Rogue waves on the periodic background

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Outline of the lecture

1. Definitions and properties of rogue waves
2. Rogue waves in the modified KdV equation
3. Algebraic construction of rogue waves
4. Rogue waves in the focusing NLS equation
5. Further problems on rogue waves

The rogue wave of the cubic NLS equation

The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \psi_{xx} + 2(|\psi|^2 - 1)\psi = 0$$

admits the exact solution

$$\psi(x,t) = 1 - \frac{4(1+4it)}{1+4x^2+16t^2}.$$

It was discovered by H. Peregrine (1983) and was labeled as *the rogue wave*.

Properties of the rogue wave:

- It is related to modulational instability of the constant wave $\psi_0(x,t) = 1$.
- It comes from nowhere: $|\psi(x,t)| \to 1$ as $|x| + |t| \to \infty$.
- It is magnified at the center: $M_0 := |\psi(0,0)| = 3$.

Main question

The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0$$

admits other wave solutions, e.g. the periodic waves

$$\psi_{\rm dn}(x,t) = {\rm dn}(x;k)e^{i(2-k^2)t}, \quad \psi_{\rm cn}(x,t) = k{\rm cn}(x;k)e^{i(2k^2-1)t}$$

or the double-periodic solutions (Akhmediev, 1987):

$$\psi(x,t) = \frac{\sqrt{k(1+k)}{\rm sn}(2t;k) - i{\rm dn}(2t;k){\rm cn}(\sqrt{2}x;\kappa)}{\sqrt{1+k} - \sqrt{k}{\rm cn}(2t;k){\rm cn}(\sqrt{2}x;\kappa)} e^{2ikt}, \quad \kappa = \frac{\sqrt{1-k}}{\sqrt{2}}.$$

where $k \in (0,1)$ is elliptic modulus.

Can we obtain the exact solution on the background ψ_0 such that

$$\inf_{x_0,t_0,\alpha_0\in\mathbb{R}} \sup_{x\in\mathbb{R}} \left|\psi(x,t) - \psi_0(x-x_0,t-t_0)e^{i\alpha_0}\right| \to 0 \quad \text{as} \quad t \to \pm\infty \quad ???$$

This corresponds to the *rogue wave on the background* ψ_0 that *appears from nowhere and disappears without trace*.

Background

- Rogue periodic waves were numerically constructed in (Kedziora–Ankiewicz–Akhmediev, 2014)
- Emergence of rogue waves from *dn*-periodic waves was numerically observed in (Agafontsev–Zakharov, 2016)
- Rogue waves on double-periodic solutions were studied numerically in (Calini–Schober, 2017)
- Magnification factor of quasi-periodic solutions were obtained from analysis of Riemann's Theta functions (Bertola–Tovbis, 2017).
- Rogue waves from a superposition of nearly identical solitons were constructed in (Slunyaev–E.Pelinovsky, 2016)
- Rogue waves were approximated by the finite-gap solutions in (Grinevich–Santini, 2017)

Rogue waves in the modified KdV equation

The modified Korteweg–de Vries (mKdV) equation

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

appears in many physical applications, e.g., in models for internal waves.

The mKdV equation admits two families of *the travelling periodic waves*:

- positive-definite periodic waves **modulationally stable**

$$u_{\rm dn}(x,t) = {\rm dn}(x - ct; k), \quad c = c_{\rm dn}(k) := 2 - k^2,$$

- sign-indefinite periodic waves **modulationally unstable**

$$u_{\rm cn}(x,t) = k{\rm cn}(x - ct; k), \quad c = c_{\rm cn}(k) := 2k^2 - 1,$$

where $k \in (0,1)$ is elliptic modulus.

Bronski–Johnson–Kapitula, 2011 and Deconinck–Nivala, 2011

As $k \to 1$, the periodic waves converge to **the soliton** $u(x,t) = {\rm sech}(x-t)$.
As $k \to 0$, the periodic waves converge to **the small-amplitude waves**.

