Self-similar solutions for reversing interfaces in slow diffusion with strong absorption

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The Diffusion Equation with Absorption

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} - h$$

The Slow Diffusion Equation with Strong Absorption

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h^m \frac{\partial h}{\partial x}\right) - h^n$$

- Slow diffusion: $m > 0$
  implies finite propagation speed for contact lines
  (Herrero-Vazquez, 1987)

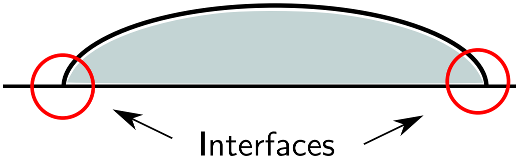
- Strong absorption: $n < 1$
  implies finite time extinction for compactly supported data
  (Kersner, 1983).

Physical Examples

The slow diffusion equation

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h^m \frac{\partial h}{\partial x}\right) - h^n$$

describes physical processes related to dynamics of interfaces.

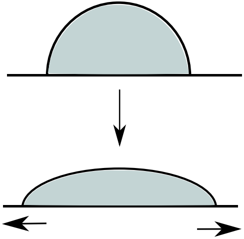


- spread of viscous films over a horizontal plate subject to gravity and constant evaporation ($m = 3$ and $n = 0$) (Acton-Huppert-Worster, 2001)
- dispersion of biological populations with a constant death rate ($m = 2$, $n = 0$)
- nonlinear heat conduction with a constant rate of heat loss ($m = 4$, $n = 0$)
- fluid flows in porous media with a drainage rate driven by gravity or background flows ($m = 1$ and $n = 1$ or $n = 0$) (Pritchard–Woods–Hogg, 2001)

Interface Dynamics

Advancing interfaces

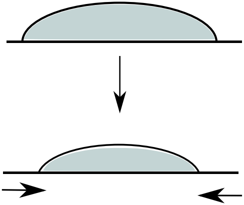
- driven by diffusion



$$h \sim (x - \ell(t))^{1/m}$$

Receding interfaces

- driven by absorption



$$h \sim (x - \ell(t))^{1/(1-n)}$$

We wish to construct a solution that exhibits reversing behaviour:

$$\text{Advancing} \rightarrow \text{Receding}$$

or anti-reversing behaviour:

$$\text{Receding} \rightarrow \text{Advancing}$$

Self-similar solutions

Consider the following self-similar reduction (Gandarias, 1994):

$$h(x,t) = (\pm t)^{\frac{1}{1-n}} H_\pm(\phi), \quad \phi = x(\pm t)^{-\frac{m+1-n}{2(1-n)}}, \quad \pm t > 0,$$

where $m > 0$ and $n < 1$. The functions $H_\pm$ satisfy the ODEs:

$$\frac{d}{d\phi}\left(H_\pm^m \frac{dH_\pm}{d\phi}\right) \pm \frac{m+1-n}{2(1-n)}\phi \frac{dH_\pm}{d\phi} = H_\pm^n \pm \frac{1}{1-n} H_\pm$$

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We seek positive solutions $H_\pm$ on the semi-infinite line $[A_\pm, \infty)$ that satisfy

(i): $H_\pm(\phi) \to 0$ as $\phi \to A_\pm$,

(ii): $H_\pm(\phi)$ is monotonically increasing for all $\phi > A_\pm$,

(iii): $H_\pm(\phi) \to +\infty$ as $\phi \to +\infty$,

(iv): $H_+(\phi) \sim H_-(\phi)$ as $\phi \to +\infty$.

