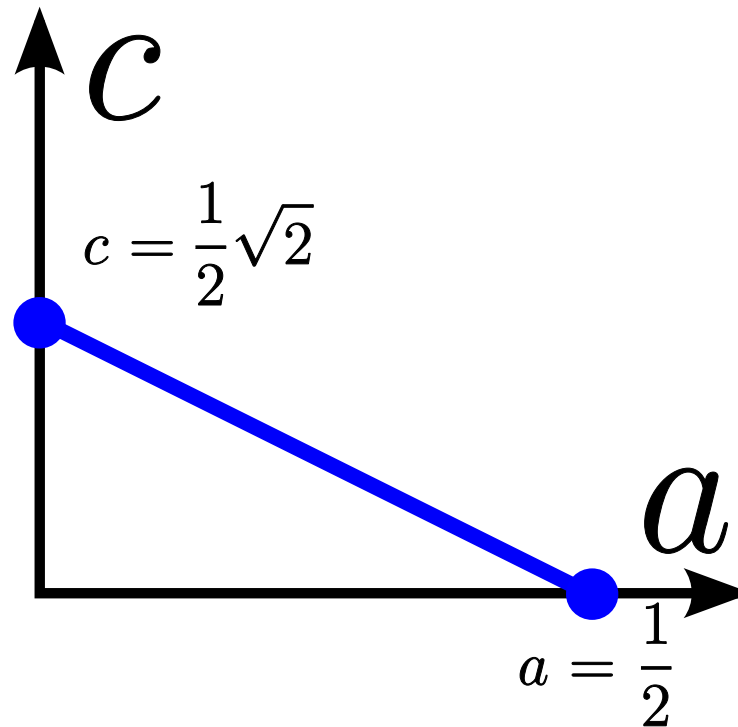
subject to

$$\lim_{\xi \rightarrow -\infty} \phi(\xi) = 0, \quad \lim_{\xi \rightarrow +\infty} \phi(\xi) = 1.$$

Explicit solutions available:

$$\phi(\xi) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{1}{4}\sqrt{2}\xi\right),$$

$$c(a) = \frac{1}{\sqrt{2}}(1 - 2a).$$



# Travelling fronts in LDE

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Back to the Nagumo LDE

$$\frac{d}{dt}U_j(t) = \frac{1}{h^2} [U_{j+1}(t) + U_{j-1}(t) - 2U_j(t)] + U_j(t)(a - U_j(t))(U_j(t) - 1), \quad j \in \mathbb{Z}.$$

Travelling front solutions  $U_j(t) = \phi(j + ct)$  must satisfy:

$$c\phi'(\xi) = \frac{1}{h^2} [\phi(\xi + h) + \phi(\xi - h) - 2\phi(\xi)] + \phi(\xi)(a - \phi(\xi))(\phi(\xi) - 1),$$

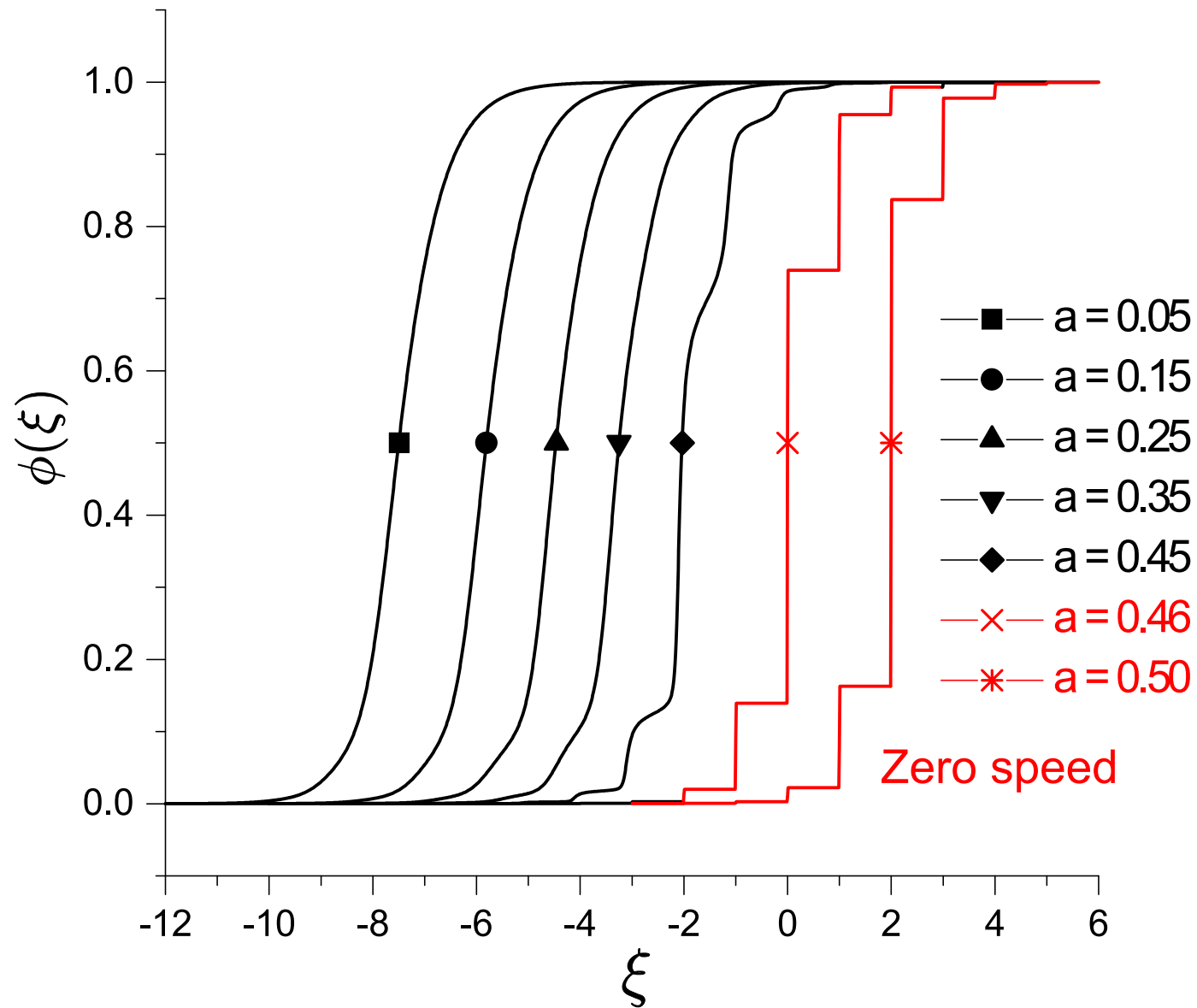
subject to

$$\lim_{\xi \rightarrow -\infty} \phi(\xi) = 0, \quad \lim_{\xi \rightarrow +\infty} \phi(\xi) = 1.$$

- When  $c \neq 0$ , this is a differential advance-delay equation.
- When  $c = 0$ , this is an advance-delay equation.
- The limit  $c \rightarrow 0$  is a singular perturbation theory.

