# Existence and stability of Klein–Gordon breathers in the small-amplitude limit

Dmitry Pelinovsky
Department of Mathematics, McMaster University, Ontario, Canada

in collaboration with
J. Cuevas (Seville, Spain), P.G. Kevrekidis (Amherst, USA),
T. Penati (Milan, Italy), and S. Paleari (Milan, Italy)

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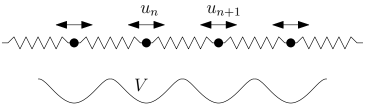
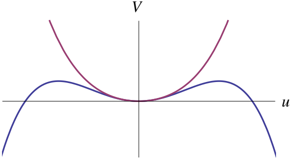
# Klein-Gordon lattice

Klein-Gordon (KG) lattice models a chain of coupled anharmonic oscillators with nearest-neighbour interactions

$$\frac{d^2 u_n}{dt^2} + V'(u_n) = \epsilon(u_{n+1} - 2u_n + u_{n-1}),$$

where $\{u_n(t)\}_{n\in\mathbb{Z}} : \mathbb{R} \to \mathbb{R}^{\mathbb{Z}}$, $\epsilon$ is the coupling constant, and $V : \mathbb{R} \to \mathbb{R}$ is the on-site potential such that $V(0) = V'(0) = 0$ and $V''(0) = 1$, e.g.,




Applications:

- dislocations in crystals (e.g. Frenkel & Kontorova '1938)
- oscillations in biological molecules (e.g. Peyrard & Bishop '1989)

# The anti-continuum limit

In the anti-continuum limit ($\epsilon = 0$), each oscillator is governed by

$$\ddot{\varphi} + V'(\varphi) = 0, \quad \Rightarrow \quad \frac{1}{2}\dot{\varphi}^2 + V(\varphi) = E,$$

where $\varphi \in H^2_{per}(0, T)$.

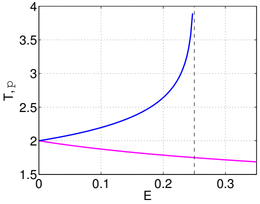



Figure: Period vs. energy in hard (magenta) and soft (blue) potential $V(u) = \frac{1}{2}u^2 \pm \frac{1}{4}u^4$.

The period of the oscillator is

$$T(E) = \sqrt{2} \int_{a_-(E)}^{a_+(E)} \frac{dx}{\sqrt{E - V(x)}},$$

where turning points $a_-(E) < 0 < a_+(E)$ are roots of $V(a) = E$.

If $V(-x) = V(x)$, then $a_-(E) = -a_+(E)$.

# Multi-breathers at the anti-continuum limit

Breathers are spatially localized time-periodic solutions. Multi-breathers are constructed by parameter continuation in $\epsilon$ from the limiting configuration:

$$u^{(0)}(t) = \sum_{k \in S} \sigma_k \varphi(t) e_k \quad \in \quad H^2_{per}((0,T); l^2(\mathbb{Z})),$$

where $S \subset \mathbb{Z}$ is a finite set of excited sites and $e_k$ is the unit vector in $l^2(\mathbb{Z})$ at the node $k$. The oscillators are in-phase if $\sigma_k = +1$ and anti-phase if $\sigma_k = -1$.

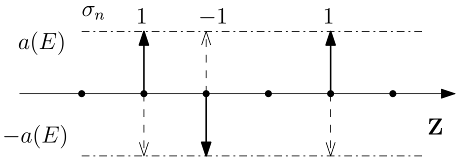



Figure: An example of a multi-site discrete breather at $\epsilon = 0$.

# Existence of multi-breathers

**Theorem (MacKay & Aubry '1994)**

*Fix the period $T \neq 2\pi n$, $n \in \mathbb{N}$ and the $T$-periodic solution $\varphi \in H^2_{per}(0,T)$ of the anharmonic oscillator equation for $T'(E) \neq 0$. There exist $\epsilon_0 > 0$ and $C > 0$ such that $\forall \epsilon \in (-\epsilon_0, \epsilon_0)$ there exists a solution $\mathrm{u}^{(\epsilon)} \in l^2(\mathbb{Z}, H^2_{per}(0,T))$ of the Klein–Gordon lattice satisfying*

$$\left\| \mathrm{u}^{(\epsilon)} - \mathrm{u}^{(0)} \right\|_{l^2(\mathbb{Z}, H^2(0,T))} \leq C\epsilon.$$