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$$\begin{aligned}
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&\text{(ii):} && H_\pm(\phi) \ \text{ is monotonically increasing for all } \ \phi > A_\pm,\\
&\text{(iii):} && H_\pm(\phi) \to +\infty \quad \text{as} \quad \phi \to +\infty,\\
&\text{(iv):} && H_+(\phi) \sim H_-(\phi) \quad \text{as} \quad \phi \to +\infty.
\end{aligned}$$

If $A_\pm > 0$, the existence of self-similar solutions imply reversing behaviour:

$$\ell(t) = A_\pm(\pm t)^{\frac{m+1-n}{2(1-n)}}, \quad \pm t > 0.$$

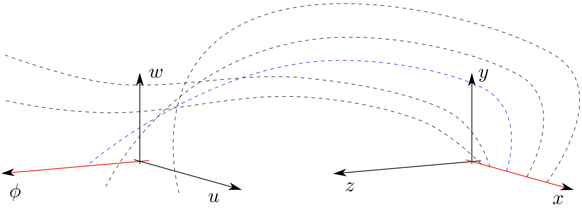
If $m + n > 1$, then $\ell'(0) = 0$.

Dynamical Systems Framework

Solutions were approximated by a naive numerical scheme
in Foster *et al.* [SIAM J. Appl. Math. **72**, 144 (2012)].

The scope of our work is to develop a “rigorous” shooting method:

- The ODEs are singular in the limits of small and large $H_\pm$
- Make transformations to change singular boundary values to equilibrium points
- Obtain near-field asymptotics (small $H_\pm$): $(\phi, u, w) = (A_\pm, 0, 0)$
- Obtain far-field asymptotics (large $H_\pm$): $(x, y, z) = (x_0, 0, 0)$
- Connect between near-field and far-field asymptotics.



Near-field asymptotics

In variables $u = H_\pm$ and $w = H_\pm^m \frac{dH_\pm}{d\phi}$, the system is non-autonomous:

$$\begin{aligned} \frac{du}{d\phi} &= \frac{w}{u^m}, \\ \frac{dw}{d\phi} &= u^n \pm \frac{1}{1-n} u \mp \frac{m+1-n}{2(1-n)} \frac{\phi w}{u^m}. \end{aligned}$$

The system is also singular at $u = 0$.

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The system is also singular at $u = 0$.

Introduce the map $\tau \mapsto \phi$ by $\frac{d\phi}{d\tau} = u^m$ for $u > 0$. Then, we obtain the 3D autonomous dynamical system

$$\begin{cases} \dot{\phi} = u^m, \\ \dot{u} = w, \\ \dot{w} = u^{m+n} \pm \frac{1}{1-n} u^{m+1} \mp \frac{m+1-n}{2(1-n)} \phi w. \end{cases}$$

Near-field asymptotics

In variables $u = H_\pm$ and $w = H_\pm^m \frac{dH_\pm}{d\phi}$, the system is non-autonomous:

$$\frac{du}{d\phi} = \frac{w}{u^m},$$
$$\frac{dw}{d\phi} = u^n \pm \frac{1}{1-n} u \mp \frac{m+1-n}{2(1-n)} \frac{\phi w}{u^m}.$$

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The set of equilibrium points is given by $(\phi, u, w) = (A, 0, 0)$, where $A \in \mathbb{R}$. If $m > 1$, each equilibrium point is associated with the Jacobian matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \mp \frac{m+1-n}{2(1-n)} A \end{bmatrix}.$$

with a double zero eigenvalue and a simple nonzero eigenvalue if $A \neq 0$.

Center manifold

For every $m > 0$, $n < 1$, and $m + n > 1$ and for every $A \neq 0$, there exists a two-dimensional center manifold near $(A, 0, 0)$, which can be parameterized by

$$W_c(A, 0, 0) = \left\{ w = \pm \frac{2(1-n)}{(m+1-n)A} u^{m+n} + \cdots, \quad \phi \in (A, A+\delta), \quad u \in (0, \delta) \right\}.$$

Dynamics on $W_c(A, 0, 0)$ is topologically equivalent to that of

$$\begin{cases} \dot{\phi} = u^m, \\ \dot{u} = \pm \frac{2(1-n)u^{m+n}}{(m+1-n)A}. \end{cases}$$

In particular, for every $A \neq 0$, there exists exactly one trajectory on $W_c(A, 0, 0)$, which approaches the equilibrium point $(A, 0, 0)$ as $\tau \to -\infty$ if $\pm A > 0$.