# Discrete Nagumo LDE - Propagation failure

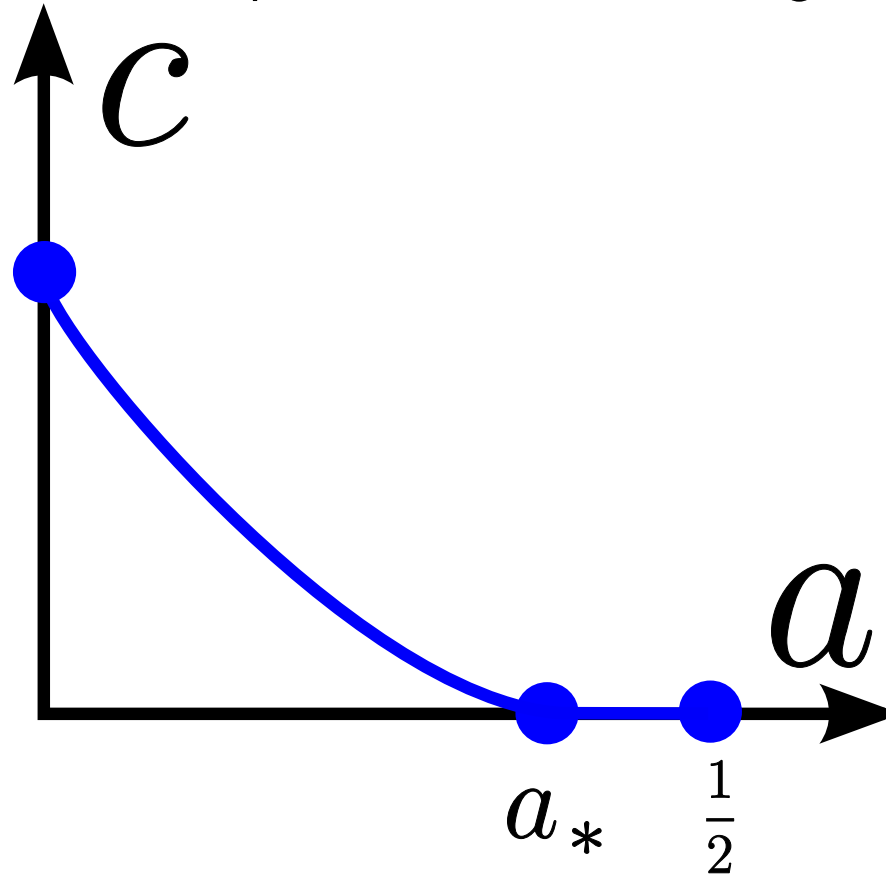
Travelling waves for the discrete Nagumo LDE connecting  $0 \rightarrow 1$ .



# Propagation failure

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Typical wave speed  $c$  versus  $a$  plot for the discrete Nagumo LDE:



We can have either  $a_* = \frac{1}{2}$  or  $a_* < \frac{1}{2}$ .

If  $a_* < \frac{1}{2}$ , we say that LDE suffers from **propagation failure**.

Propagation failure widely studied; pioneered by [Keener].



# Propagation failure

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Consider travelling wave MFDE with **saw-tooth** nonlinearity

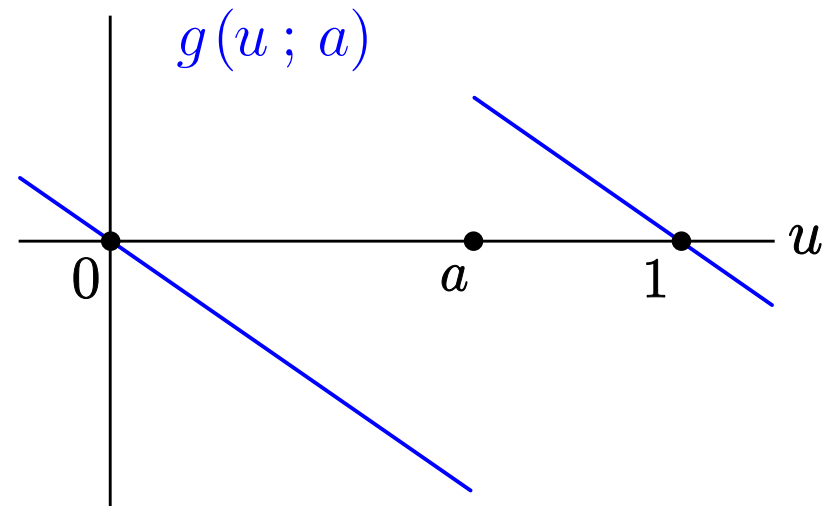
$$c\phi'(\xi) = \frac{1}{h^2}[\phi(\xi + h) + \phi(\xi - h) - 2\phi(\xi)] + g(\phi(\xi); a),$$

subject to

$$\lim_{\xi \rightarrow -\infty} \phi(\xi) = 0, \quad \lim_{\xi \rightarrow +\infty} \phi(\xi) = 1.$$

Propagation failure for all  $h > 0$   
[Cahn, Mallet-Paret, Van Vleck] (1999)

Linear analysis with Fourier series.



# Propagation failure

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Consider travelling wave MFDE with **zig-zag** bistable nonlinearity

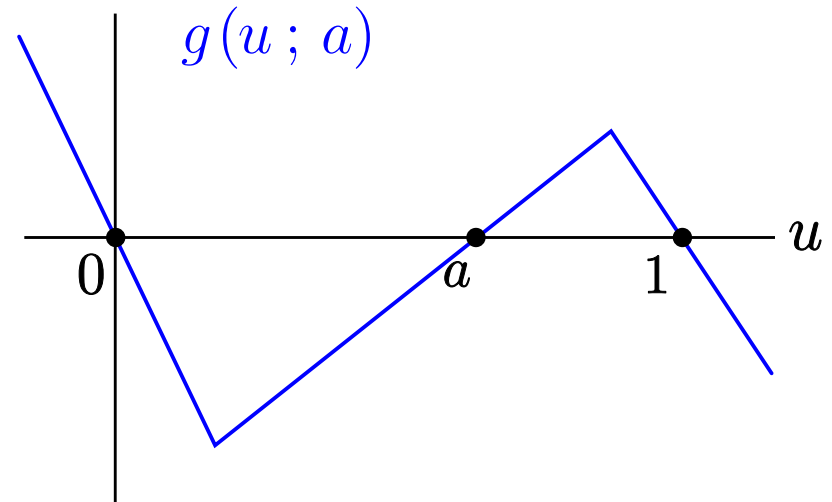
$$c\phi'(\xi) = \frac{1}{h^2}[\phi(\xi + h) + \phi(\xi - h) - 2\phi(\xi)] + g(\phi(\xi); a),$$

subject to

$$\lim_{\xi \rightarrow -\infty} \phi(\xi) = 0, \quad \lim_{\xi \rightarrow +\infty} \phi(\xi) = 1.$$

There exist countably many  $h$  for which there is **no** propagation failure.

[Elmer] (2006)



# Propagation failure for the Klein–Gordon equation

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In a similar context of the Klein-Gordon equation,

$$u_{tt} = u_{xx} + g(u; a),$$

many researchers were looking for other discretizations of  $g$  that admit a continuous (“translationally invariant”) branch of stationary solutions [Speight](1999); [Kevrekidis](2003); [Barashenkov, Oxtoby, Pelinovsky](2005); [Dmitriev, Kevrekidis, Yoshikawa](2005).

**Main Question:** Does the existence of continuous (“translationally invariant”) stationary solutions imply the existence of continuously differentiable traveling solutions?

**The Answer is NO** for the discrete Klein–Gordon equation.

## Example

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The discrete Nagumo equation

$$c\phi'(\xi) = \frac{1}{h^2}[\phi(\xi + h) + \phi(\xi - h) - 2\phi(\xi)] + g(\phi(\xi); a)$$

$$g(u; a) = \frac{2(1 - u)u(u - a)(1 + h^2(1 + au))}{(1 + h^2(1 - u)u)(1 + h^2(1 - a)a)},$$

admits an exact traveling front solution,

$$\phi(z) = \frac{1}{2}(1 + \tanh(bz - s)), \quad b = \frac{\operatorname{arcsinh}(h)}{h}, \quad c = \frac{2a - 1}{b(1 + h^2(1 - a)a)}, \quad s \in \mathbb{R}.$$

If  $a = \frac{1}{2}$ , then  $c = 0$ , and the stationary front is “translationally invariant”  $\phi(z) = \tanh(bz - s)$  with arbitrary parameter  $s \in \mathbb{R}$  (the same for KG equation).

We can see that stationary front becomes a traveling front without a propagation failure.

**Question:** Is this a coincidence?

# Formulation of the problem

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Recall the differential advance-delay equation for travelling waves:

$$c\phi'(\xi) = \frac{1}{h^2}[\phi(\xi + h) + \phi(\xi - h) - 2\phi(\xi)] + \phi(\xi)(a - \phi(\xi))(\phi(\xi) - 1).$$

When  $c = 0$ , we can restrict to  $\xi \in \mathbb{Z}$  and obtain a difference equation.