The proof is based on the Implicit Function Theorem and uses invertibility of the linearization operators

$$\begin{aligned} \mathcal{L}_0 &= \partial_t^2 + 1 : \quad H^2_{per}(0,T) \to L^2_{per}(0,T) \quad \text{if } T \neq 2\pi n, \\ \mathcal{L}_e &= \partial_t^2 + V''(\varphi(t)) : H^2_{per,even}(0,T) \to L^2_{per,even}(0,T) \quad \text{if } T'(E) \neq 0. \end{aligned}$$

# Stability of multi-breathers

- Archilla, Cuevas, Sánchez-Rey, Alvarez '2003
- Koukouloyannis, Kevrekidis '2009
- Pelinovsky, Sakovich '2012
- Yoshimura '2012

Short summary of stability results near the anti-continuum limit:

- Single-site breather - spectrally stable
- Two-site breathers at two adjacent sites:
  - spectrally unstable if in-phase (soft) or anti-phase (hard)
  - spectrally stable if anti-phase (soft) or in-phase (hard)

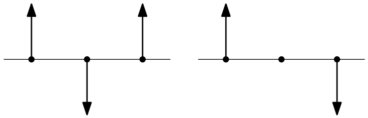
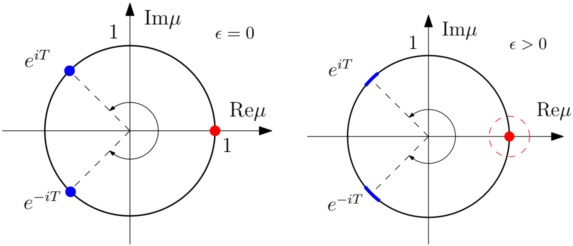


Figure: Stable configuration in soft potential: $T'(E) > 0$.

# Spectral stability via Floquet multipliers

For $\epsilon > 0$, Floquet multipliers split as follows:




Single-site breathers have a double Floquet multiplier at $\mu = 1$ if $\epsilon = 0$ and remain stable for small $\epsilon \neq 0$.

Two-site breathers have one split pair of Floquet multipliers:

- the pair is on the unit circle if the breathers are spectrally stable
- the pair is on the real line if the breathers are unstable

# Different limit: reduction to the discrete NLS equation

Consider the power model of the Klein–Gordon lattice:

$$\frac{d^2 u_n}{dt^2} + u_n + u_n^{1+2k} = \epsilon(u_{n+1} - 2u_n + u_{n-1}),$$

where the onsite (hard) potential $V(u)$ is symmetric and $k \in \mathbb{N}$.

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where the onsite (hard) potential $V(u)$ is symmetric and $k \in \mathbb{N}$.

Using the asymptotic multi-scale expansion in the small-amplitude limit

$$u_n(t) = \epsilon^{\frac{1}{2k}} \left[ a_n(\epsilon t) e^{it} + \bar{a}_n(\epsilon t) e^{-it} \right] + \text{smaller errors},$$

yields formally the discrete NLS equation at the order $\mathcal{O}(\epsilon^{1+\frac{1}{2k}})$

$$2i \frac{da_n}{d\tau} + \gamma_k |a_n|^{2k} a_n = a_{n+1} - 2a_n + a_{n-1},$$

where $\tau = \epsilon t$ and $\gamma_k = \frac{(2k+1)!}{k!(k+1)!}$.

Discrete Klein–Gordon breathers correspond to the discrete NLS solitons.

# Justification of the dNLS approximation

**Theorem (Pelinovsky–Penati–Paleari, 2016)**

*For every $\tau_0 > 0$, there are positive constants $\epsilon_0$ and $C_0$ such that for every $\epsilon \in (0, \epsilon_0)$ and for every initial data*

$$\|\mathrm{u}(0) - \epsilon^{\frac{1}{2k}} \mathrm{U}(0)\|_{l^2} \leq \epsilon^{1+\frac{1}{2k}},$$

*the solution of the dKG equation satisfies for every $t \in [-\tau_0 \epsilon^{-1}, \tau_0 \epsilon^{-1}]$,*

$$\|\mathrm{u}(t) - \epsilon^{\frac{1}{2k}} \mathrm{U}(t)\|_{l^2} \leq C_0 \epsilon^{1+\frac{1}{2k}},$$