If $\pm A_\pm > 0$, *the unique solution has the following asymptotic behaviour*

$$H_\pm(\phi) = \left[\pm \frac{2(1-n)^2}{(m+1-n)A_\pm} (\phi - A_\pm) \right]^{1/(1-n)} + \cdots, \quad \text{as} \quad \phi \to A_\pm.$$

Unstable manifold

If $\pm A < 0$, the center manifold is attracting (no trajectories leave $(A, 0, 0)$). However, there is an unstable manifold.

For every $m > 1$, $n < 1$, and $m + n > 1$ and for every $\pm A < 0$, there exists a one-dimensional unstable manifold near $(A, 0, 0)$, which can be parameterized as follows:

$$W_u(A,0,0) = \left\{ \phi = A + \mathcal{O}(u^m), \quad w = \mp \frac{m+1-n}{2(1-n)} Au + \mathcal{O}(u^{m+n}), \quad u \in (0,\delta) \right\}.$$

Dynamics on $W_u(A, 0, 0)$ is topologically equivalent to that of

$$\dot{u} = \mp \frac{m+1-n}{2(1-n)} Au.$$

If $\mp A_\pm > 0$, the unique solution has the following asymptotic behaviour

$$H_\pm(\phi) = \left(\mp \frac{m(m+1-n)A_\pm}{2(1-n)} (\phi - A_\pm) \right)^{1/m} [1 + \cdots], \quad \text{as} \quad \phi \to A_\pm.$$

Far-field asymptotics

If a trajectory departs from the point $(\phi, u, w) = (A, 0, 0)$, how does it arrive to infinity: $\phi \to \infty$, $u \to \infty$?

Far-field asymptotics

If a trajectory departs from the point $(\phi, u, w) = (A, 0, 0)$, how does it arrive to infinity: $\phi \to \infty$, $u \to \infty$?

Let us change the variables

$$\phi = \frac{x}{y^{\frac{m+1-n}{2(1-n)}}}, \quad u = \frac{1}{y^{\frac{1}{1-n}}}, \quad w = \frac{z}{y^{\frac{m+3-n}{2(1-n)}}}$$

and re-parameterize the time τ with new time s by

$$\frac{d\tau}{ds} = y^{\frac{m+1-n}{2(1-n)}}, \quad y \geq 0.$$

The 3D autonomous dynamical system is rewritten as a smooth system

$$\begin{cases} x' = y - \frac{m+1-n}{2} xz, \\ y' = -(1-n)zy, \\ z' = \pm\frac{1}{1-n} y + y^2 \mp \frac{m+1-n}{2(1-n)} xz - \frac{m+3-n}{2} z^2, \end{cases}$$

The 3D smooth dynamical system is

$$\left\{\begin{array}{l} x' = y - \frac{m+1-n}{2} xz, \\ y' = -(1-n) zy, \\ z' = \pm \frac{1}{1-n} y + y^2 \mp \frac{m+1-n}{2(1-n)} xz - \frac{m+3-n}{2} z^2, \end{array}\right.$$

The set of equilibrium points is given by $(x, y, z) = (x_0, 0, 0)$, where $x_0 \in \mathbb{R}$. Each equilibrium point is associated with the Jacobian matrix

$$\left[\begin{array}{ccc} 0 & 1 & -\frac{m+1-n}{2} x_0 \\ 0 & 0 & 0 \\ 0 & \pm\frac{1}{1-n} & \mp\frac{m+1-n}{2(1-n)} x_0 \end{array}\right].$$

with a double zero eigenvalue and a simple nonzero eigenvalue if $x_0 \neq 0$. Only $x_0 > 0$ is relevant for the asymptotics as $\phi \to +\infty$.

- Two-dimensional center manifold associated with the double zero eigenvalue.
- A stable curve for the upper sign and an unstable curve for the lower sign.