With  $p_j = \phi(j)$  and  $r_j = \phi(j + 1)$ , we find

$$\begin{aligned} p_{j+1} &= r_j \\ r_{j+1} &= -p_j + 2r_j - h^2 r_j (r_j - a)(1 - r_j). \end{aligned}$$

Two fixed point  $(0, 0)$  and  $(1, 1)$  are saddles. Generally, two heteroclinic orbits exist for  $a = \frac{1}{2}$  (symmetric case):

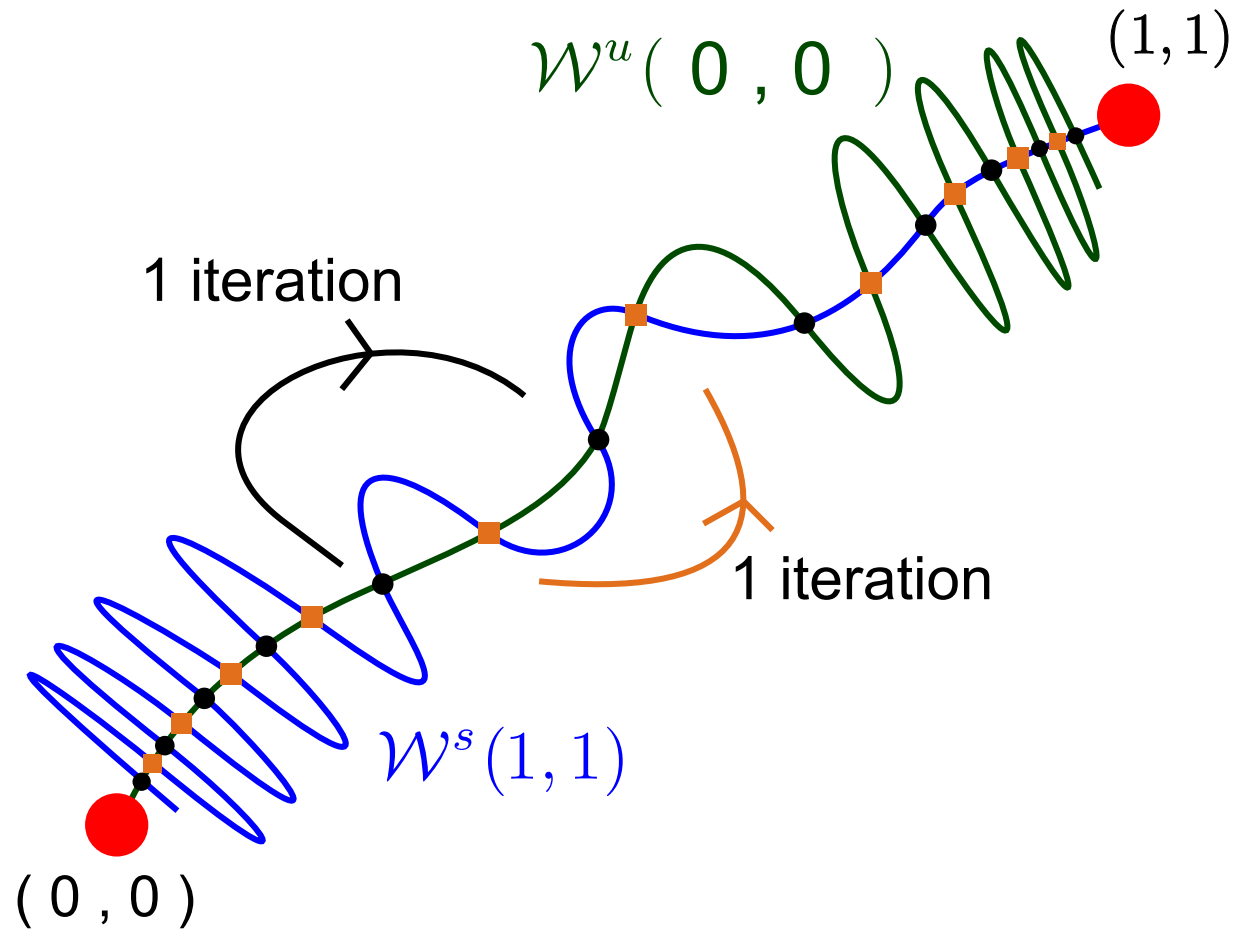
$$p_{-j}^{(s)} = -p_j^{(s)}, \quad p_{-j+1}^{(b)} = -p_j^{(b)},$$

called site-symmetric and bond-symmetric fronts.

# Formulation of the problem

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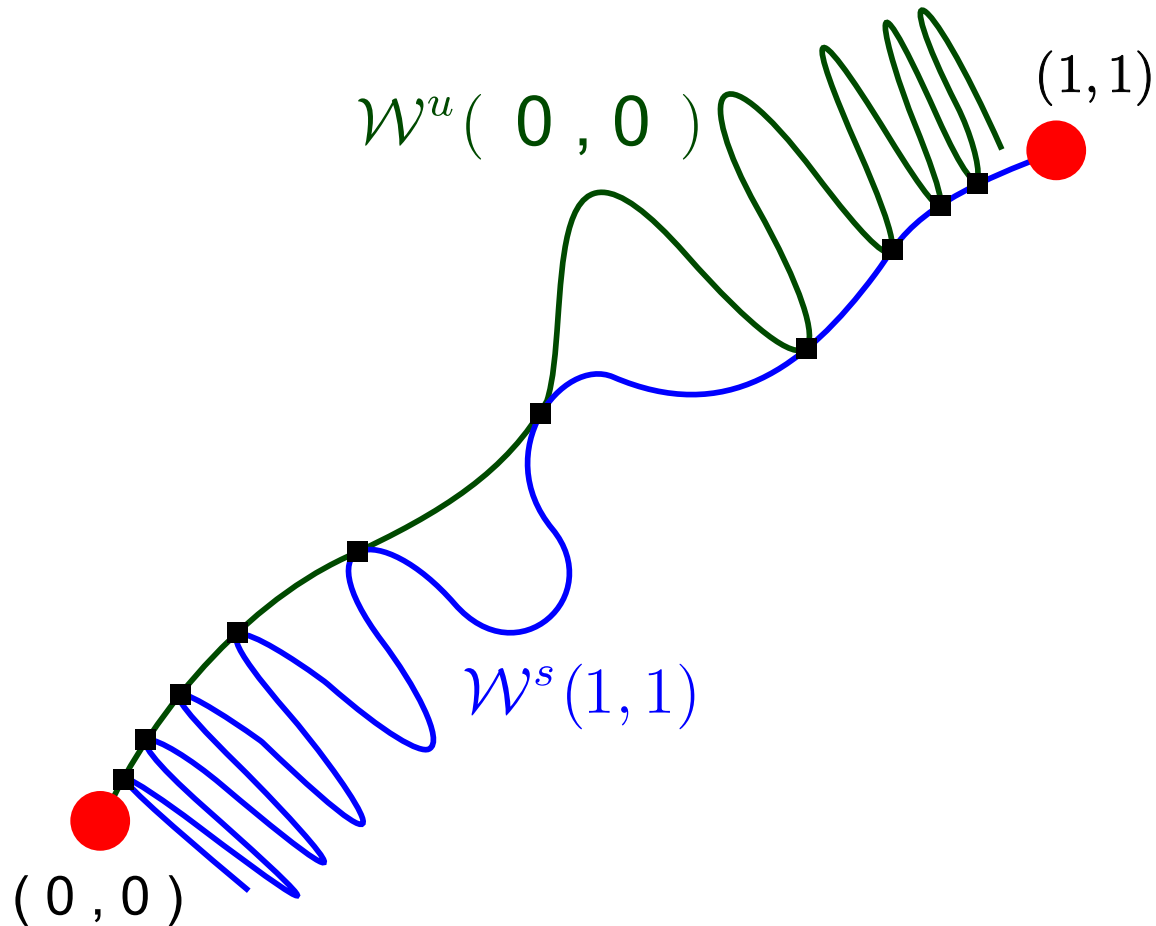
For  $a = \frac{1}{2}$ , site-symmetric (orange) and bond-symmetric (black) solutions:



## Formulation of the problem

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For  $a < \frac{1}{2}$ , the distance between nodes decreases. At  $a = a_* < \frac{1}{2}$ , two branches of stationary front solutions coincide and annihilate via a saddle-node bifurcation.