Rogue waves on the periodic background

The mKdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

is a compatibility condition of the Lax pair $\varphi(x,t) \in \mathbb{C}^2$:

$$\varphi_x = U(\lambda, u)\varphi, \quad \varphi_t = V(\lambda, u)\varphi.$$

Main question: Can we obtain the exact solution on the periodic wave background u_0 s.t.

$$\inf_{x_0, t_0 \in \mathbb{R}} \sup_{x \in \mathbb{R}} |u(x,t) - u_0(x - x_0, t - t_0)| \to 0 \quad \text{as} \quad t \to \pm\infty \quad ???$$

1. For a periodic wave u_0, we construct the periodic eigenfunctions φ for particular eigenvalues λ.
2. For each periodic eigenfunction φ, we construct the second linearly independent non-periodic solution ψ for the same value of λ.
3. Darboux transformation with a non-periodic function ψ, yields the rogue wave u on the periodic background u_0.

Rogue wave on the *cn*-periodic background

For *cn*-periodic waves

$$u_{\rm cn}(x,t) = k{\rm cn}(x-ct;k), \quad c = c_{\rm cn}(k) := 2k^2 - 1,$$

the magnification factor is

$$M_{\rm dn}(k) = 3, \quad k \in [0,1].$$

The new solution is a rogue wave created because of the modulational instability of the *cn*-periodic wave.

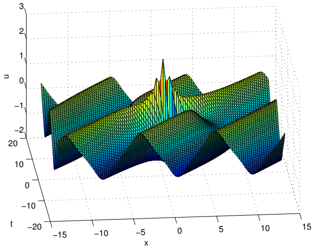

Figure: The rogue *cn*-periodic wave of the mKdV for $k = 0.95$.

Rogue wave on the *dn*-periodic background

For *dn*-periodic waves

$$u_{\rm dn}(x,t) = {\rm dn}(x - ct; k), \quad c = c_{\rm dn}(k) := 2 - k^2,$$

the magnification factor is

$$M_{\rm dn}(k) = 2 + \sqrt{1 - k^2}, \quad k \in [0,1].$$

The new solution is a superposition of the (modulationally stable) *dn*-periodic wave and a travelling algebraic soliton.

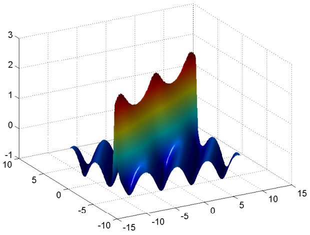

Figure: Algebraic soliton on the *dn*-periodic wave for $k = 0.95$.

Algebraic method - Step 1

1. For a periodic wave u, we compute the periodic eigenfunctions φ for particular eigenvalues λ.

The AKNS spectral problem for $\varphi(x,t) \in \mathbb{C}^2$:

$$\varphi_x = U(\lambda, u)\varphi, \quad U(\lambda, u) := \left(\begin{array}{cc} \lambda & u \\ -u & -\lambda \end{array} \right),$$

where $u(x,t) \in \mathbb{R}$ is any solution of the mKdV.

We use an algebraic technique based on the “nonlinearization” of Lax pair: Cao–Geng, 1990; Cao–Wu–Geng, 1999; Zhou, 2009; Chen, 2012.

Relations between the potential $u(x,t)$ and the squared eigenfunctions $\varphi(x,t)$ for some eigenvalues λ have been known since the original paper of Gardner–Green–Kruskal–Miura (1974).

Nonlinear Hamiltonian system from Lax operator

Fix $\lambda = \lambda_1 \in \mathbb{C}$ with an eigenfunction $\varphi = (\varphi_1, \varphi_2) \in \mathbb{C}^2$. Set

$$u = \varphi_1^2 + \varphi_2^2 \in \mathbb{R}$$

and consider the Hamiltonian system

$$\left\{ \begin{array}{l} \frac{d\varphi_1}{dx} = \lambda_1 \varphi_1 + (\varphi_1^2 + \varphi_2^2)\varphi_2 = \frac{\partial H}{\partial \varphi_2}, \\ \frac{d\varphi_2}{dx} = -\lambda_1 \varphi_2 - (\varphi_1^2 + \varphi_2^2)\varphi_1, = -\frac{\partial H}{\partial \varphi_1} \end{array} \right.$$

related to the Hamiltonian function

$$H(\varphi_1, \varphi_2) = \frac{1}{4}(\varphi_1^2 + \varphi_2^2)^2 + \lambda_1 \varphi_1 \varphi_2.$$