*where $U_n(t) := a_n(\epsilon t) e^{it} + \bar{a}_n(\epsilon t) e^{-it}$.*

**Remark:** The constant $C_0$ may grow exponentially in $\tau_0$.

# Steps in the proof of justification

1. Using decomposition $\mathrm{u} = \epsilon^{\frac{1}{2k}} [\mathrm{U} + \mathrm{y}]$ yields

$$\ddot{y}_n + y_n + \epsilon \left[ (2k+1) U_n^{2k} y_n + N(y_n) \right] + \mathrm{Res}_n = \epsilon (\Delta y)_n,$$

where $N(y_n) = \mathcal{O}(y_n^2)$ and

$$\mathrm{Res}_n = \epsilon^2 \underbrace{\left( \ddot{a}_n e^{it} + \ddot{\bar{a}}_n e^{-it} \right)}_{\textit{due to second order}} + \epsilon \underbrace{\left[ a_n^{2k+1} e^{i(2k+1)t} + \dots + \bar{a}_n^{2k+1} e^{-i(2k+1)t} \right]}_{\textit{due to nonlinearity}; \ \textit{no resonances at } e^{\pm it}}.$$

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2. Energy for the approximation error

$$E(t) := \frac{1}{2}\sum_{n\in\mathbb{Z}} \dot{y}_n^2 + y_n^2 + \epsilon(2k+1)U_n^2 y_n^2 + \epsilon(y_{n+1} - y_n)^2,$$

such that $\|\dot{\mathrm{y}}\|_{\ell^2}^2 + \|\mathrm{y}\|_{\ell^2}^2 \leq 4E(t)$ and

$$\frac{dE(t)}{dt} = -\langle \mathrm{y}, \mathrm{Res} + (2k+1)\epsilon \mathrm{U}\dot{\mathrm{U}}\,\mathrm{y} + \epsilon N(\mathrm{y})\rangle.$$

# Steps in the proof of justification

3. For every $a_0 \in \ell^2$, there exists a unique global solution $a(t) \in C(\mathbb{R}, \ell^2)$ of the discrete NLS equation, where $\ell^2$ forms a Banach algebra with respect to multiplication.

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4. With near-identity transformation, non-resonant terms in $\epsilon$ can be removed by $X = X^{(0)} + \epsilon X^{(1)}$ such that

$$\mathrm{Res}_n = \epsilon^2 \underbrace{(\ddot{a}_n e^{it} + \ddot{\bar{a}}_n e^{-it})}_{due\ to\ second\ order} + \epsilon^2 \underbrace{\cdots}_{after\ near-identity\ transformation}.$$

such that $\|\mathrm{Res}\|_{\ell^2} \leq C\epsilon^2$.

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such that $\|\mathrm{Res}\|_{\ell^2} \leq C\epsilon^2$.

5. Gronwall's inequality is used in the energy estimates for $E(t) = Q(t)^2$:

$$\frac{dE}{dt} \leq C\epsilon^2 E^{1/2} + C\epsilon E \quad \Rightarrow \quad \frac{dQ}{dt} \leq C\epsilon^2 + C\epsilon Q$$

such that $|Q(t)| \leq C\epsilon e^{C\tau_0}$ for $t \in [0, \tau_0\epsilon^{-1}]$.

# 1. Justification of the reduction on the extended time scale

**Theorem (Pelinovsky–Penati–Paleari, 2016)**

*For every $\alpha \in (0,1)$, there are positive constants $\epsilon_0$ and $C_0$ such that for every $\epsilon \in (0,\epsilon_0)$ and for every initial data*

$$\|\mathrm{u}(0) - \epsilon^{\frac{1}{2k}} \mathrm{U}(0)\|_{l^2} \leq \epsilon^{1+\frac{1}{2k}},$$

*the solution of the dKG equation satisfies for every $t \in [-\alpha|\log(\epsilon)|\epsilon^{-1}, \alpha|\log(\epsilon)|\epsilon^{-1}]$,*

$$\|\mathrm{u}(t) - \epsilon^{\frac{1}{2k}} \mathrm{U}(t)\|_{l^2} \leq C_0 \epsilon^{1-\alpha+\frac{1}{2k}},$$

*where $U_n(t) := a_n(\epsilon t) e^{it} + \bar{a}_n(\epsilon t) e^{-it}$.*

**Remark:** Global well-posedness of the DNLS equation is used since the solution of DNLS is defined in $\tau = \epsilon t$ on $[-\alpha|\log(\epsilon)|, \alpha|\log(\epsilon)|]$.