# Formulation of the problem

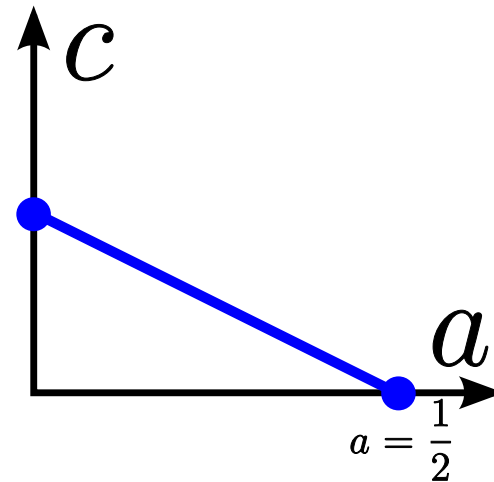
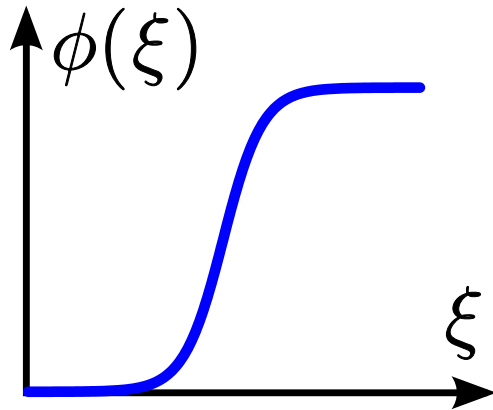
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Special discretizations of  $g$  may also involve multiple lattice sites:

$$\frac{d}{dt}U_j = \frac{1}{h^2}[U_{j-1} + U_{j+1} - 2U_j] + \frac{1}{2}U_j(U_{j+1} + U_{j-1} - 2a)(1 - U_j).$$

Explicit solutions available:

$$U_j(t) = \frac{1}{2} + \frac{1}{2} \tanh \left( \operatorname{arcsinh} \left( \frac{1}{4} \sqrt{2} h \right) (j + ct) \right), \quad c(a) = \frac{(1 - 2a)}{4 \operatorname{arcsinh} \left( \frac{1}{4} \sqrt{2} h \right)}.$$



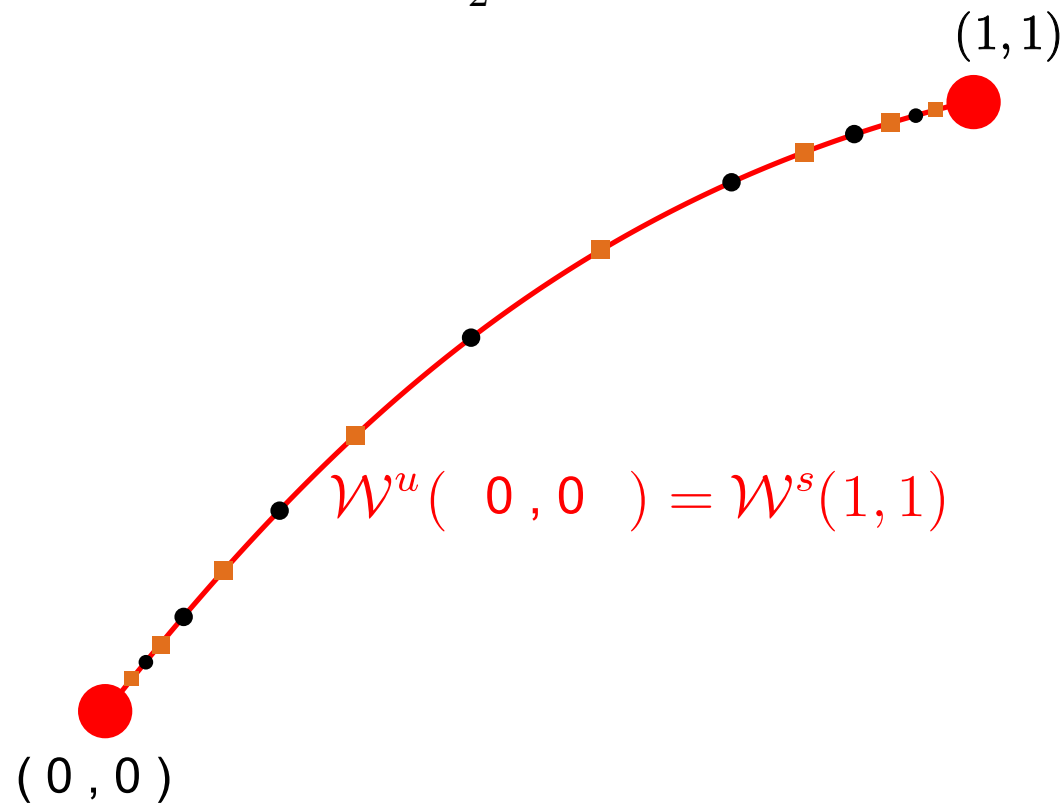
No propagation failure; smooth wave profile.



# Formulation of the problem

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Smooth standing wave profile at  $a = \frac{1}{2}$  correspond to:



Site-symmetric and bond-symmetric solutions are connected by a continuous branch of “translationally invariant” standing waves.

**Main Question:** What happens to manifolds when  $a \neq \frac{1}{2}$ ?

Do intersections disappear (no prop failure) or survive (prop failure)?

# Lattice point of view

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Let us write LDE as:

$$\frac{d}{dt}U(t) = \mathcal{F}(U(t); a),$$

with  $U(t) \in \ell^\infty$  and  $\mathcal{F} : \ell^\infty \times [0, 1] \rightarrow \ell^\infty$ .

Travelling waves  $U_j(t) = \phi(j + ct)$  satisfy the differential advance-delay equation,

$$c\phi'(\xi) = \mathcal{G}\left(\phi(\xi - 1), \phi(\xi), \phi(\xi + 1); a\right)$$

Suppose at  $a = \frac{1}{2}$  we have a **smooth** solution  $p(\xi)$  to

$$0 = \mathcal{G}\left(p(\xi - 1), p(\xi), p(\xi + 1); a = \frac{1}{2}\right), \quad \xi \in \mathbb{R}.$$

Then for every  $\vartheta \in \mathbb{R}$ , we have equilibrium solution  $p^{(\vartheta)} \in \ell^\infty$  to our LDE:

$$\mathcal{F}\left(p^{(\vartheta)}; \frac{1}{2}\right) = 0, \quad p_j^{(\vartheta)} = p(\vartheta + j).$$

# Invariant Manifold

Recall  $p^{(\vartheta)} \in \ell^\infty$  with  $p_j^{(\vartheta)} = p(\vartheta + j)$ .

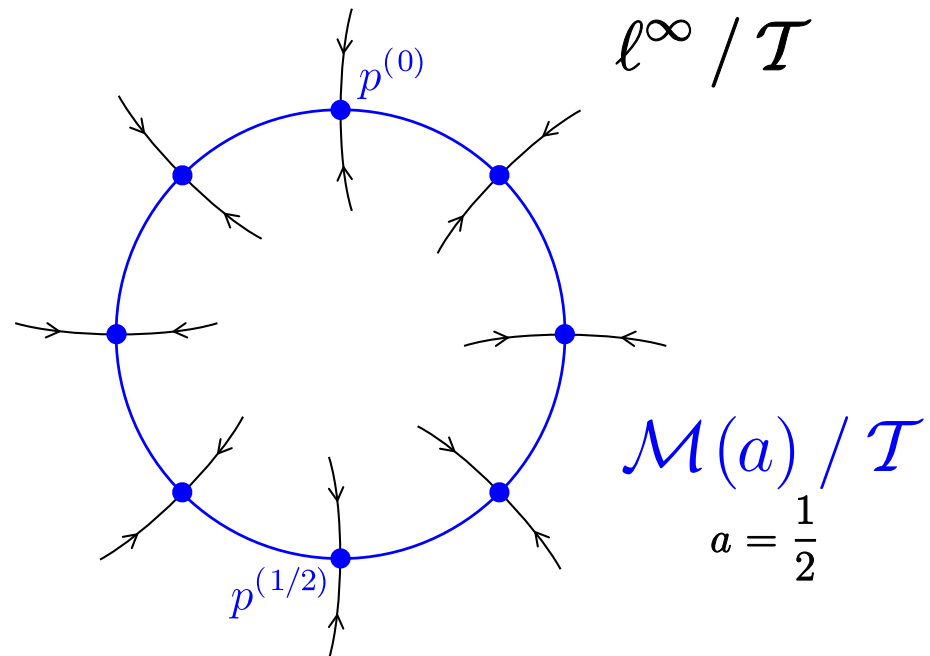
Notice that

$$p^{(\vartheta)} = \mathcal{T}p^{(\vartheta+1)},$$

where  $\mathcal{T} : \ell^\infty \rightarrow \ell^\infty$  is right-shift operator  $(\mathcal{T}u)_j = u_{j-1}$ .