Besides $u = \varphi_1^2 + \varphi_2^2$, we also have constraints

$$\frac{du}{dx} = 2\lambda_1(\varphi_1^2 - \varphi_2^2)$$

and

$$E_0 - u^2 = 4\lambda_1 \varphi_1 \varphi_2,$$

where $E_0 = 4H(\varphi_1, \varphi_2)$ is conserved.

Nonlinear Hamiltonian system from Lax operator

Fix $\lambda = \lambda_1 \in \mathbb{C}$ with an eigenfunction $\varphi = (\varphi_1, \varphi_2) \in \mathbb{C}^2$. Set

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Besides $u = \varphi_1^2 + \varphi_2^2$, we also have constraints

$$\frac{du}{dx} = 2\lambda_1(\varphi_1^2 - \varphi_2^2)$$

and

$$E_0 - u^2 = 4\lambda_1 \varphi_1 \varphi_2,$$

where $E_0 = 4H(\varphi_1, \varphi_2)$ is conserved.

Integrability of the Hamiltonian system

The Hamiltonian system is a compatibility condition of the Lax equation

$$\frac{d}{dx} W(\lambda) = Q(\lambda) W(\lambda) - W(\lambda) Q(\lambda),$$

where

$$Q(\lambda) = \left(\begin{array}{cc} \lambda & u \\ -u & -\lambda \end{array} \right), \quad W(\lambda) = \left(\begin{array}{cc} W_{11}(\lambda) & W_{12}(\lambda) \\ W_{12}(-\lambda) & -W_{11}(-\lambda) \end{array} \right),$$

with

$$\begin{aligned} W_{11}(\lambda) &= 1 - \frac{\varphi_1 \varphi_2}{\lambda - \lambda_1} + \frac{\varphi_1 \varphi_2}{\lambda + \lambda_1} = 1 - \frac{E_0 - u^2}{2(\lambda^2 - \lambda_1^2)}, \\ W_{12}(\lambda) &= \frac{\varphi_1^2}{\lambda - \lambda_1} + \frac{\varphi_2^2}{\lambda + \lambda_1} = \frac{2\lambda u + u_x}{2(\lambda^2 - \lambda_1^2)}, \end{aligned}$$

Differential relations on u

The $(1,2)$-element of the Lax equation is equivalent to

$$\frac{d^2u}{dx^2} + 2u^3 = cu, \quad c = 2E_0 + 4\lambda_1^2.$$

The determinant equation

$$\det[W(\lambda)] = -[W_{11}(\lambda)]^2 - W_{12}(\lambda)W_{21}(\lambda) = -1 + \frac{E_0}{\lambda^2 - \lambda_1^2}$$

yields

$$\left(\frac{du}{dx}\right)^2 + u^4 - cu^2 = d, \quad d = -E_0^2.$$

The differential equations on u are satisfied if u is the periodic wave of the mKdV equation. Moreover, if $u(x-ct)$, then $\varphi(x-ct)$ is compatible with the time evolution of the Lax pair.

dn-periodic waves

The connection formulas:

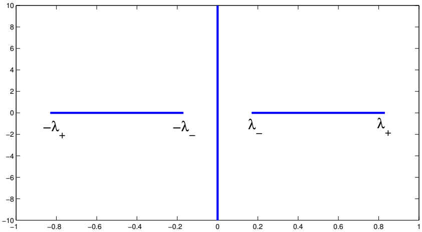
$$c = 4\lambda_1^2 + 2E_0, \quad d = -E_0^2.$$

For *dn*-periodic waves

$$u_{\rm dn}(x,t) = {\rm dn}(x - ct; k), \quad c = c_{\rm dn}(k) := 2 - k^2,$$

we have $d = k^2 - 1 \leq 0$. Hence $E_0 = \pm\sqrt{1-k^2}$ and

$$\lambda_1^2 = \frac{1}{4}\left[2 - k^2 \mp 2\sqrt{1-k^2}\right].$$



cn-periodic waves

The connection formulas:

