Rogue periodic waves for mKdV and NLS equations

Jinbing Chen and Dmitry Pelinovsky

Department of Mathematics, McMaster University, Hamilton, Ontario, Canada
http://dmpeli.math.mcmaster.ca

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The rogue wave of the cubic NLS equation

The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \psi_{xx} + 2(|\psi|^2 - 1)\psi = 0$$

admits the exact solution

$$\psi(x,t) = 1 - \frac{4(1+4it)}{1+4x^2+16t^2}.$$

It was discovered by H. Peregrine (1983) and was labeled as *the rogue wave*.

Properties of the rogue wave:

- It is developed due to modulational instability of the constant wave background $\psi_0(x,t) = 1$.
- It comes from nowhere: $|\psi(x,t)| \to 1$ as $|x| + |t| \to \infty$.
- It is magnified at the center: $M_0 := |\psi(0,0)| = 3$.

Periodic waves of the modified KdV equation

The modified Korteweg–de Vries (mKdV) equation

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

appears in many physical applications, e.g., in models for internal waves.

The mKdV equation admits two families of *the travelling periodic waves*:

- positive-definite periodic waves

$$u_{\rm dn}(x,t) = {\rm dn}(x - ct; k), \quad c = c_{\rm dn}(k) := 2 - k^2,$$

- sign-indefinite periodic waves

$$u_{\rm cn}(x,t) = k{\rm cn}(x - ct; k), \quad c = c_{\rm cn}(k) := 2k^2 - 1,$$

where $k \in (0,1)$ is elliptic modulus.

As $k \to 1$, the periodic waves converge to **the soliton** $u(x,t) = {\rm sech}(x - t)$.
As $k \to 0$, the periodic waves converge to **the small-amplitude waves**.

Modulation theory for the Gardner equation

References: E. Parkes, J. Phys. A 20, 2025-2036 (1987);
R. Grimshaw *et al.*, Physica D 159, 35-57 (2001)

Start with the following Gardner equation with the parameter α:

$$u_t + \alpha u u_x + 6u^2 u_x + u_{xxx} = 0$$

and use the small-amplitude slowly-varying approximation

$$u(x,t) = \epsilon^{1/2}\left[\psi(\sqrt{\epsilon}(x + c_0 t), \epsilon t) e^{i(k_0 x + \omega_0 t)} + \text{c.c}\right] + \mathcal{O}(\epsilon),$$

where $\omega_0 = \omega(k_0) = k_0^3$, $c_0 = \omega'(k_0) = 3k_0^2$, and ψ in scaled variables X and T satisfies the cubic NLS equation

$$i\psi_T + \frac{1}{2}\omega''(k_0)\psi_{XX} + \beta|\psi|^2\psi = 0,$$

where $\omega''(k_0) = 6k_0$ and $\beta = 6k_0 - \frac{\alpha^2}{6k_0}$.

Application of the modulation theory

The cubic NLS equation with $\omega''(k_0) = 6k_0$ and $\beta = 6k_0 - \frac{\alpha^2}{6k_0}$:

$$i\psi_T + \frac{1}{2}\omega''(k_0)\psi_{XX} + \beta|\psi|^2\psi = 0.$$

- sign-indefinite periodic waves

$$u_{\rm cn}(x,t) = k{\rm cn}(x - ct; k), \quad c = c_{\rm cn}(k) := 2k^2 - 1,$$

As $k \to 0$, $u_{\rm cn}(x,t) \sim k cos(x+t)$, hence $k_0 = 1$, $\alpha = 0$ and $\beta > 0$.
***cn* periodic waves are modulationally unstable.**

- positive-definite periodic waves

$$u_{\rm dn}(x,t) = {\rm dn}(x - ct; k), \quad c = c_{\rm dn}(k) := 2 - k^2,$$

As $k \to 0$, $u_{\rm dn}(x,t) \sim 1 + k^2 cos2(x - 2t)$, hence $k_0 = 2$, $\alpha = 12$, $\beta = 0$.
***dn* periodic waves are modulationally stable.**
(See also Bronski–Johnson–Kapitula, 2011 and Deconinck–Nivala, 2011)

Main questions

1. Can one construct rogue waves for periodic waves of mKdV?
2. Can one compute the magnification factor for such rogue waves?