Combining these equilibria gives a smooth manifold

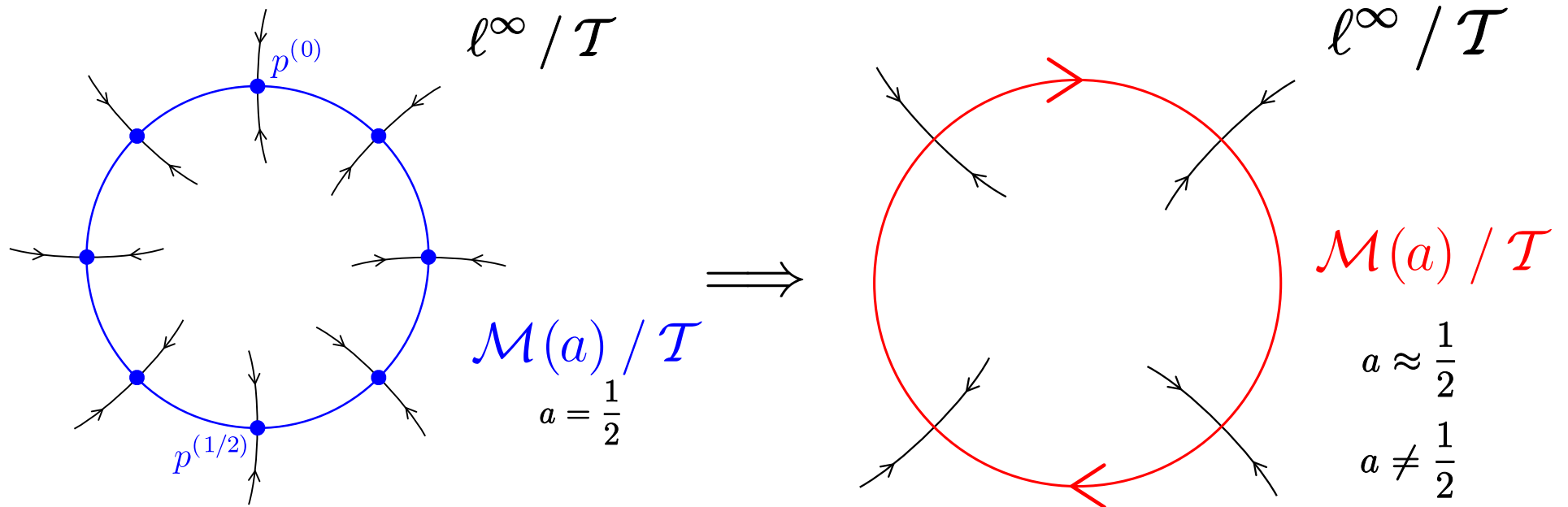
$$\mathcal{M}\left(a = \frac{1}{2}\right) = \{p^{(\vartheta)}\}_{\vartheta \in \mathbb{R}}.$$



Based on spectral stability of equilibria  $p^{(\vartheta)}$  [Chow, Mallet-Paret, Shen, 1998] and comparison principles, we can prove that the manifold  $\mathcal{M}(a = \frac{1}{2})$  is normally hyperbolic.

# Invariant Manifold - Scenario #1

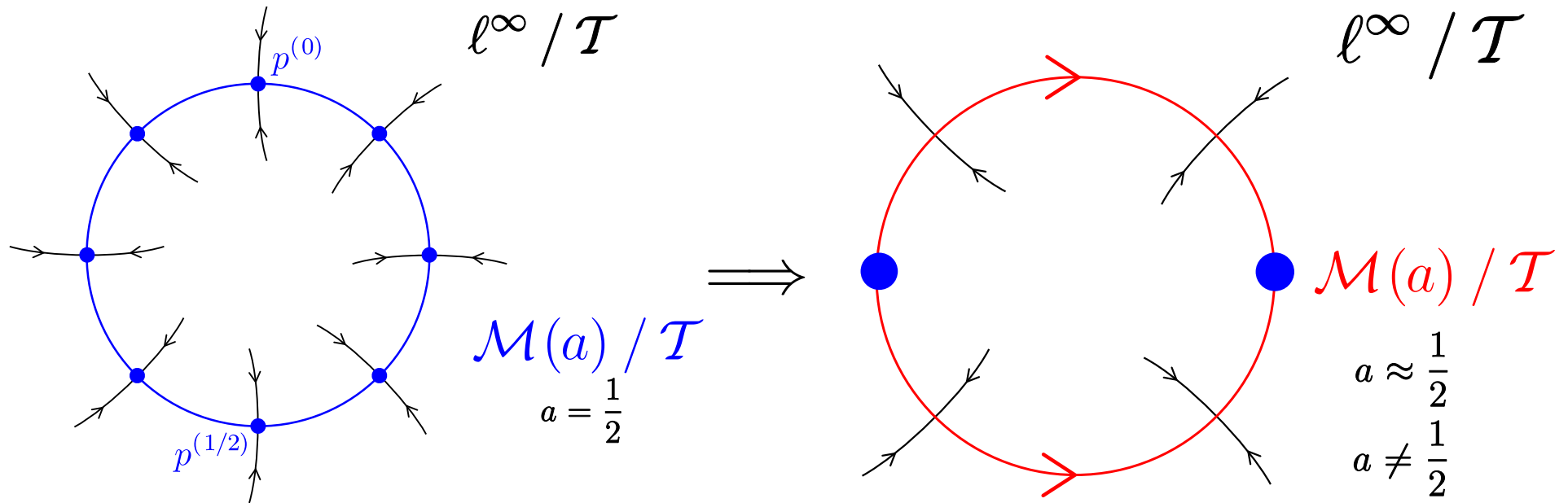
Possible scenario #1 for persistence of  $\mathcal{M}(a)$  with  $a \neq \frac{1}{2}$ :



No equilibria survive;  $\mathcal{M}(a)$  is orbit of travelling wave. **No Propagation Failure.**

# Invariant Manifold - Scenario #2

Possible scenario #2 for persistence of  $\mathcal{M}(a)$  with  $a \neq \frac{1}{2}$ :



One or more equilibria survive. **Propagation Failure.**

## Dynamics at $\mathcal{M}(a)$

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Angular coordinate  $\theta$  measures position along  $\mathcal{M}(a)$ . Dynamics at  $\mathcal{M}(a)$  for  $a \approx \frac{1}{2}$  is given by

$$\frac{d}{dt}\theta = (a - \frac{1}{2})\Psi(\theta) + O\left(\left|a - \frac{1}{2}\right|^2\right),$$

in which  $\Psi(\theta)$  given by

$$\Psi(\vartheta) = \sum_{j \in \mathbb{Z}} q_j^{(\vartheta)} \partial_a \mathcal{G}\left(p_{j-1}^{(\vartheta)}, p_j^{(\vartheta)}, p_{j+1}^{(\vartheta)}; a = \frac{1}{2}\right).$$