Center manifold

Assume $m > 0$, $n < 1$, and $m + n > 1$. For every $x_0 > 0$, there exists a two-dimensional center manifold near $(x_0, 0, 0)$, which can be parameterized as follows:

$$W_c(x_0, 0, 0) = \left\{ y = \frac{m+1-n}{2} xz + \mathcal{O}(z^2), \quad x \in (x_0 - \delta, x_0 + \delta), \;\; z \in (-\delta, \delta) \right\}.$$

The dynamics on $W_c(x_0, 0, 0)$ is topologically equivalent to that of

$$\begin{cases} x' = \pm(1-n)\left(\frac{m+n+1}{2} - \frac{(m+1-n)^2}{4} x_0^2\right) z^2, \\ z' = -(1-n) z^2. \end{cases}$$

In particular, there exists exactly one trajectory on $W_c(x_0, 0, 0)$, which approaches the equilibrium point $(x_0, 0, 0)$ as $s \to +\infty$.

The solution at infinity satisfies the asymptotic behaviour

$$H_\pm(\phi) \sim \left(\frac{\phi}{x_0}\right)^{\frac{2}{m+1-n}} \quad \text{as} \quad \phi \to +\infty.$$

The family of diverging solutions is 1D for H_- and 2D for H_+.

Back to the plan

We are developing "rigorous" shooting method:

- The ODEs are singular in the limits of small and large $H_\pm$
- Make transformations to change singular boundary values to equilibrium points
- Obtain near-field asymptotics (small $H_\pm$): $(\phi, u, w) = (A_\pm, 0, 0)$
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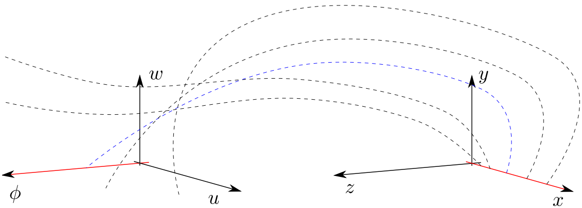

Connection results for H_+ (after reversing)

- Trajectory that departs from $(\phi, u, w) = (A_+, 0, 0)$ is 1D
- Trajectory that arrives to $(x, y, z) = (x_0, 0, 0)$ is 2D.

Fix $A_+ \in \mathbb{R}\backslash\{0\}$ and consider a 1D trajectory such that $(\phi, u, w) \to (A_+, 0, 0)$ as $\tau \to -\infty$ and $u > 0$. Then, there exists a $\tau_0 \in \mathbb{R}$ such that $\phi(\tau) \to +\infty$ and $u(\tau) \to +\infty$ as $\tau \to \tau_0$.

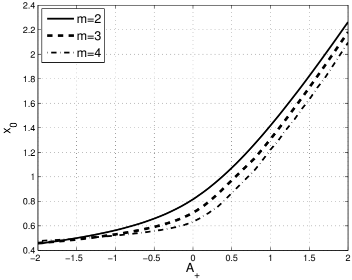

Figure: Plots of the variation of x_0 with A_+ for various different values of $m = 2,\ 3$ and 4.

Connection results for H_- (before reversing)

- Trajectory that departs from $(\phi, u, w) = (A_-, 0, 0)$ is 1D
- Trajectory that arrives to $(x, y, z) = (x_0, 0, 0)$ is 1D.

If we shoot from $(A_-, 0, 0)$, then the trajectory does not generally reach $(x_0, 0, 0)$

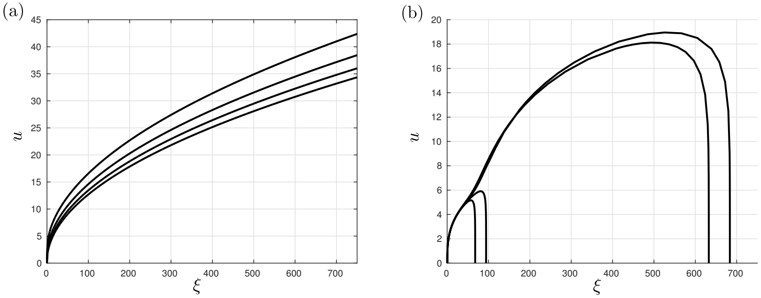

Figure: Panels (a) and (b) show trajectories with $m = 3$ and $n = 0$ for H_+ and H_- respectively.

