Self-similar solutions for reversing interfaces in slow diffusion with strong absorption

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The Diffusion Equation with Absorption

\[ \frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} - h \]

The Slow Diffusion Equation with Strong Absorption

\[ \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( h^m \frac{\partial h}{\partial x} \right) - h^n \]

- Slow diffusion: \( m > 0 \)
  implies finite propagation speed for contact lines
  (Herrero-Vazquez, 1987)

- Strong absorption: \( n < 1 \)
  implies finite time extinction for compactly supported data
  (Kersner, 1983).
Physical Examples

The slow diffusion equation

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( h^m \frac{\partial h}{\partial x} \right) - h^n$$

describes physical processes with dynamics of interfaces.

- spread of viscous films over a horizontal plate subject to gravity and constant evaporation ($m = 3$ and $n = 0$) (Acton-Huppert-Worster, 2001)
- dispersion of biological populations with a constant death rate ($m = 2$, $n = 0$)
- nonlinear heat conduction with a constant rate of heat loss ($m = 4$, $n = 0$)
- fluid flows in porous media with a drainage rate driven by gravity or background flows ($m = 1$ and $n = 1$ or $n = 0$) (Pritchard–Woods–Hogg, 2001)
Interface Dynamics

Advancing interfaces
▶ driven by diffusion

\[ h \sim (x - \ell(t))^{1/m} \]

Receding interfaces
▶ driven by absorption

\[ h \sim (x - \ell(t))^{1/(1-n)} \]

We wish to construct a solution that exhibits **reversing** behaviour:

Advancing \(\rightarrow\) Receding

or **anti-reversing** behaviour:

Receding \(\rightarrow\) Advancing
Results in mathematical literature

- Kawohl–Kersner (1992) - weak formulation for $n \leq 0$

$$
\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( h^m \frac{\partial h}{\partial x} \right) - h^n \chi_{h > 0}
$$

and existence of solutions (uniqueness is open).


- Galaktionov-Shmarev-Vazquez (1999) - position of the interface, $\ell(t)$, is a Lipschitz continuous function of time if $m + n > 1$, found from integrating the interface equations

$$
\frac{d\ell}{dt} = \begin{cases} 
-h^{m-1} \left. \frac{\partial h}{\partial x} \right|_{x=\ell(t)+} & \text{if } \dot{\ell} \leq 0, \\
-h^n \left. \left( \frac{\partial h}{\partial x} \right)^{-1} \right|_{x=\ell(t)+} & \text{if } \dot{\ell} \geq 0.
\end{cases}
$$

For technical reasons, $n \in (0, 1)$ was assumed.
Self-similar solutions

Consider the following self-similar reduction (Gandarias, 1994):

\[ h(x, t) = (\pm t)^{\frac{1}{1-n}} H_\pm(\phi), \quad \phi = x(\pm t)^{-\frac{m+1-n}{2(1-n)}}, \quad \pm t > 0, \]

where \( m > 0 \) and \( n < 1 \). The functions \( H_\pm \) satisfy the ODEs:

\[
\frac{d}{d\phi} \left( H_\pm^m \frac{dH_\pm}{d\phi} \right) \pm \frac{m + 1 - n}{2(1 - n)} \phi \frac{dH_\pm}{d\phi} = H_\pm^n \pm \frac{1}{1 - n} H_\pm
\]
Self-similar solutions

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\]

We seek positive solutions \( H_{\pm} \) on the semi-infinite line \([A_{\pm}, \infty)\) that satisfy

(i): \( H_{\pm}(\phi) \to 0 \) as \( \phi \to A_{\pm} \),

(ii): \( H_{\pm}(\phi) \) is monotonically increasing for all \( \phi > A_{\pm} \),

(iii): \( H_{\pm}(\phi) \to +\infty \) as \( \phi \to +\infty \),

(iv): \( H_+(\phi) \sim H_-(\phi) \) as \( \phi \to +\infty \).
Self-similar solutions

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(iii): \( H_{\pm}(\phi) \to +\infty \) as \( \phi \to +\infty \),

(iv): \( H_{+}(\phi) \sim H_{-}(\phi) \) as \( \phi \to +\infty \).