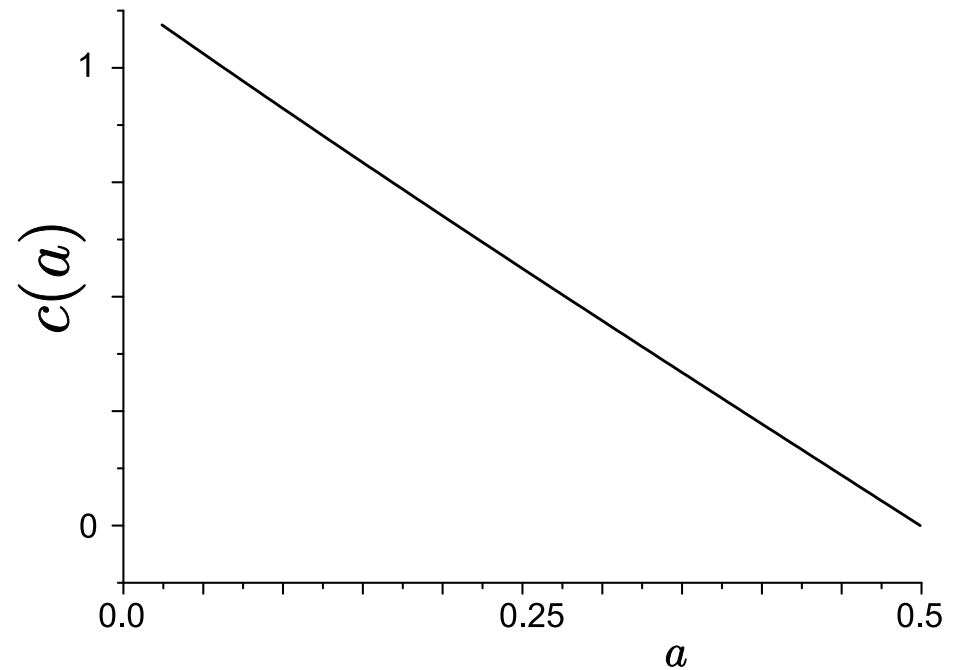
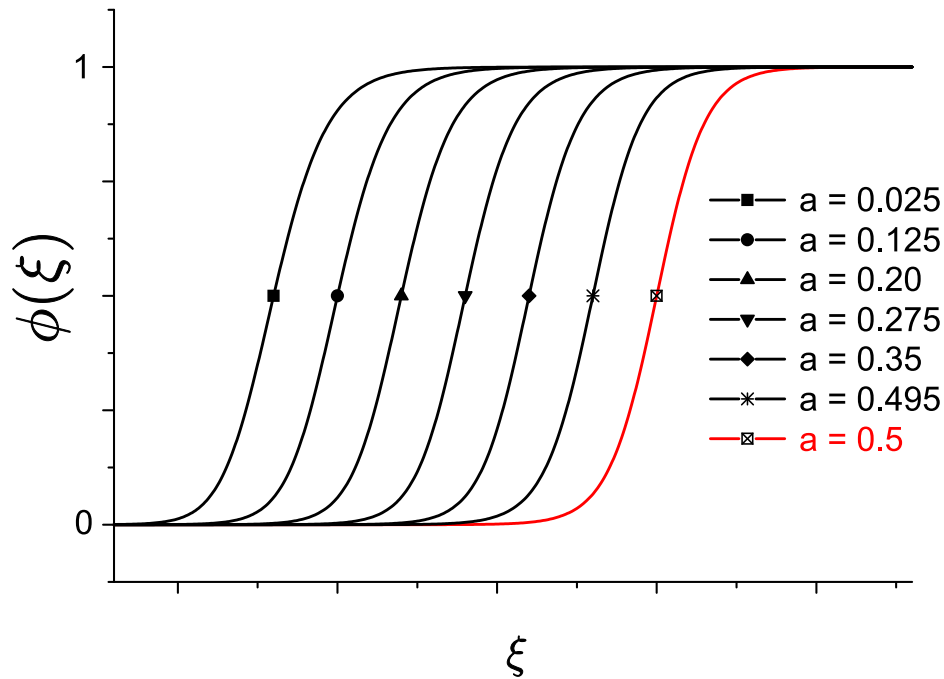
Here  $q^{(\vartheta)}$  is adjoint eigenvector; i.e. solves  $L^{(\vartheta)*} q^{(\vartheta)} = 0$ .

**Known:**  $q_j^{(\vartheta)} > 0$  for all  $j \in \mathbb{Z}$  and  $\vartheta \in \mathbb{R}$ . So  $\partial_a \mathcal{G} < 0$  guarantees **no** prop failure.

# Example 1

No prop failure for LDE

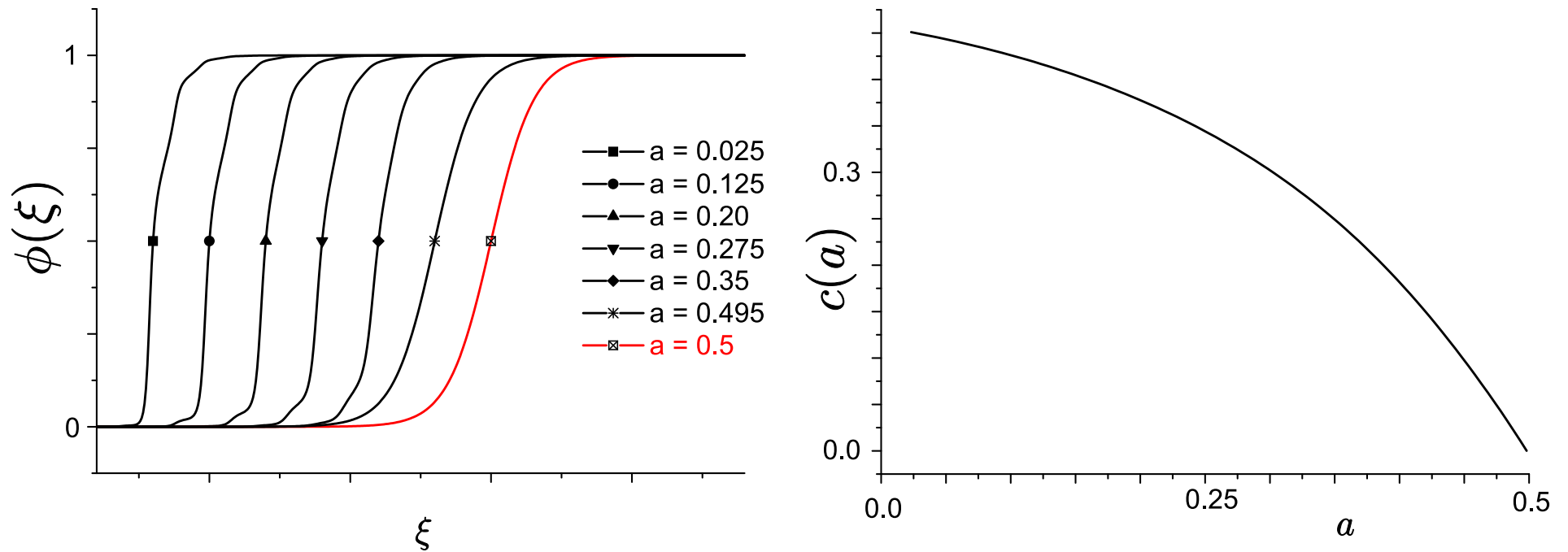
$$\frac{d}{dt}u_j = u_{j-1} + u_{j+1} - 2u_j + (u_j - a) \left( u_{j-1}(1 - u_{j+1}) + u_{j+1}(1 - u_{j-1}) \right)$$



## Example 2

No prop failure for LDE

$$\frac{d}{dt}u_j = u_{j-1} + u_{j+1} - 2u_j + (u_j - a) \left( u_{j-1}(1 - u_{j+1}) + u_{j+1}(1 - u_{j-1}) \right) - \frac{5}{4} \left( a - \frac{1}{2} \right) \sin(2\pi u_j).$$