# 2. Existence of breathers from existence of solitons

The discrete NLS equation

$$2i\frac{da_n}{d\tau} + \gamma_k |a_n|^{2k} a_n = a_{n+1} - 2a_n + a_{n-1}, \quad \gamma_k > 0,$$

has standing wave solutions $a_n(\tau) = A_n e^{-\frac{i}{2}\Omega\tau}$ (bright solitons), e.g. for $\Omega < -4$. These solitons can be characterized as minimizers of the constrained variational problem (M. Weinstein, 1999)

$$\inf_{A \in \ell^2} \left\{ E(A) : \quad P(A) = P_0 > 0 \right\},$$

where $P(A) = \|A\|_{\ell^2}^2$ is conserved mass and $E(A)$ is conserved energy of the discrete NLS equation.

Does there exist a discrete breather (spatially localized, time-periodic solution of dKG) near each soliton of dNLS for which the Jacobian operator is invertible?

$$\mathcal{J} := \Omega + (2k+1)\gamma_k |A|^{2k} - \Delta.$$

## Theorem (Pelinovsky–Penati–Paleari, 2020)

*Assume the existence of* $A \in \ell^2$ *in dNLS equation for some* $\Omega < -4d$ *such that* $\mathcal{J}$ *is invertible. There are positive constants* $\epsilon_0$ *and* $C_0$ *such that the dKG equation for* $\epsilon \in (0, \epsilon_0)$ *admits the unique breather solution* $u \in H^2_{\rm per}((0,T), \ell^2(\mathbb{Z}))$ *with breather frequency* $\omega = \frac{2\pi}{T}$ *satisfying*

$$\|\mathrm{u}(t) - \epsilon^{\frac{1}{2k}} \mathrm{U}(t)\|_{l^2} \leq C_0 \epsilon^{1+\frac{1}{2k}}, \quad |\omega - 1 + \frac{\epsilon\Omega}{2}| \leq C_0 \epsilon^2,$$

*where* $U_n(t) := A_n(\epsilon t) e^{i(1-\frac{\epsilon\Omega}{2})t} + \bar{A}_n(\epsilon t) e^{-i(1-\frac{\epsilon\Omega}{2})t}$.

**Remark:** The proof is based on the Fourier series decomposition

$$\mathrm{u}(t) = \sum_{j\in\mathbb{Z}} \mathrm{A}^{(m)} e^{im\omega t}$$

and Lyapunov–Schmidt reduction in $H^2_{\rm per}((0,T), \ell^2(\mathbb{Z}))$ with

$$\|A^{(0)}\|_{\ell^2} + \|\|A^{(1)} - \epsilon^{\frac{1}{2k}} A\|_{\ell^2} + \|A^{m\geq 2}\|_{\ell^2} \leq C_0 \epsilon^{1+\frac{1}{2k}}.$$

# 3. Stability of breathers from stability of solitons

The KG lattice

$$\frac{d^2 u_n}{dt^2} + V'(u_n) = \epsilon(u_{n+1} - 2u_n + u_{n-1})$$

has the conserved energy

$$H(u) = \sum_{n\in\mathbb{Z}} \frac{1}{2}\left(\frac{du_n}{dt}\right)^2 + V(u_n) + \frac{1}{2}\epsilon(u_{n+1} - u_n)^2.$$

Breathers are not characterized variationally from the energy function $H$. Nevetherless, the energy function gives the criterion of their stability. [Kevrekidis–Cuevas–Pelinovsky, Phys. Rev. Lett. **117** (2016), 094101]

Let $u \in H^2_{\rm per}((0,T),\ell^2(\mathbb{Z}))$ be the breather solution and compute $\mathcal{H}(\omega) := H(u)$, where $\omega = 2\pi/T$ is breather frequency. Breathers with increasing (decreasing) $\mathcal{H}(\omega)$ are unstable in soft (hard) potentials $V(u)$.