Connection results for H_- (before reversing)

- Trajectory that departs from $(\phi, u, w) = (A_-, 0, 0)$ is 1D
- Trajectory that arrives to $(x, y, z) = (x_0, 0, 0)$ is 1D.

Therefore, we shoot from $(x_0, 0, 0)$ trying to reach $(A_-, 0, 0)$.

Lemma

Fix $x_0 > 0$ and consider a 1D trajectory such that $(x, y, z) \to (x_0, 0, 0)$ as $s \to +\infty$ and $y > 0$. Then, there exists an $s_0 \in \mathbb{R}$ such that

(i) *either $w = 0$ and $u \geq 0$ as $s \to s_0$*

(ii) *or $u = 0$ and $w \geq 0$ as $s \to s_0$.*

In both cases, if $(u, w) \neq (0, 0)$ as $s \to s_0$, then $|\phi| < \infty$ as $s \to s_0$.

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In both cases, if $(u, w) \neq (0, 0)$ as $s \to s_0$, then $|\phi| < \infty$ as $s \to s_0$.

Open ends:

- Do the two piecewise C^1 maps intersect?

$$\text{(i)} \quad \mathbb{R}^+ \ni x_0 \mapsto (\phi, u) \in \mathbb{R} \times \mathbb{R}^+ \quad \text{and} \quad \text{(ii)} \quad \mathbb{R}^+ \ni x_0 \mapsto (\phi, w) \in \mathbb{R} \times \mathbb{R}^+.$$

- If they do, does ϕ remain bounded at the intersection point?

And here the numerical approximation kicks in...

Finding the intersection points $x_0 = x_*$

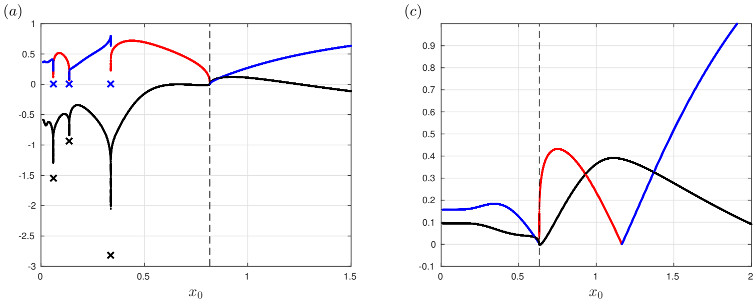
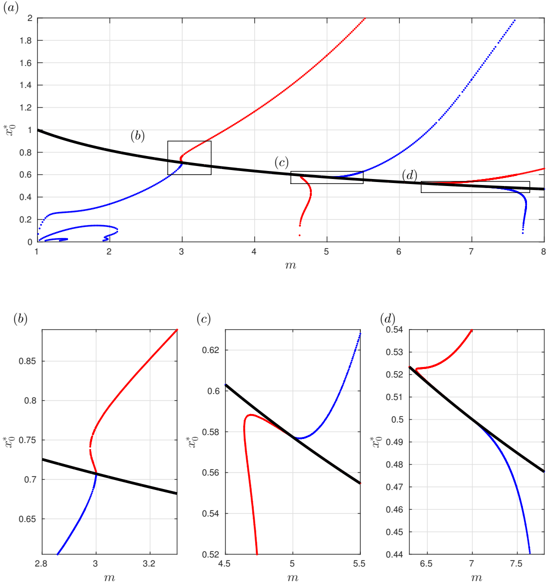
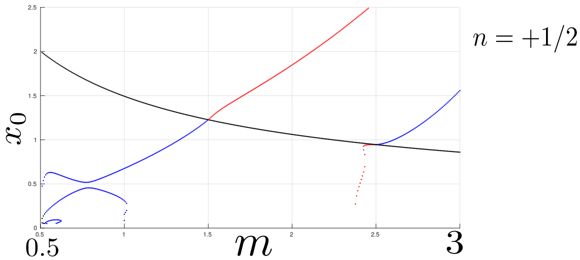