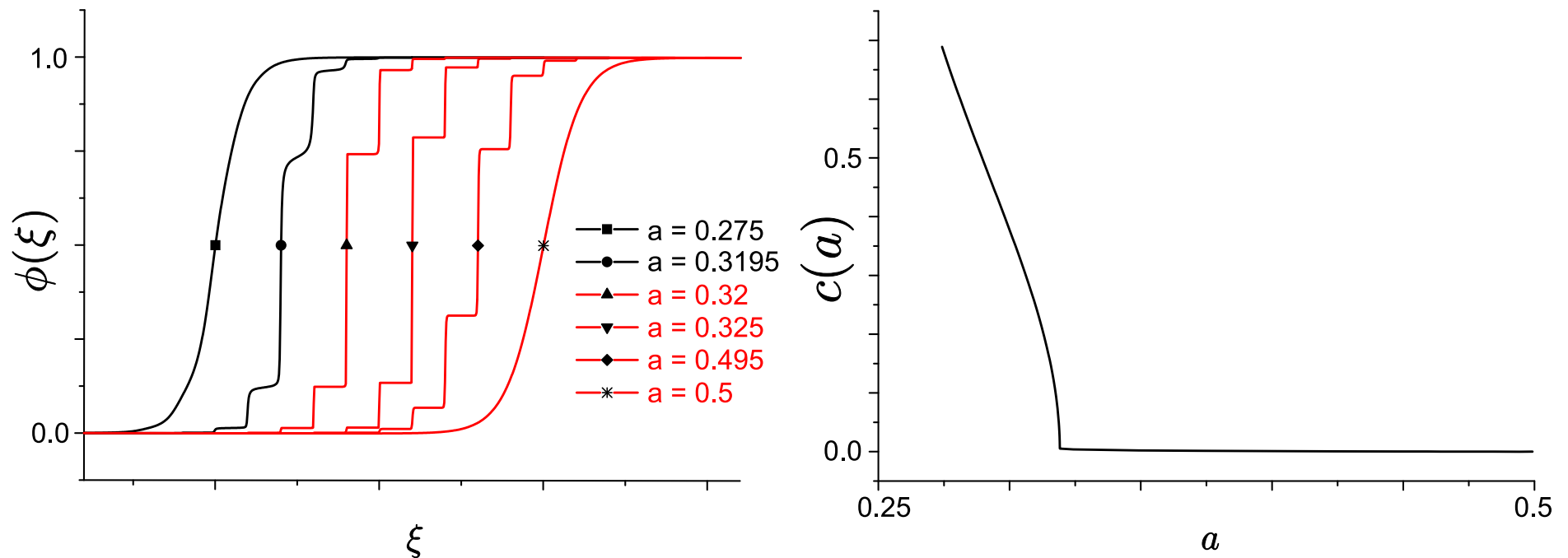
Here  $\partial_a \mathcal{G}$  may have both signs, but (numerically)  $\Psi(\theta) < 0$  for all  $\theta$ .



## Example 3

Do have prop failure for LDE

$$\begin{aligned} \frac{d}{dt}u_j = & u_{j-1} + u_{j+1} - 2u_j + 4u_j(1 - u_j)(u_{j-1} + u_{j+1} - 2a) \\ & - 5(a - \frac{1}{2}) \sin(2\pi u_j)(\frac{6}{5} + \frac{8}{5}u_j). \end{aligned}$$



Numerically computed:  $\Psi(\theta = 0) < 0 < \Psi(\theta = \frac{1}{2})$ .

# Differential advance-delay equation point of view

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Let us write the traveling wave problem as

$$c\phi'(\xi) = \frac{1}{h^2}[\phi(\xi + h) + \phi(\xi - h) - 2\phi(\xi)] + g(\phi(\xi); a),$$

with  $\phi(\xi) \in H^1(\mathbb{R})$  and  $g : H^1(\mathbb{R}) \times [0, 1] \rightarrow H^1(\mathbb{R})$ .

Differential advance-delay operator  $L_c : H^1(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  is

$$(L_c\psi)(z) := -c\psi'(\xi) + \frac{1}{h^2}[\psi(\xi + h) + \psi(\xi - h) - 2\psi(\xi)] + g'(\phi(\xi); a)\psi(\xi),$$

Under the same assumptions, we have at  $a = \frac{1}{2}$ ,

$$\text{Ker}(L_0) = \text{span} \left\{ \varphi'(\xi) e^{i\kappa m \xi} \right\}_{m \in \mathbb{Z}}, \quad \kappa = \frac{2\pi}{h},$$

where  $\varphi(\xi)$  is the stationary front solution for  $c = 0$  and  $a = \frac{1}{2}$ .

# Differential advance-delay equation point of view

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Perturbation theory for small  $c \neq 0$  and  $a = \frac{1}{2}$  gives:

a unique real eigenvalue  $\lambda_c$  such that

$$\lambda_c = \mathcal{O}(c^2) \quad \text{as } c \rightarrow 0.$$

the corresponding eigenfunction  $\chi_c \in H^1(\mathbb{R})$  such that

$$\|\chi_c - \varphi'\|_{L^2} \geq C > 0 \quad \text{as } c \rightarrow 0.$$

a countable set of simple eigenvalues

$$\lambda_c^{(m)} = \lambda_c - i\kappa m c, \quad \chi_c^{(m)}(\xi) = \chi_c(\xi) e^{i\kappa m \xi}, \quad m \in \mathbb{Z}.$$

# Numerical approximations for small $c$

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$$(L_0\psi)(z) := \frac{1}{h^2}[\psi(\xi+h)+\psi(\xi-h)-2\psi(\xi)] + \frac{2(2 - 3\operatorname{sech}^2(b\xi) - h^2\operatorname{sech}^4(b\xi))}{(1 + h^2\operatorname{sech}^2(b\xi))^2}\psi(\xi),$$

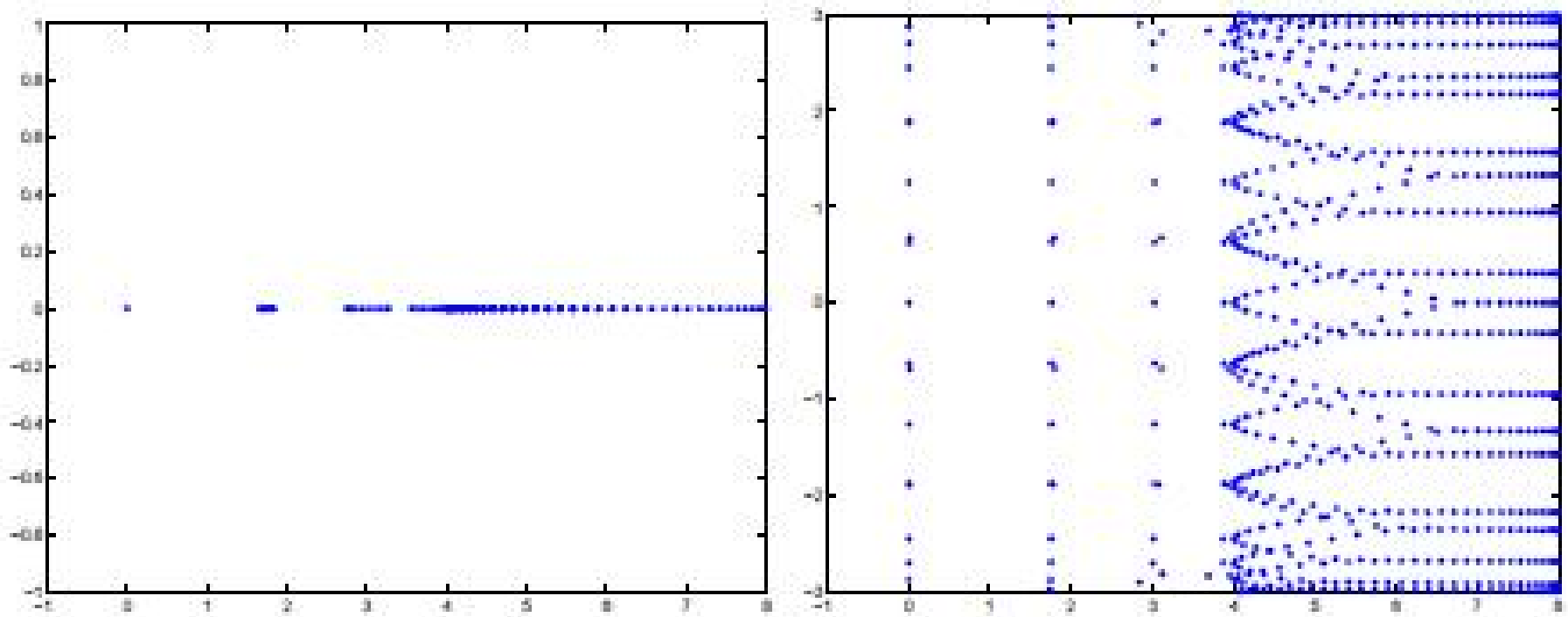


Figure 1: Numerical approximation of spectrum of  $L_c$  for  $c = 0$  (left) and  $c = 0.1$  (right).