# A simple argument of why critical points of $\mathcal{H}(\omega)$ matter

Normalized breather profile $U(\tau) \in H^2_{\rm per}((0,2\pi),\ell^2(\mathbb{Z}))$ satisfies

$$\omega^2 U_n''(\tau) + V'(U_n(\tau)) = \epsilon(\Delta U)_n(\tau), \quad n \in \mathbb{Z}.$$

Linearized equations for small perturbations $w \in C^2(\mathbb{R},\ell^2(\mathbb{Z}))$ are given by

$$\ddot{w}_n + V''(U_n)w_n = \epsilon(\Delta w)_n, \quad n \in \mathbb{Z}. \tag{1}$$

With Floquet theory,

$$w(t) = W(\tau)e^{\lambda t}, \quad \tau = \omega t, \quad W(\tau + 2\pi) = W(\tau),$$

the spectral stability problem is formulated by

$$(LW)(\tau) = 2\lambda\omega W'(\tau) + \lambda^2 W(\tau),$$

where $L = \epsilon\Delta - V''(U(\tau)) - \omega^2\partial_\tau^2$ acts on $H^2_{\rm per}((0,2\pi),\ell^2(\mathbb{Z}))$.

# A simple argument of why critical points of $\mathcal{H}(\omega)$ matter

$\lambda = 0$ is at least a double eigenvalue because of the translational invariance:

$$LU'(\tau) = 0, \quad L\partial_\omega U(\tau) = 2\omega U''(\tau).$$

**Assumptions:**

- $\lambda = 0$ is bounded away from the spectral bands of $L$.
- $\mathrm{Ker}(L)$ is exactly one-dimensional with the eigenvector $U'(\tau)$.
- The mapping $\omega \mapsto \mathcal{H}(\omega)$ is $C^1$.

Perturbation expansion in powers of $\lambda$:

$$W(\tau) = U'(\tau) + \lambda \partial_\omega U(\tau) + \lambda^2 Y(\tau) + \mathcal{O}(\lambda^3).$$

yields the inhomogeneous equation for $Y(\tau) \in H^2_{\rm per}((0,2\pi),\ell^2(\mathbb{Z}))$:

$$(LY)(\tau) = 2\omega \partial_\omega U'(\tau) + U'(\tau).$$

The Fredholm condition yields

$$0 = \int_0^{2\pi} \sum_{n\in\mathbb{Z}} U_n'(\tau) \left[2\omega\partial_\omega U_n'(\tau) + U_n'(\tau)\right] d\tau = T\mathcal{H}'(\omega).$$

# Energy stability criterion for discrete breathers

When $\mathcal{H}'(\omega) = 0$, $\lambda = 0$ is a quadruple eigenvalue.

Extending the perturbation extensions in powers of $\lambda$:

$$W(\tau) = U'(\tau) + \lambda \partial_\omega U(\tau) + \lambda^2 Y(\tau) + \lambda^3 Z(\tau) + \mathcal{O}(\lambda^4)$$

and using Fredholm conditions yields the dispersion relation

$$0 = \lambda^2 T \mathcal{H}'(\omega) + \lambda^4 M(\omega) + \mathcal{O}(\lambda^6),$$

where $M(\omega)$ is computed in terms of $U$ and $Y$.

The sign of $M(\omega)$ is not generally defined...

However, in the dNLS approximation limit, one can show that
$M(\omega) > 0$ for hard potentials [breathers are unstable for $\mathcal{H}'(\omega) < 0$];
$M(\omega) < 0$ for soft potentials [breathers are unstable for $\mathcal{H}'(\omega) > 0$].