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

Background for these questions:

- Numerically constructed rogue periodic waves for NLS (Kedziora–Ankiewicz–Akhmediev, 2014)
- Numerically constructed rogue waves for two-phase solutions of NLS (Calini–Schober, 2017)
- Rogue waves from a superposition of nearly identical solitons for mKdV
  (Shurgalina–E.Pelinovsky, 2016) - two solitons
  (Slunyaev–E.Pelinovsky, 2016) - N solitons
  The magnification factor for such N-soliton rogue waves $= N$.

Main results

MKdV equation for $u(x,t) \in \mathbb{R}$

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

is a compatibility condition of the Lax pair $\varphi(x,t) \in \mathbb{C}^2$:

$$\varphi_x = U(\lambda, u)\varphi, \quad \varphi_t = V(\lambda, u)\varphi.$$

1. For periodic waves u, we compute explicitly the periodic eigenfunctions φ for four particular eigenvalues λ.
2. For each periodic eigenfunction φ, we construct the second linearly independent non-periodic solution ψ for the same values of λ.
3. By using Darboux transformations of mKdV with non-periodic function ψ, we define the rogue periodic waves in the closed (implicit) form.
4. From the implicit solutions, we compute the magnification factor explicitly.

“Rogue” *dn*-periodic waves

For *dn*-periodic waves

$$u_{\rm dn}(x,t) = {\rm dn}(x - ct; k), \quad c = c_{\rm dn}(k) := 2 - k^2,$$

the magnification factor is

$$M_{\rm dn}(k) = 2 + \sqrt{1 - k^2}, \quad k \in [0,1].$$

The “rogue” *dn*-periodic wave is a superposition of the (modulationally stable) *dn*-periodic wave and a travelling algebraic soliton.

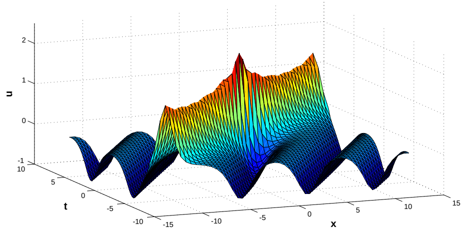

Figure: The “rogue” *dn*-periodic wave of the mKdV for $k = 0.99$.

Rogue *cn*-periodic waves

For *cn*-periodic waves

$$u_{\rm cn}(x,t) = k{\rm cn}(x - ct; k), \quad c = c_{\rm cn}(k) := 2k^2 - 1,$$

the magnification factor is

$$M_{\rm dn}(k) = 3, \quad k \in [0,1].$$

The rogue *cn*-periodic wave is a result of the modulational instability of the *cn*-periodic wave.

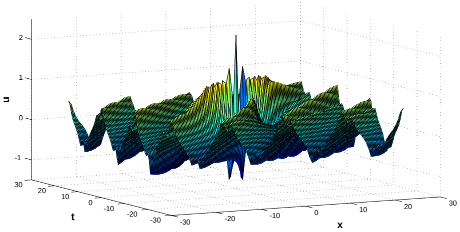

Figure: The rogue *cn*-periodic wave of the mKdV for $k = 0.99$.

How did we complete the job?

1. For periodic waves u, we compute explicitly the periodic eigenfunctions φ for four particular eigenvalues λ.

The AKNS spectral problem for $\varphi(x,t) \in \mathbb{C}^2$:

$$\varphi_x = U(\lambda, u)\varphi, \quad U(\lambda, u) := \left(\begin{array}{cc} \lambda & u \\ -u & -\lambda \end{array} \right),$$

where $u \in \mathbb{R}$ is any solution of the mKdV.