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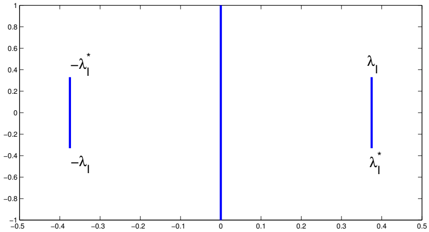
For *cn*-periodic waves

$$u_{\rm cn}(x,t) = k{\rm cn}(x - ct; k), \quad c = c_{\rm cn}(k) := 2k^2 - 1,$$

we have $d = k^2(1-k^2) \geq 0$. Hence $E_0 = \pm ik\sqrt{1-k^2}$ and

$$\lambda_1^2 = \frac{1}{4}\left[2k^2 - 1 \mp 2ik\sqrt{1-k^2}\right]$$



Algebraic method - Step 2

2. For each periodic eigenfunction φ, we construct the second linearly independent non-periodic solution ψ for the same value of λ.

For $\lambda = \lambda_1 \in \mathbb{C}$, we have one periodic solution $\varphi = (\varphi_1, \varphi_2)$ of

$$\varphi_x = U(\lambda, u)\varphi, \quad U(\lambda, u) := \begin{pmatrix} \lambda & u \\ -u & -\lambda \end{pmatrix},$$

where $u \in \mathbb{R}$ is any solution of the mKdV.

Let us define the second solution $\psi = (\psi_1, \psi_2)$ by

$$\psi_1 = \frac{\theta - 1}{\varphi_2}, \quad \psi_2 = \frac{\theta + 1}{\varphi_1},$$

such that $\varphi_1\psi_2 - \varphi_2\psi_1 = 2$ (Wronskian is constant). Then, θ satisfies the first-order reduction

$$\frac{d\theta}{dx} = u\theta\frac{\varphi_2^2 - \varphi_1^2}{\varphi_1\varphi_2} + u\frac{\varphi_1^2 + \varphi_2^2}{\varphi_1\varphi_2}.$$

Non-periodic solutions

Because $u = \varphi_1^2 + \varphi_2^2$, $u_x = 2\lambda_1(\varphi_1^2 - \varphi_2^2)$, and $E_0 - u^2 = 4\lambda_1\varphi_1\varphi_2$, we can rewrite the ODE for θ as

$$\frac{d\theta}{dx} = \theta \frac{2uu'}{u^2 - E_0} - \frac{4\lambda_1 u^2}{u^2 - E_0},$$

where $u^2 - E_0 \neq 0$ is assumed. Integration yields

$$\theta(x) = -4\lambda_1(u(x)^2 - E_0)\int_0^x \frac{u(y)^2}{(u(y)^2 - E_0)^2} dy.$$

Moreover, if $u(x - ct)$ and $\varphi(x - ct)$, then the time evolution yields

$$\theta(x,t) = -4\lambda_1(u(x-ct)^2 - E_0)\left[\int_0^{x-ct} \frac{u(y)^2}{(u(y)^2 - E_0)^2} dy - t\right].$$

up to translation in t.

Algebraic method - Step 3

3. Darboux transformation with the non-periodic function ψ yields a rogue wave u on the periodic background u_0.

One-fold Darboux transformation:

$$u = u_0 + \frac{4\lambda_1 pq}{p^2 + q^2},$$

where u_0 and u are solutions of the mKdV and (p, q) is a nonzero solution of the Lax pair with $\lambda = \lambda_1$ and u_0.

Two-fold Darboux transformation:

$$u = u_0 + \frac{4(\lambda_1^2 - \lambda_2^2)\left[\lambda_1 p_1 q_1 (p_2^2 + q_2^2) - \lambda_2 p_2 q_2 (p_1^2 + q_1^2)\right]}{(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2) - 2\lambda_1\lambda_2\left[4 p_1 q_1 p_2 q_2 + (p_1^2 - q_1^2)(p_2^2 - q_2^2)\right]}$$

where (p_1, q_1) and (p_2, q_2) are nonzero solutions of the Lax pair with λ_1 and λ_2 such that $\lambda_1 \neq \pm\lambda_2$.