# Energy stability criterion in the dNLS approximation

For the power nonlinearity in the dKG equation,

$$\frac{d^2 u_n}{dt^2} + u_n + u_n^{1+2k} = \epsilon(u_{n+1} - 2u_n + u_{n-1}),$$

and the small-amplitude approximation of the dNLS equation,

$$U_n(\tau) = \epsilon^{\frac{1}{2k}} \left[A_n e^{it} + \bar{A}_n e^{-it}\right] + \mathcal{O}(\epsilon^{1+\frac{1}{2k}}),$$

with the correspondence $\omega = 1 - \frac{\epsilon\Omega}{2} + \mathcal{O}(\epsilon^2)$, it follows that

$$\mathcal{H}(\omega) = 2\epsilon^{\frac{1}{k}} \|A\|_{\ell^2}^2 + \mathcal{O}(\epsilon^{1+\frac{1}{k}}).$$

The energy stability criterion becomes the Vakhitov–Kolokolov slope condition:

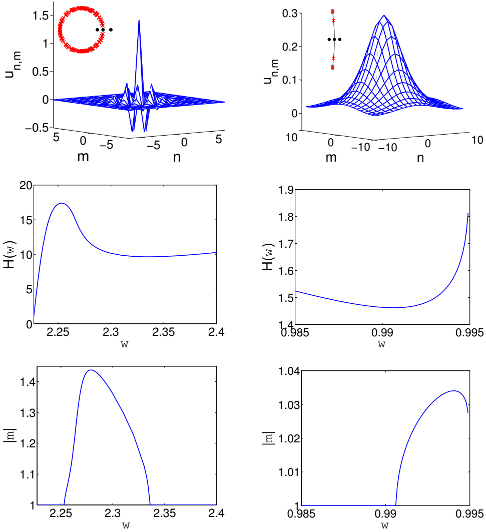
$$\mathcal{H}'(\omega) < 0 \quad \Leftrightarrow \quad \frac{d}{d\Omega}\|A\|_{\ell^2}^2 > 0$$

is the instability criterion for the hard potentials.

# Numerical illustration: 2D KG lattice.

Left - hard $\phi^4$ potential with $\epsilon = 0.5$.

Right - soft Morse potential with $\epsilon = 0.2$.



# 4. Instability of two-site breathers

Consider the discrete KG equation

$$\frac{d^2 u_n}{dt^2} + V'(u_n) = \varepsilon(u_{n+1} - 2u_n + u_{n-1}), \quad n \in \mathbb{Z},$$

where $V$ is smooth and $V = \frac{1}{2}u^2 + \mathcal{O}(u^3)$.

Assumptions:

- The double eigenvalue $\lambda = 0$ is isolated from the spectral bands.
- There exists a pair of eigenvalues at $\lambda = \pm i\Omega$ isolated from the spectral bands.
- The double eigenvalue $\lambda = \pm 2i\Omega$ belongs to the spectral bands.

Dynamics of the dNLS equation suggests the following conclusion:

If Krein signature of eigenvalues at $\lambda = \pm i\Omega$ is opposite to that of the spectral bands, the breather is spectrally stable and nonlinearly unstable.

[Cuevas–Kevrekidis–Pelinovsky, Stud. Appl. Math. **137** (2016), 214]

# Krein quantity

Linearized equations for small perturbations are given by

$$\ddot{w}_n + V''(u_n) w_n = \epsilon (\Delta w)_n, \quad n \in \mathbb{Z}. \tag{2}$$

The symplectic structure is given by

$$\frac{dw_n}{dt} = \frac{\partial H}{\partial p_n}, \quad \frac{dp_n}{dt} = -\frac{\partial H}{\partial w_n}, \quad n \in \mathbb{Z}$$

The Krein quantity $K$ is real and constant in time $t$:

$$K = i \sum_{n \in \mathbb{Z}} (\bar{p}_n w_n - p_n \bar{w}_n) = 2\Omega \sum_{n \in \mathbb{Z}} |W_n|^2 + i \sum_{n \in \mathbb{Z}} \left( \dot{\bar{W}}_n W_n - \dot{W}_n \bar{W}_n \right),$$

where

$$w(t) = W(t) e^{i\Omega t}, \quad W(t + T) = W(\tau),$$

is the Floquet mode.