If \( A_{\pm} > 0 \), the existence of self-similar solutions imply reversing behaviour:

\[ \ell(t) = A_{\pm}(\pm t)^{\frac{m+1-n}{2(1-n)}}, \quad \pm t > 0. \]

If \( m + n > 1 \), then \( \ell'(0) = 0 \).
Dynamical Systems Framework

Solutions were approximated by a naive numerical scheme in Foster et al. [SIAM J. Appl. Math. 72, 144 (2012)].

The scope of our work is to develop a “rigorous” shooting method:

- The ODEs are singular in the limits of small and large $H_\pm$
- Make transformations to change singular boundary values to equilibrium points
- Obtain near-field asymptotics (small $H_\pm$): $(\phi, u, w) = (A_\pm, 0, 0)$
- Obtain far-field asymptotics (large $H_\pm$): $(x, y, z) = (x_0, 0, 0)$
- Connect between near-field and far-field asymptotics.
Near-field asymptotics

In variables $u = H_{\pm}$ and $w = H_{\pm}^m H'_{\pm}(\phi)$, the system is non-autonomous:

$$\frac{du}{d\phi} = \frac{w}{u^m},$$
$$\frac{dw}{d\phi} = u^n \mp \frac{1}{1-n} u \mp \frac{m + 1 - n}{2(1-n)} \phi w.$$ 

The system is singular at $u = 0$. 
Near-field asymptotics

In variables \( u = H_{\pm} \) and \( w = H^m_{\pm}H'_\pm(\phi) \), the system is non-autonomous:

\[
\frac{du}{d\phi} = \frac{w}{u^n}, \quad \frac{dw}{d\phi} = u^n \pm \frac{1}{1-n}u \mp \frac{m + 1 - n}{2(1-n)} \phi w.
\]

The system is singular at \( u = 0 \).

Introduce the map \( \tau \mapsto \phi \) by \( \frac{d\phi}{d\tau} = u^m \) for \( u > 0 \). Then, we obtain the 3D autonomous dynamical system

\[
\begin{cases}
\dot{\phi} = u^m, \\
\dot{u} = w, \\
\dot{w} = u^{m+n} \pm \frac{1}{1-n}u^{m+1} \mp \frac{m+1-n}{2(1-n)} \phi w.
\end{cases}
\]
Near-field asymptotics

In variables \( u = H_\pm \) and \( w = H_\pm^m H'_\pm(\phi) \), the system is non-autonomous:

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\frac{du}{d\phi} = \frac{w}{u^m},
\]
\[
\frac{dw}{d\phi} = u^n \pm \frac{1}{1-n} u \mp \frac{m+1-n}{2(1-n)} \phi w.
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\end{array}
\right.
\]

The set of equilibrium points is given by \( (\phi, u, w) = (A, 0, 0) \), where \( A \in \mathbb{R} \). If \( m > 1 \), each equilibrium point is associated with the Jacobian matrix

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & \mp \frac{m+1-n}{2(1-n)} A
\end{bmatrix}.
\]

with a double zero eigenvalue and a simple nonzero eigenvalue if \( A \neq 0 \).
Center manifold

For every $m > 0$, $n < 1$, and $m + n > 1$ and for every $A \neq 0$, there exists a two-dimensional center manifold near $(A, 0, 0)$, which can be parameterized by

$$W_c(A, 0, 0) = \left\{ w = \pm \frac{2(1-n)}{(m+1-n)A} u^{m+n} + \cdots, \; \phi \in (A, A + \delta), \; u \in (0, \delta) \right\}.$$ 

Dynamics on $W_c(A, 0, 0)$ is topologically equivalent to that of

$$\begin{cases} 
\dot{\phi} = u^m, \\
\dot{u} = \pm \frac{2(1-n)u^{m+n}}{(m+1-n)A}.
\end{cases}$$

In particular, for every $A \neq 0$, there exists exactly one trajectory on $W_c(A, 0, 0)$, which approaches the equilibrium point $(A, 0, 0)$ as $\tau \to -\infty$ if $\pm A > 0$. 