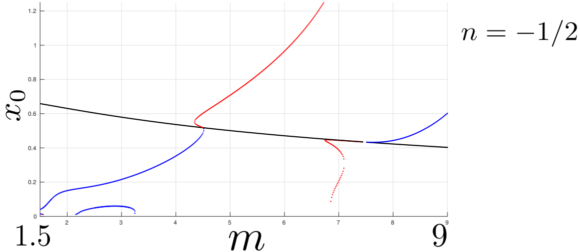
Figure: Panels (a)-(b) show plots of the piecewise C^1 maps for $m = 2$ and $m = 4$. In all cases the blue, red and black curves show the value of w at $u = 0$, the value of u at $w = 0$ and the value of ξ at the termination point respectively.

Self-similar solutions for $n = 0$ and bifurcations



Self-similar solutions for other values of n





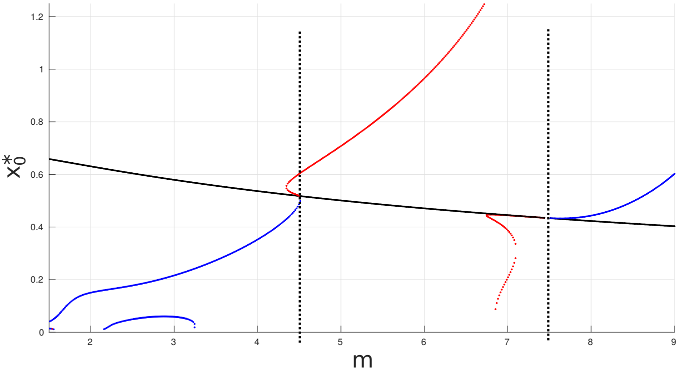
Location of Bifurcations

The black curve corresponds to the exact solution with $A_+ = A_- = 0$:

$$H_\pm(\phi) = \left(\frac{\phi}{x_*}\right)^{\frac{2}{m+1-n}}, \quad x_*^2 = \frac{2(m+1+n)}{(m+1-n)^2}.$$

After substituting self-similar variables, it is a static solution $h(x,t) = h(x)$. New self-similar solutions bifurcate from the static solutions at

$$m = m_k = (1-n)(2k-1), \quad k = 1, 2, 3, \ldots$$



Analysis of Bifurcations ($n = 0$)

Write H_- as a perturbation to the exact solution

$$H_- = x^{\frac{2}{m+1}} + u(x).$$

The bifurcation problem is related to the linear equation $Lu = 0$, where

$$Lu = \frac{m+1}{2}\frac{d^2}{dx^2}\left(x^{\frac{2m}{m+1}}u(x)\right) - \frac{m+1}{2}x\frac{du}{dx} + u(x) = 0, \quad x \in (0, \infty).$$

The boundary conditions for admissible self-similar solutions are

$$u(x) \sim x^{\frac{2}{m+1}} \quad \text{as} \quad x \to \infty$$

Near $x = 0$, the self-similar solutions satisfy

$$u(x) \sim c_1 x^{\frac{1-m}{1+m}} + c_2 x^{\frac{-2m}{1+m}} \quad \text{as} \quad x \to 0.$$

Analysis of Bifurcations

After a coordinate transformation, the homogeneous equation $Lu = 0$ becomes the Kummer's differential equation (1837),

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} + aw(z) = 0, \quad z \in (0,\infty),$$

where

$$a := -\frac{m+1}{2}, \quad b := \frac{m+3}{2}.$$

The power series solution is given by Kummer's function

$$M(z;a,b) = 1 + \frac{a}{b}\frac{z}{1!} + + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \cdots$$

The other solution is singular as $z \to 0$.