An algebraic technique based on the “nonlinearization” of Lax pair.
Cao–Geng, 1990; Cao–Wu–Geng, 1999; Zhou, 2009; Chen, 2012;

Fix $\lambda = \lambda_1 \in \mathbb{C}$ with an eigenfunction $\varphi = (\varphi_1, \varphi_2) \in \mathbb{C}^2$. Set $u = \varphi_1^2 + \varphi_2^2 \in \mathbb{R}$ and consider the Hamiltonian system

$$\left\{ \begin{array}{l} \frac{d\varphi_1}{dx} = \lambda_1 \varphi_1 + (\varphi_1^2 + \varphi_2^2)\varphi_2 = \frac{\partial H}{\partial \varphi_2}, \\ \frac{d\varphi_2}{dx} = -\lambda_1 \varphi_2 - (\varphi_1^2 + \varphi_2^2)\varphi_1, = -\frac{\partial H}{\partial \varphi_1} \end{array} \right.$$

related to the Hamiltonian function $H(\varphi_1, \varphi_2) = \frac{1}{4}(\varphi_1^2 + \varphi_2^2)^2 + \lambda_1 \varphi_1 \varphi_2$.

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Fix $\lambda = \lambda_1 \in \mathbb{C}$ with an eigenfunction $\varphi = (\varphi_1, \varphi_2) \in \mathbb{C}^2$. Set $u = \varphi_1^2 + \varphi_2^2 \in \mathbb{R}$ and consider the Hamiltonian system

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Periodic eigenfunctions

Besides $u = \varphi_1^2 + \varphi_2^2$, we also have constraints $u_x = 2\lambda_1(\varphi_1^2 - \varphi_2^2)$ and $E_0 - u^2 = 4\lambda_1\varphi_1\varphi_2$, where $E_0 = 4H(\varphi_1, \varphi_2)$ is conserved.

Moreover, the nonlinear Hamiltonian system satisfies the compatibility condition of the Lax pair if and only if $\mu := -\frac{u_x}{2u}$ satisfies the following (Dubrovin ?) equation

$$\frac{1}{4}\left(\frac{d\mu}{dx}\right)^2 = (\mu^2 - \lambda_1^2)(\mu^2 - \lambda_1^2 - E_0).$$

This ODE is satisfied if u satisfies the travelling wave reduction of the mKdV:

$$\frac{d^2u}{dx^2} + 2u^3 = cu, \quad \left(\frac{du}{dx}\right)^2 + u^4 = cu^2 + d,$$

where with real constants c and d given by

$$c = 4\lambda_1^2 + 2E_0, \quad d = -E_0^2.$$

Moreover, if $u(x - ct)$, then $\varphi(x - ct)$ is compatible with the time evolution.

dn-periodic waves

The connection formulas:

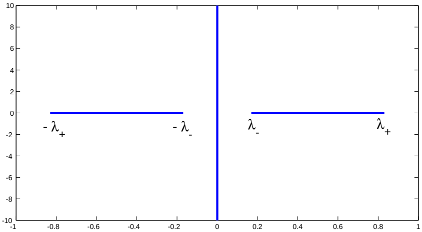
$$c = 4\lambda_1^2 + 2E_0, \quad d = -E_0^2.$$

For *dn*-periodic waves

$$u_{\rm dn}(x,t) = {\rm dn}(x - ct; k), \quad c = c_{\rm dn}(k) := 2 - k^2,$$

we have $d = k^2 - 1 \leq 0$. Hence $E_0 = \pm\sqrt{1-k^2}$ and

$$\lambda_1^2 = \frac{1}{4}\left[2 - k^2 \mp 2\sqrt{1-k^2}\right].$$



cn-periodic waves

The connection formulas:

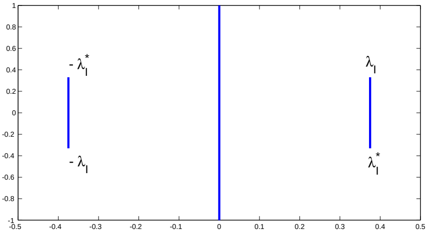
$$c = 4\lambda_1^2 + 2E_0, \quad d = -E_0^2.$$

For *cn*-periodic waves

$$u_{\rm cn}(x,t) = k{\rm cn}(x-ct;k), \quad c = c_{\rm cn}(k) := 2k^2 - 1,$$

we have $d = k^2(1-k^2) \geq 0$. Hence $E_0 = \pm ik\sqrt{1-k^2}$ and

$$\lambda_1^2 = \frac{1}{4}\left[2k^2 - 1 \mp 2ik\sqrt{1-k^2}\right]$$



How did we complete the job?