Algebraic soliton on the *dn*-periodic wave

The *dn*-periodic wave is $u_0 = \mathrm{dn}(x - ct; k)$. Using one-fold transformation with periodic eigenfunction (φ_1, φ_2) yields

$$u = u_0 + \frac{4\lambda_1 \varphi_1 \varphi_2}{\varphi_1^2 + \varphi_2^2} = -\frac{\sqrt{1-k^2}}{\mathrm{dn}(x - ct; k)} = -\mathrm{dn}(x - ct + K(k); k),$$

which is a translation of the *dn*-periodic wave.

Using one-fold transformation with non-periodic (ψ_1, ψ_2) yields

$$u = u_0 + \frac{4\lambda_1 \psi_1 \psi_2}{\psi_1^2 + \psi_2^2} = u_0 + \frac{4\lambda_1 \varphi_1 \varphi_2 (\theta^2 - 1)}{(\varphi_1^2 + \varphi_2^2)(1 + \theta^2) - 2(\varphi_1^2 - \varphi_2^2)\theta},$$

which is not a translation of the *dn*-periodic wave.

- As $|\theta| \to \infty$ (as $|x| + |t| \to \infty$ almost everywhere):

$$u(x,t) \sim -\frac{\sqrt{1-k^2}}{\mathrm{dn}(x - ct; k)} = -\mathrm{dn}(x - ct + K(k); k).$$

- At $\theta = 0$ (at $(x,t) = (0,0)$), the rogue wave is at the maximum point:

$$u(0,0) = 2 + \sqrt{1-k^2}.$$

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Algebraic soliton on the *dn*-periodic wave

For *dn*-periodic waves

$$u_{\rm dn}(x,t) = {\rm dn}(x - ct; k), \quad c = c_{\rm dn}(k) := 2 - k^2,$$

the magnification factor is

$$M_{\rm dn}(k) = 2 + \sqrt{1 - k^2}, \quad k \in [0,1].$$

The new solution is a superposition of the (modulationally stable) *dn*-periodic wave and a travelling algebraic soliton.

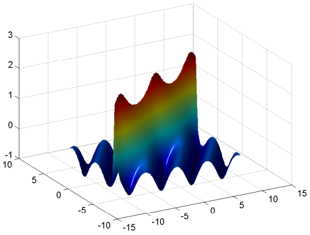

Figure: Algebraic soliton on the *dn*-periodic wave for $k = 0.95$.

Rogue wave on the *cn*-periodic wave

The *cn*-periodic wave is $u_0 = k\,\mathrm{cn}(x - ct; k)$. Since $\lambda_1 \notin \mathbb{R}$, one-fold transformation yields complex solutions of the mKdV. Using two-fold transformation with periodic (φ_1, φ_2) and its conjugate yields

$$u = u_0 + \frac{4k^2(1-k^2)u_0}{(2k^2-1)u_0^2 - u_0^4 - k^2(1-k^2) - (u_0')^2} = -u_0,$$

which is a translation of the *cn*-periodic wave.

Using two-fold transformation with non-periodic (ψ_1, ψ_2) and its conjugate:

$$u = u_0 + \frac{4(\lambda_I^2 - \overline{\lambda}_I^2)\left[\lambda_I \psi_1 \psi_2 (\overline{\psi}_1^2 + \overline{\psi}_2^2) - \overline{\lambda}_I \overline{\psi}_1 \overline{\psi}_2 (\psi_1^2 + \psi_2^2)\right]}{(\lambda_I^2 + \overline{\lambda}_I^2)|\psi_1^2 + \psi_2^2|^2 - 2|\lambda_I|^2\left[4|\psi_1|^2|\psi_2|^2 + |\psi_1^2 - \psi_2^2|^2\right]}.$$

- As $|\theta| \to \infty$ (as $|x| + |t| \to \infty$ everywhere):

$$u(x,t) \sim -u_0(x,t).$$

- At $\theta = 0$ (at $(x,t) = (0,0)$), the rogue wave is at the maximum point:

$$u(0,0) = 3k.$$

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$$u = u_0 + \frac{4k^2(1-k^2)u_0}{(2k^2-1)u_0^2 - u_0^4 - k^2(1-k^2) - (u_0')^2} = -u_0,$$

which is a translation of the *cn*-periodic wave.