**Theorem**

If $\pm A > 0$, the unique solution has the following asymptotic behaviour

$$H_{\pm}(\phi) = \left[ \pm 2 \frac{(1-n)}{(m+1-n)A} u^{m+n} + \cdots, \; \phi \to A_{\pm} \right].$$
Center manifold

For every $m > 0$, $n < 1$, and $m + n > 1$ and for every $A \neq 0$, there exists a two-dimensional center manifold near $(A, 0, 0)$, which can be parameterized by

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\end{array} \right.$$

In particular, for every $A \neq 0$, there exists exactly one trajectory on $W_c(A, 0, 0)$, which approaches the equilibrium point $(A, 0, 0)$ as $\tau \to -\infty$ if $\pm A > 0$.

Theorem

*If $\pm A_\pm > 0$, the unique solution has the following asymptotic behaviour*

$$H_{\pm}(\phi) = \left[ \pm \frac{2(1 - n)^2}{(m + 1 - n)A_\pm} (\phi - A_\pm) \right]^{1/(1-n)} + \cdots, \quad \text{as} \quad \phi \to A_\pm.$$
Unstable manifold

If $\pm A < 0$, the center manifold is attracting (no trajectories leave $W_c(A, 0, 0)$). However, there is an unstable manifold.

For every $m > 1$, $n < 1$, and $m + n > 1$ and for every $\pm A < 0$, there exists a one-dimensional unstable manifold near $(A, 0, 0)$, which can be parameterized as follows:

$$W_u(A, 0, 0) = \left\{ \phi = A + \mathcal{O}(u^m), \quad w = \mp \frac{m + 1 - n}{2(1 - n)} Au + \mathcal{O}(u^{m+n}), \quad u \in (0, \delta) \right\}.$$

Dynamics on $W_u(A, 0, 0)$ is topologically equivalent to that of

$$\dot{u} = \mp \frac{m + 1 - n}{2(1 - n)} Au.$$
Unstable manifold

If $\pm A < 0$, the center manifold is attracting (no trajectories leave $W_c(A, 0, 0)$). However, there is an unstable manifold.

For every $m > 1$, $n < 1$, and $m + n > 1$ and for every $\pm A < 0$, there exists a one-dimensional unstable manifold near $(A, 0, 0)$, which can be parameterized as follows:

$$W_u(A, 0, 0) = \left\{ \phi = A + \mathcal{O}(u^m), \quad w = \mp \frac{m + 1 - n}{2(1 - n)}Au + \mathcal{O}(u^{m+n}), \quad u \in (0, \delta) \right\}.$$  

Dynamics on $W_u(A, 0, 0)$ is topologically equivalent to that of

$$\dot{u} = \mp \frac{m + 1 - n}{2(1 - n)}Au.$$

Theorem

If $\mp A_\pm > 0$, the unique solution has the following asymptotic behaviour

$$H_{\pm}(\phi) = \left( \mp \frac{m(m + 1 - n)A_{\pm}}{2(1 - n)}(\phi - A_{\pm}) \right)^{1/m} [1 + \cdots], \quad \text{as} \quad \phi \to A_{\pm}.$$
Far-field asymptotics

If a trajectory departs from the point \((\phi, u, w) = (A, 0, 0)\), how does it arrive to infinity: \(\phi \to \infty, u \to \infty\)?
Far-field asymptotics

If a trajectory departs from the point \((\phi, u, w) = (A, 0, 0)\), how does it arrive to infinity: \(\phi \rightarrow \infty, u \rightarrow \infty\)?

Let us change the variables

\[
\phi = \frac{x}{m+1-n} y^{2(1-n)/2}, \quad u = \frac{1}{y^{1-n}}, \quad w = \frac{z}{m+3-n} y^{2(1-n)/2}
\]

and re-parameterize the time \(\tau\) with new time \(s\) by

\[
\frac{d\tau}{ds} = y^{m+1-n/2(1-n)}, \quad y \geq 0.
\]