2. For each periodic eigenfunction φ, we construct the second linearly independent non-periodic solution ψ for the same values of λ.

For $\lambda = \lambda_1 \in \mathbb{C}$, we have one periodic solution $\varphi = (\varphi_1, \varphi_2)$ of

$$\varphi_x = U(\lambda, u)\varphi, \quad U(\lambda, u) := \begin{pmatrix} \lambda & u \\ -u & -\lambda \end{pmatrix},$$

where $u \in \mathbb{R}$ is any solution of the mKdV.

Let us define the second solution $\psi = (\psi_1, \psi_2)$ by

$$\psi_1 = \frac{\theta - 1}{\varphi_2}, \quad \psi_2 = \frac{\theta + 1}{\varphi_1},$$

such that $\varphi_1\psi_2 - \varphi_2\psi_1 = 2$ (Wronskian is constant). Then, θ satisfies the first-order reduction

$$\frac{d\theta}{dx} = u\theta\frac{\varphi_2^2 - \varphi_1^2}{\varphi_1\varphi_2} + u\frac{\varphi_1^2 + \varphi_2^2}{\varphi_1\varphi_2}.$$

Non-periodic solutions

Because $u = \varphi_1^2 + \varphi_2^2$, $u_x = 2\lambda_1(\varphi_1^2 - \varphi_2^2)$, and $E_0 - u^2 = 4\lambda_1\varphi_1\varphi_2$, we can rewrite the ODE for θ as

$$\frac{d\theta}{dx} = \theta\frac{2uu'}{u^2 - E_0} - \frac{4\lambda_1 u^2}{u^2 - E_0},$$

where $u^2 - E_0 \neq 0$ is assumed. Integration yields

$$\theta(x) = -4\lambda_1(u(x)^2 - E_0)\int_0^x \frac{u(y)^2}{(u(y)^2 - E_0)^2}dy.$$

Moreover, if $u(x - ct)$ and $\varphi(x - ct)$, then the time evolution yields

$$\theta(x,t) = -4\lambda_1(u(x - ct)^2 - E_0)\left[\int_0^{x-ct} \frac{u(y)^2}{(u(y)^2 - E_0)^2}dy - t\right].$$

up to translation in t.

How did we complete the job?

3. By using Darboux transformations of mKdV with non-periodic function ψ, we define the rogue periodic waves in the closed form.

One-fold Darboux transformation:

$$\widetilde{u} = u + \frac{4\lambda_1 pq}{p^2 + q^2},$$

where u and $\widetilde{u}$ are solutions of the mKdV and $\varphi = (p, q)$ is a nonzero solution of the Lax pair with $\lambda = \lambda_1$ and u.

Two-fold Darboux transformation:

$$\tilde{u} = u + \frac{4(\lambda_1^2 - \lambda_2^2)\left[\lambda_1 p_1 q_1 (p_2^2 + q_2^2) - \lambda_2 p_2 q_2 (p_1^2 + q_1^2)\right]}{(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2) - 2\lambda_1\lambda_2\left[4p_1 q_1 p_2 q_2 + (p_1^2 - q_1^2)(p_2^2 - q_2^2)\right]}$$

where (p_1, q_1) and (p_2, q_2) are nonzero solutions of the Lax pair with λ_1 and λ_2 such that $\lambda_1 \neq \pm\lambda_2$.