# Numerical approximations for small $c$

$$\lambda_c = \mathcal{O}(c^2), \quad \|\chi_c - \varphi'\|_{L^2} = \mathcal{O}(1), \quad \text{as } c \rightarrow 0.$$

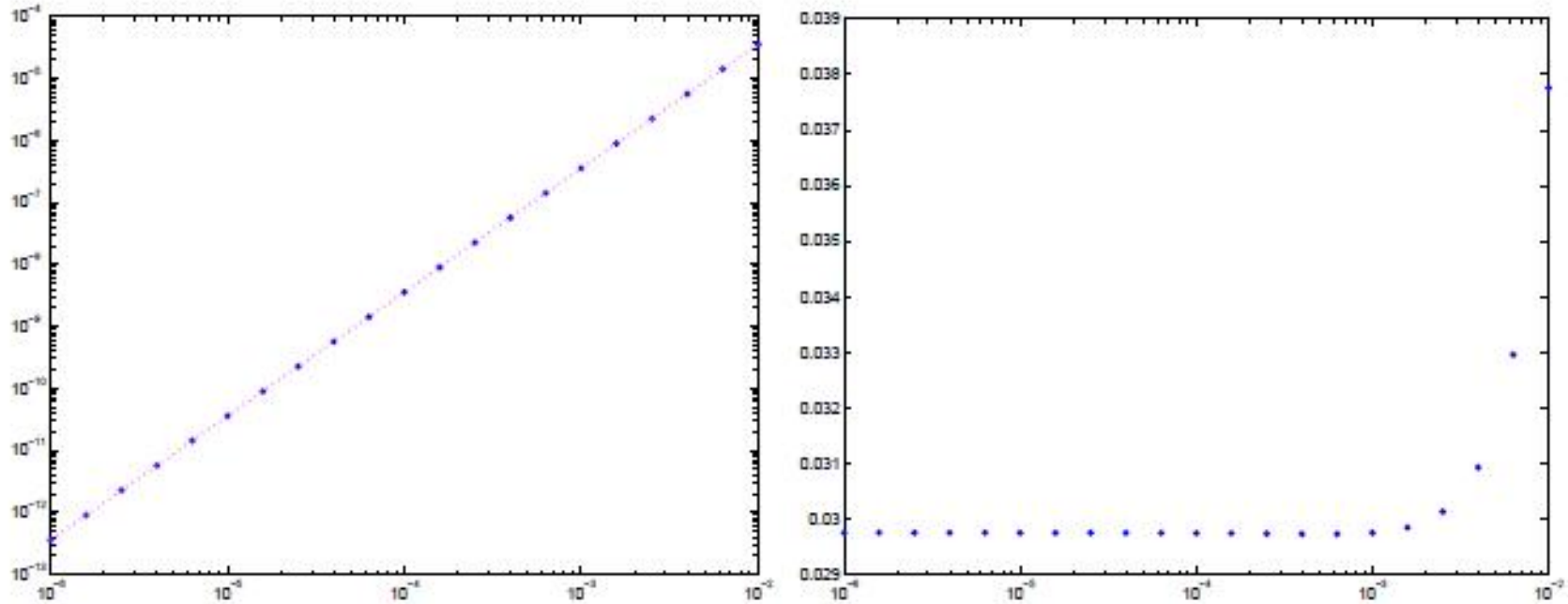


Figure 2: Left: convergence of the smallest eigenvalue of  $L_c$  as  $c \rightarrow 0$ . The dotted curve shows the power fit with  $c^{1.9997}$ . Right: the norm  $\|\chi_c - \chi\|_{L^2}$  versus  $c$  for the corresponding eigenvector.

# Numerical approximations for small $c$

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$$(L_c - \lambda_c I)\psi = f_c : \quad \langle \theta_c, f_c \rangle_{L^2} = 0,$$

where  $\theta_c$  is the eigenvector of the adjoint operator  $L_c^*$ .

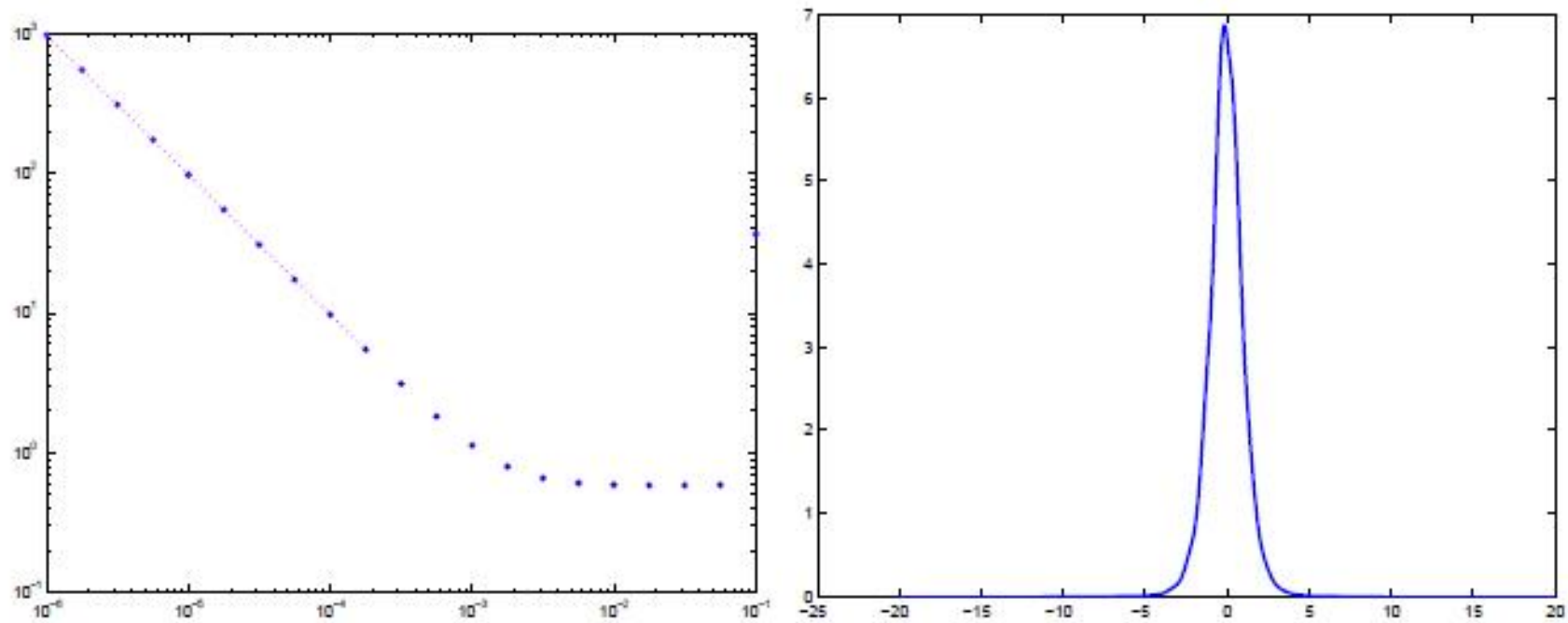


Figure 3: Left: the norm  $\|\psi\|_{L^2}$  versus  $c$ . The dotted curve shows the power fit with  $c^{-0.9993}$ . Right: the solution  $\psi(z)$  for  $c = 0.1$ .

# Projection method

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Differential advance-delay equation,

$$c\phi'(\xi) = \frac{1}{h^2}[\phi(\xi + h) + \phi(\xi - h) - 2\phi(\xi)] + g(\phi(\xi); a),$$

The decomposition

$$\phi(z) = \varphi(z) + \psi(z), \quad \langle \varphi', \psi \rangle_{L^2} = 0$$

is not sufficient because of the singular behavior  $\|\psi\|_{H^1} = \mathcal{O}(c^{-1})$  as  $c \rightarrow 0$ .

If  $\varphi(z)$  is a solution and  $g(z+h) = g(z)$  is any  $C^1$  function such that  $\|g'\|_{L^\infty} < 1$ , then  $\tilde{\varphi}(\tilde{z})$  is also a solution of the advanced-delay equation with  $c = 0$ , where

$$z = \tilde{z} - g(\tilde{z}) \quad \Rightarrow \quad \frac{d\tilde{z}}{dz} = 1 + \sum_{m \in \mathbb{Z}} b_m e^{i\kappa m z}.$$

Coefficients  $\{b_m\}_{m \in \mathbb{Z}}$  can be chosen to remove singular projections and to prove  $c(a) = c_1(a - \frac{1}{2}) + \mathcal{O}(a - \frac{1}{2})$ .