The 3D autonomous dynamical system is rewritten as a smooth system

\[
\begin{cases}
x' = y - \frac{m+1-n}{2} x z, \\
y' = -(1 - n) z y, \\
z' = \pm \frac{1}{1-n} y + y^2 \mp \frac{m+1-n}{2(1-n)} x z - \frac{m+3-n}{2} z^2,
\end{cases}
\]
The 3D smooth dynamical system is
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\begin{align*}
    x' &= y - \frac{m+1-n}{2} xz, \\
    y' &= -(1-n) z y, \\
    z' &= \pm \frac{1}{1-n} y + y^2 \mp \frac{m+1-n}{2(1-n)} xz - \frac{m+3-n}{2} z^2,
\end{align*}
\]

The set of equilibrium points is given by \((x, y, z) = (x_0, 0, 0)\), where \(x_0 \in \mathbb{R}\). Each equilibrium point is associated with the Jacobian matrix
\[
\begin{bmatrix}
    0 & 1 & -\frac{m+1-n}{2} x_0 \\
    0 & 0 & 0 \\
    0 & \pm \frac{1}{1-n} & \mp \frac{m+1-n}{2(1-n)} x_0
\end{bmatrix}.
\]

with a double zero eigenvalue and a simple nonzero eigenvalue if \(x_0 \neq 0\). Only \(x_0 > 0\) is relevant for the asymptotics as \(\phi \to +\infty\).

In addition to the two-dimensional center manifold, there is a stable (attracting) curve for the upper sign and an unstable (repelling) curve for the lower sign.
Center manifold

Assume \( m > 0, n < 1, \) and \( m + n > 1. \) For every \( x_0 > 0, \) there exists a two-dimensional center manifold near \((x_0, 0, 0)\), which can be parameterized as follows:

\[
W_c(x_0, 0, 0) = \left\{ y = \frac{m + 1 - n}{2} x z + O(z^2), \quad x \in (x_0 - \delta, x_0 + \delta), \quad z \in (-\delta, \delta) \right\}.
\]

The dynamics on \( W_c(x_0, 0, 0) \) is topologically equivalent to that of

\[
\begin{align*}
x' &= \pm (1 - n) \left( \frac{m+n+1}{2} - \frac{(m+1-n)^2}{4} x_0^2 \right) z^2, \\
z' &= -(1 - n) z^2.
\end{align*}
\]

In particular, there exists exactly one trajectory on \( W_c(x_0, 0, 0) \), which approaches the equilibrium point \((x_0, 0, 0)\) as \( s \to +\infty. \)
Center manifold

Assume $m > 0$, $n < 1$, and $m + n > 1$. For every $x_0 > 0$, there exists a two-dimensional center manifold near $(x_0, 0, 0)$, which can be parameterized as follows:

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The dynamics on $W_c(x_0, 0, 0)$ is topologically equivalent to that of

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In particular, there exists exactly one trajectory on $W_c(x_0, 0, 0)$, which approaches the equilibrium point $(x_0, 0, 0)$ as $s \to +\infty$.

**Theorem**

*The solution at infinity satisfies the asymptotic behaviour*

$$H_{\pm}(\phi) \sim \left( \frac{\phi}{x_0} \right)^\frac{2}{m+1-n} \quad \text{as} \quad \phi \to +\infty.$$

*The family of diverging solutions is 1D for $H_-$ and 2D for $H_+$.***
Back to the plan

We are developing “rigorous” shooting method:

- The ODEs are singular in the limits of small and large $H_{\pm}$
- Make transformations to change singular boundary values to equilibrium points
- Obtain near-field asymptotics (small $H_{\pm}$): $(\phi, u, w) = (A_{\pm}, 0, 0)$
- Obtain far-field asymptotics (large $H_{\pm}$): $(x, y, z) = (x_0, 0, 0)$
- Connect between near-field and far-field asymptotics.
Connection results for $H_+$ (after reversing)

- Trajectory that departs from $(\phi, u, w) = (A_+, 0, 0)$ is 1D
- Trajectory that arrives to $(x, y, z) = (x_0, 0, 0)$ is 2D.

Lemma

Fix $A_+ \in \mathbb{R}\{0\}$ and consider a 1D trajectory such that $(\phi, u, w) \to (A_+, 0, 0)$ as $\tau \to -\infty$ and $u > 0$. Then, there exists a $\tau_0 \in \mathbb{R}$ such that $\phi(\tau) \to +\infty$ and $u(\tau) \to +\infty$ as $\tau \to \tau_0$. 