# Krein quantity for two-site breathers

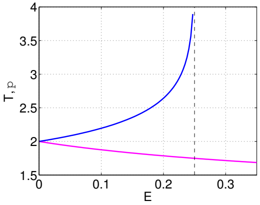
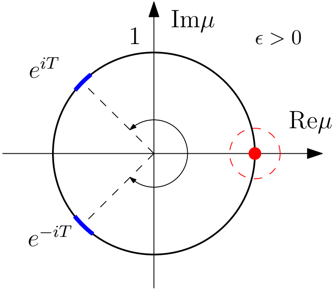


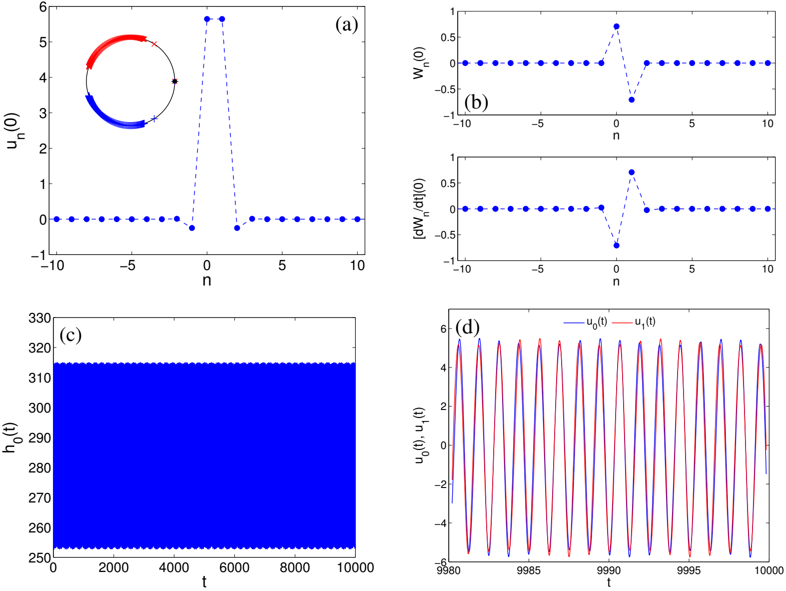

Figure: Period vs. energy in hard (magenta) and soft (blue) potential $V(u) = \frac{1}{2}u^2 \pm \frac{1}{4}u^4$.



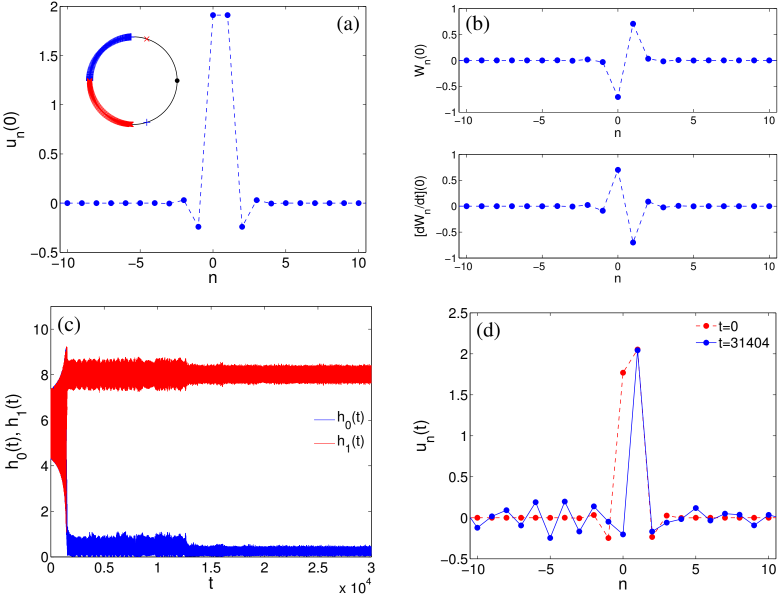

For the hard potential with $T'(E) < 0$:

- $0 < T < \pi$: the Krein signatures of the internal mode and the wave spectrum in the upper semi-circle coincide;
- $\pi \leq T < 2\pi$: the Krein signatures of the internal mode and the wave spectrum in the upper semi-circle are opposite to each other.

# Hard $\phi^4$ potential $T < \pi$ (stable case)



# Hard $\phi^4$ potential $T > \pi$ (unstable case)



# Conclusions

- Breathers of the discrete Klein–Gordon equation can be characterized in the anti-continuum limit and in the limit of small amplitudes.
- The validity of the discrete NLS equation has been justified to control dynamics of discrete breathers in the discrete KG equation.
- Existence and spectral stability of dKG breathers are handled with the method of Lyapunov–Schmidt reductions from those of dNLS solitons.
- The energy stability criterion and the relevance of Krein signatures are similar between the dKG and dNLS models.