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Rogue *cn*-periodic waves

For *cn*-periodic waves

$$u_{\rm cn}(x,t) = k\,{\rm cn}(x-ct;k), \quad c = c_{\rm cn}(k) := 2k^2 - 1,$$

the magnification factor is

$$M_{\rm cn}(k) = 3, \quad k \in [0,1].$$

The new solution is a rogue wave on the background of the modulationally unstable *cn*-periodic wave.

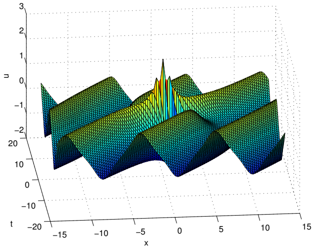


Figure: The rogue *cn*-periodic wave for $k = 0.95$.

Rogue periodic waves in NLS

The NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = 0$$

has a similar Lax pair, e.g.

$$\varphi_x = U\varphi, \qquad U = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix}.$$

The NLS equation admits two families of *the periodic waves*:

- positive-definite periodic waves

$$u_{\mathrm{dn}}(x,t) = \mathrm{dn}(x;k)e^{ict}, \quad c = 2 - k^2,$$

- sign-indefinite periodic waves

$$u_{\mathrm{cn}}(x,t) = k\mathrm{cn}(x;k)e^{ict}, \quad c = 2k^2 - 1,$$

where $k \in (0,1)$ is elliptic modulus.

Both periodic waves are modulationally unstable.

Rogue *dn*-periodic waves

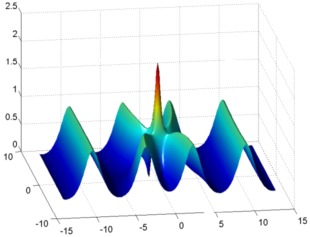
For *dn*-periodic waves

$$u_{\rm dn}(x,t) = {\rm dn}(x;k) e^{ict}, \quad c = 2 - k^2,$$

the magnification factor is still

$$M_{\rm dn}(k) = 2 + \sqrt{1-k^2}, \quad k \in [0,1].$$

The rogue *dn*-periodic wave is a generalization of Peregrine's breather. Exact solutions are computed compared to the numerical approximation in (Kedziora–Ankiewicz–Akhmediev, 2014).



Rogue *cn*-periodic waves

For *cn*-periodic waves

$$u_{\rm cn}(x,t) = k{\rm cn}(x;k)e^{ict}, \quad c = 2k^2 - 1,$$

we employ the one-fold transformation and obtain the magnification factor $M_{\rm cn}(k) = 2$ for every $k \in (0,1)$.

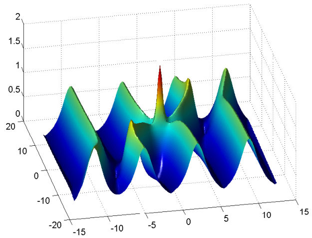


Figure: The rogue *cn*-periodic wave of the NLS for $k = 0.99$.

Rogue *cn*-periodic waves

For *cn*-periodic waves

$$u_{\rm cn}(x,t) = k{\rm cn}(x;k)e^{ict}, \quad c = 2k^2 - 1,$$

we employ the two-fold transformation and obtain the magnification factor $M_{\rm cn}(k) = 3$ for every $k \in (0,1)$.

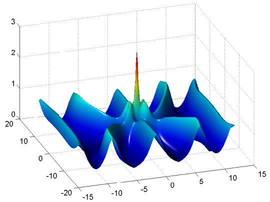


Figure: The rogue *cn*-periodic wave of the NLS for $k = 0.99$.

Summary and open problems

Summary:

- New method is developed for computations of eigenfunctions of the periodic spectral problem associated with the periodic waves.
- New exact solutions are obtained for rogue waves which generalize Peregrine's breathers in the context of *dn* and *cn* periodic waves.