The proof is based on the energy method

$E(w, u) := \frac{1}{2}w^2 - \frac{1}{2}m + n + \frac{1}{2}u^{m+n+1} - \frac{1}{2}((m+2)(1-n))u^{m+2}$

with the rate of change along the trajectory given by

$\frac{dE}{d\tau} = -\frac{m+1}{2}(1-n)\phi w^2$. 

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The proof is based on the energy method

$$E(w, u) := \frac{1}{2}w^2 - \frac{1}{m + n + 1}u^{m+n+1} - \frac{1}{(m + 2)(1 - n)}u^{m+2}$$

with the rate of change along the trajectory given by

$$\frac{dE}{d\tau} = -\frac{m + 1 - n}{2(1 - n)}\phi w^2.$$
Connection results for $H_-$ (before reversing)

- Trajectory that departs from $(\phi, u, w) = (A_-, 0, 0)$ is 1D
- Trajectory that arrives to $(x, y, z) = (x_0, 0, 0)$ is 1D.

If we shoot from $(A_-, 0, 0)$, then the trajectory does not generally reach $(x_0, 0, 0)$

Figure: Panels (a) and (b) show trajectories with $m = 3$ and $n = 0$ for $H_+$ and $H_-$ respectively.
Connection results for $H_-$ (before reversing)

- Trajectory that departs from $(\phi, u, w) = (A_-, 0, 0)$ is 1D.
- Trajectory that arrives to $(x, y, z) = (x_0, 0, 0)$ is 1D.

Therefore, we shoot from $(x_0, 0, 0)$ trying to reach $(A_-, 0, 0)$.

**Lemma**

*Fix $x_0 > 0$ and consider a 1D trajectory such that* $(x, y, z) \to (x_0, 0, 0)$ as $s \to +\infty$ and $y > 0$. *Then, there exists an $s_0 \in \mathbb{R}$ such that*

(i) either $w = 0$ and $u \geq 0$ as $s \to s_0$
(ii) or $u = 0$ and $w \geq 0$ as $s \to s_0$.

*In both cases, if* $(u, w) \neq (0, 0)$ as $s \to s_0$, *then* $|\phi| < \infty$ as $s \to s_0$.
Connection results for $H_-$ (before reversing)

- Trajectory that departs from $(\phi, u, w) = (A_-, 0, 0)$ is 1D
- Trajectory that arrives to $(x, y, z) = (x_0, 0, 0)$ is 1D.

Therefore, we shoot from $(x_0, 0, 0)$ trying to reach $(A_-, 0, 0)$.

**Lemma**

Fix $x_0 > 0$ and consider a 1D trajectory such that $(x, y, z) \to (x_0, 0, 0)$ as $s \to +\infty$ and $y > 0$. Then, there exists an $s_0 \in \mathbb{R}$ such that

(i) either $w = 0$ and $u \geq 0$ as $s \to s_0$
(ii) or $u = 0$ and $w \geq 0$ as $s \to s_0$.

In both cases, if $(u, w) \neq (0, 0)$ as $s \to s_0$, then $|\phi| < \infty$ as $s \to s_0$.

**Open ends:**

- It is unknown if the two piecewise $C^1$ maps intersect:
  
  (i) $\mathbb{R}^+ \ni x_0 \mapsto (\xi, u) \in \mathbb{R} \times \mathbb{R}^+$ and (ii) $\mathbb{R}^+ \ni x_0 \mapsto (\xi, w) \in \mathbb{R} \times \mathbb{R}^+$.

- It is unknown if $\phi$ remains bounded at the intersection point.