The only solution with the correct boundary condition at infinity,

$$U(z;a,b) \sim z^{-a}\left[1 + \mathcal{O}(z^{-1})\right] \quad \text{as} \quad z \to \infty,$$

was characterized by Tricomi (1947).

When $a = -k$ or $m = m_k = (2k-1)$, $k \in \mathbb{N}$, Kummer's power series $M(z;a,b)$ becomes a polynomial which connects to the Tricomi's function $U(z;a,b)$.

Connection problem ($n = 0$)

The *inner* solution near the interface:

$$\phi = A + |A|^{\frac{m+1}{m-1}} \eta, \quad H(\phi) = |A|^{\frac{2}{m-1}} \mathcal{H}(\eta),$$

satisfying

$$\mathcal{H}(\eta) \sim \left(\frac{m(m+1)}{2} \eta \right)^{\frac{1}{m}} \quad \text{as} \quad \eta \to 0; \quad \sim \left(\frac{m+1}{2} \eta^2 \right)^{\frac{1}{m+1}} \quad \text{as} \quad \eta \to \infty.$$

The *outer* solution in the far field:

$$H(\phi) = x^{\frac{2}{m+1}} + \alpha u_1(x) + \alpha^2 u_2(x) + \mathcal{O}(\alpha^3), \quad x := \phi / x_*,$$

where u_1 is Tricomi's function

$$u_1(x) = x^{\frac{1-m}{1+m}} U \left(\frac{m+1}{2} x^{\frac{2}{m+1}}; -\frac{m+1}{2}, \frac{m+3}{2} \right),$$

whereas u_2 is a solution of the inhomogeneous equation

$$L u_2 = R_2 := -\frac{m(m+1)}{4} \frac{d^2}{dx^2} \left[x^{\frac{2(m-1)}{m+1}} u_1^2 \right].$$

Matching conditions as $\eta \to \infty$ and $x \to 0$ determine α and $x_0 - x_*$ in terms of A, and A in terms of $m - m_k$, where $m = m_k = (2k-1)$, $k \in \mathbb{N}$ is the bifurcation point.

Numerical confirmations

Bifurcation at $m = 5$ and $n = 0$:

$$A_- = -\frac{40}{9}(x_0 - x_*) + \cdots$$

and

$$5 - m = \frac{27\sqrt{3}}{4}A_- + \cdots$$

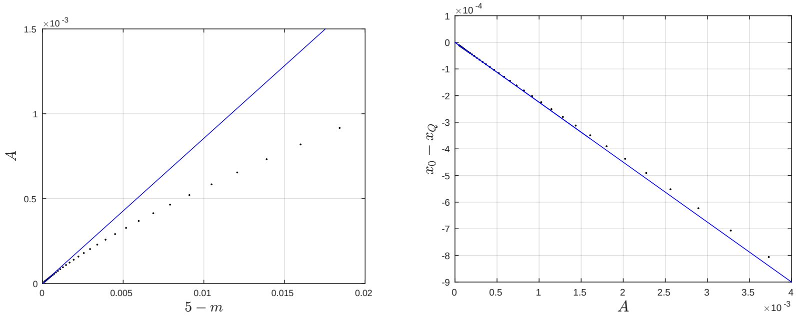


Figure: Left: The variation of the parameter A_- with $5 - m$, and; right: the variation of $x_0 - x_*$ with A_- local to $m = 5$. The black dots are numerics, the blue lines are asymptotics.

Conclusion

- For every $m > 0$, $n < 1$ and $m + n > 1$ a pair of solutions H_+ and H_- can be constructed numerically and then converted to $h(x, t)$
  - Solutions with $A_\pm > 0$ correspond to reversing interfaces
  - Solutions with $A_\pm < 0$ correspond to anti-reversing interfaces

- The behaviour of the self-similar solution at zero and infinity is justified by the dynamical system theory.

- Bifurcations of self-similar solutions are predicted from analysis of the classical Kummer's differential equation.

- Relevance of the self-similar solutions for the slow diffusion equation is confirmed numerically.

References

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