Open problems:

- Extend this approach to the quasi-periodic solutions such as the double-periodic wave patterns.
- Characterize squared eigenfunctions and the location of spectral bands for the quasi-periodic solutions.
- Understand the connections between parameters of the higher-order differential equations and parameters of the algebraic method.

Hamiltonian system of degree two

Fix $\lambda_1, \lambda_2 \in \mathbb{C}$ with eigenfunctions $(p_1, q_1) \in \mathbb{C}^2$ and $(p_2, q_2) \in \mathbb{C}^2$. Set

$$u = p_1^2 + q_1^2 + p_2^2 + q_2^2$$

and consider the Hamiltonian system

$$\left\{ \begin{array}{l} \frac{dp_j}{dx} = \frac{\partial H}{\partial q_j}, \\ \frac{dq_j}{dx} = -\frac{\partial H}{\partial p_j}, \end{array} \right. \quad j = 1, 2,$$

related to the Hamiltonian function

$$H = \frac{1}{4}(p_1^2 + q_1^2 + p_2^2 + q_2^2)^2 + \lambda_1 p_1 q_1 + \lambda_2 p_2 q_2.$$

and higher-order conserved energy

$$\begin{array}{rcl} H_1 & = & 4(\lambda_1^3 p_1 q_1 + \lambda_2^3 p_2 q_2) - 4(\lambda_1 p_1 q_1 + \lambda_2 p_2 q_2)^2 \\ & & +2(p_1^2 + q_1^2 + p_2^2 + q_2^2)(\lambda_1^2(p_1^2 + q_1^2) + \lambda_2^2(p_2^2 + q_2^2)) \\ & & -(\lambda_1(p_1^2 - q_1^2) + \lambda_2(p_2^2 - q_2^2))^2. \end{array}$$

Differential relations on u

Parameters λ_1, λ_2, $E_0 = 4H$, and $E_1 = 4H_1$. By differentiating in x, we obtain

$$\frac{du}{dx} = 2\lambda_1(p_1^2 - q_1^2) + 2\lambda_2(p_2^2 - q_2^2),$$

$$\frac{d^2u}{dx^2} + 2u^3 - cu = -4\lambda_2^2(p_1^2 + q_1^2) - 4\lambda_1^2(p_2^2 + q_2^2),$$

$$\frac{d^3u}{dx^3} + 6u^2\frac{du}{dx} - c\frac{du}{dx} = -8\lambda_1\lambda_2\left[\lambda_2(p_1^2 - q_1^2) + \lambda_1(p_2^2 - q_2^2)\right],$$

and

$$\frac{d^4u}{dx^4} + 10u^2\frac{d^2u}{dx^2} + 10u\left(\frac{du}{dx}\right)^2 + 6u^5 - c\left(\frac{d^2u}{dx^2} + 2u^3\right) = 2du,$$

where

$$c = 2E_0 + 4\lambda_1^2 + 4\lambda_2^2, \quad d = E_1 + E_0^2 - 4E_0(\lambda_1^2 + \lambda_2^2) - 8\lambda_1^2\lambda_2^2.$$

Main question: is to characterize location of (λ_1, λ_2) in terms of solutions u to the fourth-order differential equation.

Very recent progress

For the differential equation

$$\frac{d^3u}{dx^3} + 6u^2\frac{du}{dx} - c\frac{du}{dx} = 0,$$

integrated as

$$\frac{d^2u}{dx^2} + 2u^3 - cu = e$$

and

$$\left(\frac{du}{dx}\right)^2 + u^4 - cu^2 + d = 2eu,$$

there exist only three pairs of eigenvalues $\pm\lambda_1$, $\pm\lambda_2$, and $\pm\lambda_3$ such that

$$\begin{aligned} c &= 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2), \\ d &= \lambda_1^4 + \lambda_2^4 + \lambda_3^4 - 2(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2), \\ e &= -4\lambda_1\lambda_2\lambda_3. \end{aligned}$$

This enables us to characterize all periodic waves of the mKdV equation and related rogue waves on the periodic background.

References

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