And here the numerical approximation kicks in...
Numerical Shooting Method

for $H_-$ (before reversing)

- Select a value $x_0$ which characterizes the far-field behaviour
- Use asymptotic behaviour to take a small step away
- Integrate in decreasing $\phi$
- Find intersections of the trajectory either with $w = 0$ or with $u = 0$
- Plot the two piecewise maps versus $x_0$
- Find the interaction points $x_0 = x_*$ between the two piecewise maps at $(u, w) = (0, 0)$ and record the corresponding value for $\phi = A_-$
Numerical Shooting Method

for $H_-$ (before reversing)

- Select a value $x_0$ which characterizes the far-field behaviour
- Use asymptotic behaviour to take a small step away
- Integrate in decreasing $\phi$
- Find intersections of the trajectory either with $w = 0$ or with $u = 0$
- Plot the two piecewise maps versus $x_0$
- Find the interaction points $x_0 = x_*$ between the two piecewise maps at $(u, w) = (0, 0)$ and record the corresponding value for $\phi = A_-$

for $H_+$ (after reversing)

- Select a value $A_+$ which characterizes the near-field behaviour
- Use asymptotic behaviour to take a small step away
- Integrate with increasing $\phi$
- Find $x_0$ from the far-field behaviour
- Plot $x_0$ versus $A_+$
- For the value $x_*$ in part $H_-$, find the value of $A_+$. 
Finding the intersection points $x_0 = x_*$

![Diagram](image)

**Figure:** Panels (a)-(b) show plots of the piecewise $C^1$ maps for $m = 2$ and $m = 4$. In all cases the blue, red and black curves show the value of $w$ at $u = 0$, the value of $u$ at $w = 0$ and the value of $\xi$ at the termination point respectively.
Finding the value of $A_+$

Figure: Panel (a): Plots of the variation of $x_0$ with $A_+$ for various different values of $m = 2, 3$ and 4. Panel (b): Plots of the trajectories emanating from $A_+ = 0.1, 0.2, 0.3, 0.4$ and $0.5$ for $m = 3$. The constant to which these trajectories tend in the far-field is selected to be the corresponding value of $x_0$. 
Self-similar solutions for $n = 0$ and bifurcations
Summary on self-similar solutions

We have demonstrated that

- For each value of $m > 1$ and $n = 0$, there exists at least one value of $x_0 = x_0^*$ that defines a trajectory emanating from $(x_0^*, 0, 0)$ and terminating at $(A_-, 0, 0)$, and thus a suitable solution for $H_-$.  

- For every value of $A_+ \in \mathbb{R}$, there exists a unique corresponding value of $x_0$, thereby defining an infinite family of suitable solutions for $H_+$.  

- Invoking the matching condition at far-field for each value of $x_0 = x_0^*$, we obtain unique values of $A_-$ and $A_+$.  

Relevance to the slow diffusion equation

The asymptotic behavior of the self-similar solutions $H_{\pm}$ is compatible with the interface conditions

$$\frac{d\ell}{dt} = \begin{cases} -h^{m-1} \frac{\partial h}{\partial x} \bigg|_{x=\ell(t)+} & \text{if } \ell'(t) \leq 0, \\ h^n \left(\frac{\partial h}{\partial x}\right)^{-1} \bigg|_{x=\ell(t)+} & \text{if } \ell'(t) \geq 0. \end{cases}$$

obtained in Galaktionov-Shmarev-Vazquez (1999).

For instance, if $A_{\pm} > 0$ and $\ell'(t) > 0$, $t > 0$, we have justified that

$$\frac{d\ell}{dt} = \lim_{x \to \ell(t)+} h^n \left(\frac{\partial h}{\partial x}\right)^{-1} = \frac{m + 1 - n}{2(1-n)} A_{+} t^{\frac{m+1-n}{2(1-n)}} - 1.$$ 

After an integration with the condition $\ell(0) = 0$, this yields the interface evolution

$$\ell(t) = A_{+} t^{\frac{m+1-n}{2(1-n)}}, \quad t > 0,$$

in agreement with the self-similar solution

$$h(x, t) = t^{\frac{1}{1-n}} H_+(\phi), \quad \phi = xt^{-\frac{m+1-n}{2(1-n)}}, \quad t > 0.$$
Simulations of the slow diffusion equation

Numerical simulations with initial data

\[ h|_{t=0} = \lambda \sin \left( \frac{\pi}{2} \left( 1 - \frac{x}{\ell_0} \right)^{1/m} \right) \quad \text{and} \quad \ell|_{t=0} = \ell_0, \]