“Rogue” *dn*-periodic waves

Using one-fold transformation with periodic eigenfunction (φ_1, φ_2) yields

$$\tilde{u} = u + \frac{4\lambda_1 \varphi_1 \varphi_2}{\varphi_1^2 + \varphi_2^2} = -\frac{\sqrt{1-k^2}}{\mathrm{dn}(x-ct;k)} = -\mathrm{dn}(x - ct + K(k); k),$$

which is a translation of the *dn*-periodic wave.

Using one-fold transformation with non-periodic (ψ_1, ψ_2) yields

$$\tilde{u} = u + \frac{4\lambda_1 \psi_1 \psi_2}{\psi_1^2 + \psi_2^2} = u + \frac{4\lambda_1 \varphi_1 \varphi_2 (\theta^2 - 1)}{(\varphi_1^2 + \varphi_2^2)(1+\theta^2) - 2(\varphi_1^2 - \varphi_2^2)\theta},$$

which is not a translation of the *dn*-periodic wave.

- As $|\theta| \to \infty$ (as $|x| + |t| \to \infty$ almost everywhere):

$$\tilde{u}(x,t) \sim -\frac{\sqrt{1-k^2}}{\mathrm{dn}(x-ct;k)} = -\mathrm{dn}(x - ct + K(k); k).$$

- At $\theta = 0$ (at $(x,t) = (0,0)$), the rogue wave is at the maximum point:

$$\tilde{u}(0,0) = 2 + \sqrt{1-k^2}.$$

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“Rogue” *dn*-periodic waves

For *dn*-periodic waves

$$u_{\mathrm{dn}}(x,t) = \mathrm{dn}(x - ct; k), \quad c = c_{\mathrm{dn}}(k) := 2 - k^2,$$

the magnification factor is

$$M_{\mathrm{dn}}(k) = 2 + \sqrt{1 - k^2}, \quad k \in [0,1].$$

The “rogue” *dn*-periodic wave is a superposition of the (modulationally stable) *dn*-periodic wave and a travelling algebraic soliton.

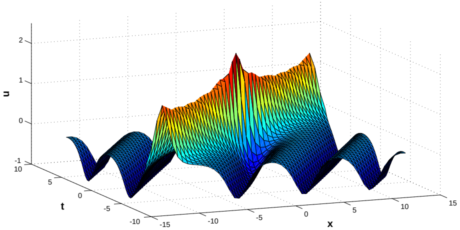

Figure: The “rogue” *dn*-periodic wave of the mKdV for $k = 0.99$.

Rogue *cn*-periodic waves

Since $\lambda_1 \notin \mathbb{R}$, one-fold transformation yields complex solutions of the mKdV. Using two-fold transformation with periodic (φ_1, φ_2) and its conjugate yields

$$\tilde{u} = u + \frac{4k^2(1-k^2)u}{(2k^2-1)u^2 - u^4 - k^2(1-k^2) - (u')^2} = -u,$$

which is a translation of the *cn*-periodic wave.

Using two-fold transformation with non-periodic (ψ_1, ψ_2) and its conjugate:

$$\tilde{u} = u + \frac{4(\lambda_I^2 - \overline{\lambda}_I^2)\left[\lambda_I \psi_1 \psi_2 (\overline{\psi}_1^2 + \overline{\psi}_2^2) - \overline{\lambda}_I \overline{\psi}_1 \overline{\psi}_2 (\psi_1^2 + \psi_2^2)\right]}{(\lambda_I^2 + \overline{\lambda}_I^2)|\psi_1^2 + \psi_2^2|^2 - 2|\lambda_I|^2 \left[4|\psi_1|^2|\psi_2|^2 + |\psi_1^2 - \psi_2^2|^2\right]}.$$

- As $|\theta| \to \infty$ (as $|x| + |t| \to \infty$ everywhere):

$$\tilde{u}(x,t) \sim -u(x,t).$$

- At $\theta = 0$ (at $(x,t) = (0,0)$), the rogue wave is at the maximum point:

$$\tilde{u}(0,0) = 3k.$$

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Rogue *cn*-periodic waves

For *cn*-periodic waves

$$u_{\rm cn}(x,t) = k{\rm cn}(x-ct;k), \quad c = c_{\rm cn}(k) := 2k^2 - 1,$$

the magnification factor is

$$M_{\rm cn}(k) = 3, \quad k \in [0,1].$$

The rogue *cn*-periodic wave is a result of the modulational instability of the *cn*-periodic wave.