Figure: Panel (a) shows a comparison between the predictions on the behaviour of \( \ell(t) \) as given by: (i) the self-similar theory (solid line) and (ii) direct numerical simulation, for \( m = 4 \) and \( n = 0 \). The dashed, dash-dotted and dotted curves show the results from direct numerical simulation with \( \lambda = 5, 10, 20 \) and \( \ell_0 = -1 \). Panel (b) shows representative plots of \( h(x, t) \) for the choice \( \lambda = 5 \) at 10 equally spaced values of \( t \). The blue curve show the solution immediately prior to the reversing event, \textit{i.e.}, when the simulation was terminated.
Self-similar solutions for other values of $n$

$n = +1/2$

$n = -1/2$
Location of Bifurcations

The black curve corresponds to the exact solution with $A_+ = A_- = 0$:

$$H_\pm(\phi) = \left(\frac{\phi}{x_Q}\right)^{\frac{2}{m+1-n}}, \quad x_Q^2 = \frac{2(m + 1 + n)}{(m + 1 - n)^2}.$$

After substituting self-similar variables, it is a static solution $h(x, t) = h(x)$. New self-similar solutions bifurcate from the static solutions at

$$m = m_k = (1 - n)(2k - 1), \quad k = 1, 2, 3, \ldots$$
Analysis of Bifurcations \( (n = 0) \)

Write \( H_- \) as a perturbation to the exact solution

\[
H_- = x^{\frac{2}{m+1}} + u(x), \quad x = \frac{\phi}{x_Q}, \quad x_Q^2 = \frac{2}{m+1}.
\]

The bifurcation problem is related to the linear equation \( Lu = 0 \), where

\[
Lu = \frac{m + 1}{2} \frac{d^2}{dx^2} \left( x^{\frac{2m}{m+1}} u(x) \right) - \frac{m + 1}{2} x \frac{du}{dx} + u(x) = 0, \quad x \in (0, \infty).
\]

The boundary conditions for admissible self-similar solutions are

\[
u(x) \sim x^{\frac{2}{m+1}} \quad \text{as} \quad x \to \infty
\]

At the bifurcation point \( m = m_k \), the self-similar solutions must satisfy the other boundary condition

\[
u(x) \sim x^{\frac{1-m}{1+m}} \quad \text{as} \quad x \to 0,
\]

related to the derivative of \( x^2/(m+1) \) in \( x \).
Analysis of Bifurcations

After a coordinate transformation, the homogeneous equation $Lu = 0$ becomes the Kummer’s differential equation (1837),

$$z \frac{d^2w}{dz^2} + (b - z) \frac{dw}{dz} + aw(z) = 0, \quad z \in (0, \infty),$$

where

$$a := -\frac{m + 1}{2}, \quad b := \frac{m + 3}{2}.$$

The only solution with the correct boundary condition at infinity was characterized by Tricomi (1947). The set of bifurcating solutions at $m = m_k, k \in \mathbb{N}$ with the correct boundary condition at zero is found by truncating the power series into a polynomial.

These solutions of the linear equation $Lu = 0$ are incorporated in the construction of self-similar solutions with a two-scale (inner and outer) matched asymptotic expansion.
Numerical confirmations

Bifurcation at $m = 5$ and $n = 0$:

$$A_- = -\frac{40}{9}(x_0 - x_Q) + \cdots$$

and

$$5 - m = \frac{27\sqrt{3}}{4}A_- + \cdots$$

Figure: Left: The variation of the parameter $A_-$ with $5 - m$, and; right: the variation of $x_0 - x_Q$ with $A_-$ local to $m = 5$. The black dots are numerics, the blue lines are asymptotics.
Conclusion

- For every \( m > 0, \ n < 1 \) and \( m + n > 1 \) a pair of solutions \( H_+ \) and \( H_- \) can be constructed numerically and then converted to \( h(x, t) \)
  - Solutions with \( A_\pm > 0 \) correspond to reversing interfaces
  - Solutions with \( A_\pm < 0 \) correspond to anti-reversing interfaces

- The behaviour of the self-similar solution at zero and infinity was justified by dynamical system theory

- Bifurcations of self-similar solutions are predicted from analysis of the classical differential equations

- Relevance of the self-similar solutions for the slow diffusion equations is confirmed numerically and from weak solutions.

References