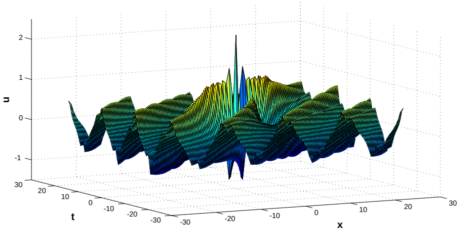

Figure: The rogue *cn*-periodic wave of the mKdV for $k = 0.99$.

Rogue periodic waves in NLS

The NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = 0$$

has a similar Lax pair, e.g.

$$\varphi_x = U\varphi, \qquad U = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix}.$$

The NLS equation admits two families of *the periodic waves*:

- positive-definite periodic waves

$$u_{\rm dn}(x,t) = {\rm dn}(x;k)e^{ict}, \quad c = 2 - k^2,$$

- sign-indefinite periodic waves

$$u_{\rm cn}(x,t) = k{\rm cn}(x;k)e^{ict}, \quad c = 2k^2 - 1,$$

where $k \in (0,1)$ is elliptic modulus.

Both periodic waves are modulationally unstable.

For *dn*-periodic waves

$$u_{\mathrm{dn}}(x,t) = \mathrm{dn}(x;k)e^{ict}, \quad c = 2 - k^2,$$

the magnification factor is still

$$M_{\mathrm{dn}}(k) = 2 + \sqrt{1-k^2}, \quad k \in [0,1].$$

The rogue *dn*-periodic wave is a generalization of the Peregrine's rogue wave.

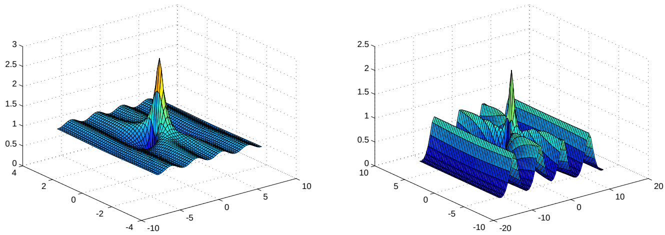

Figure: The rogue *dn*-periodic wave of the NLS for $k = 0.5$ and $k = 0.99$.

Rogue *cn*-periodic waves

For *cn*-periodic waves

$$u_{\mathrm{cn}}(x,t) = k\,\mathrm{cn}(x;k)e^{ict}, \quad c = 2k^2 - 1,$$

the magnification factor is $M_{\mathrm{cn}}(k) = 2$ for every $k \in (0,1)$ as the rogue wave is obtained from the one-fold Darboux transformation. Exact solutions are computed compared to the numerical approximation in (Kedziora–Ankiewicz–Akhmediev, 2014).

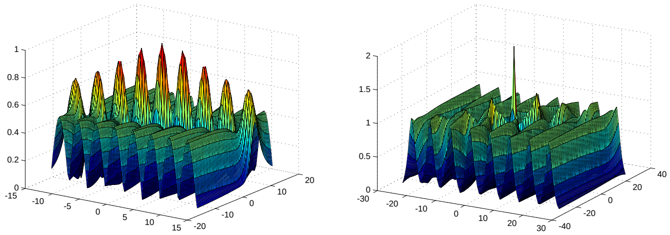

Figure: The rogue *cn*-periodic wave of the NLS for $k = 0.5$ and $k = 0.99$.

Summary

- New method to obtain eigenfunctions of the periodic spectral (AKNS) problem associated with the periodic waves.
- New exact solutions to generalize the Peregrine's rogue waves to the *dn*-periodic and *cn*-periodic waves in mKdV and NLS.