Rogue waves on the background of periodic solutions of the DNLS equation

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Introduction

I am mainly interested in stability of periodic wave solutions to nonlinear integrable Hamiltonian systems.

Integrable models (mKdV, Gardner, NLS, DNLS, KP) feature rogue waves on complex wave backgrounds, can be treated mathematically, and are still relevant for modelling of important physical processes.

The derivative nonlinear Schrödinger (DNLS) equation

$$i\psi_t + \psi_{xx} + i(|\psi|^2\psi)_x = 0$$

where $\psi(x,t)$ is complex valued. It is one of the basic models for Alfvén waves propagating along the magnetic field in cold plasmas **[E. Mjolhus, 1976]**

Periodic waves and linear stability

The DNLS equation admits the periodic traveling and standing wave solution

$$\psi(x,t) = e^{4ibt} u(x+2ct)$$

with two parameters b (frequency) and c (speed). **[A. Kamchatnov, 1990]**

Linear stability of such solutions is defined by the linearized equation

$$iw_t - 4bw + 2icw_x + w_{xx} + i[2|u|^2 w_x + u^2 \bar{w}_x + 2(u\bar{u}_x + \bar{u}u_x)w + 2uu_x\bar{w}] = 0$$

for the perturbation w in $\psi(x,t) = e^{4ibt}\left[u(x+2ct) + w(x+2ct,t)\right]$.

Separating variables by $w(x,t) = w_1(x)e^{t\Lambda}$ and $\bar{w}(x,t) = w_2(x)e^{t\Lambda}$ results in the spectral problem for the eigenvector $\vec{w} := (w_1, w_2)^T$ and eigenvalue Λ:

$$\mathcal{L}\left[\begin{array}{c} w_1 \\ w_2 \end{array}\right] = \Lambda \left[\begin{array}{c} w_1 \\ w_2 \end{array}\right].$$

Stability of periodic waves

If $u(x)$ is periodic, one can consider periodic $\vec{w}(x)$ with the same period.

On the other hand, by Floquet theorem, $\vec{w}(x) = \vec{p}(x)e^{ikx}$, where $\vec{p}$ is periodic and k is real, then, $\vec{w} \in L^\infty(\mathbb{R})$ forms a basis of functions in $L^2(\mathbb{R})$.

Stability spectrum is the union of all Λ for which $\vec{w} \in L^\infty(\mathbb{R})$.

Definition

The periodic wave u is spectrally unstable if there exists Λ with $\mathrm{Re}(\Lambda) > 0$ such that $\vec{w} \in L^\infty(\mathbb{R})$.

How to study stability of periodic waves?

Stability spectrum Λ can be characterized from the linear Lax system representing the DNLS equation for $\phi(x,t) = e^{2bt\sigma_3}\varphi(x+2ct,t)$:

$$\varphi_x = U(u,\lambda)\varphi, \quad \varphi_t + 2ib\sigma_3\varphi + 2c\varphi_x = V(u,\lambda)\varphi,$$

where

$$U = \begin{pmatrix} -i\lambda^2 & \lambda u \\ -\lambda\bar{u} & i\lambda^2 \end{pmatrix}, \quad V = \begin{pmatrix} -2i\lambda^4 + i\lambda^2|u|^2 & 2\lambda^3 u + \lambda(iu_x - |u|^2 u) \\ -2\lambda^3\bar{u} + \lambda(i\bar{u}_x + |u|^2\bar{u}) & 2i\lambda^4 - i\lambda^2|u|^2 \end{pmatrix}$$

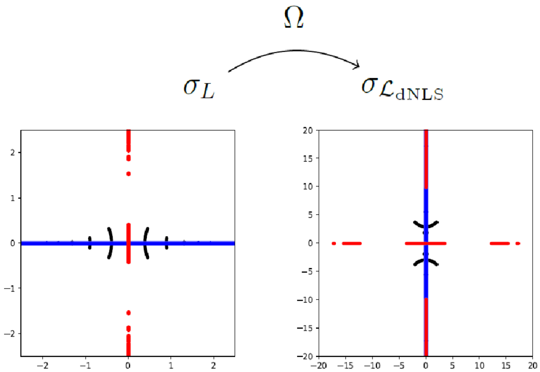
If $\lambda \in \mathbb{C}$ belongs to Lax spectrum with $\chi \in L^\infty(\mathbb{R})$ of $\chi_x = U(u,\lambda)\chi$, then the time evolution yields $\varphi(x,t) = \chi(x)e^{t\Omega(\lambda)}$ for some $\Omega(\lambda)$. Moreover, $\vec{w} = (w_1, w_2)^T$ and Λ in the spectral stability problem are given by

$$w_1 = \partial_x \chi_1^2, \quad w_2 = \partial_x \chi_2^2, \quad \Lambda = 2\Omega(\lambda).$$

Squared eigenfunctions for DNLS were found in **[X.G. Chen-J.Yang (2002)]**. Recent study of DNLS was done in **[J. Upsal-B. Deconinck (2020)]**.

Main result from **J. Upsal-B. Deconinck (2020)**:

If $\Lambda \in i\mathbb{R}$ for a given $\lambda \in \mathbb{R} \cup i\mathbb{R}$, then $\lambda \in \mathbb{R} \cup i\mathbb{R}$ belongs to the Lax spectrum.



Spectral stability of **some** periodic waves in DNLS in $L^2_{\rm per}$ was also studied in **[S.Hakkaev-A.Stefanov-M.Stanislavova (2021)]**.

Our methods and results

- We explore construction of periodic waves of integrable equations by using complex-valued Hamiltonian systems arising in the nonlinearization of the Lax equations **[Cao–Geng, 1990]** Also **(Z.Qiao; R.Zhou; J.Chen)**
- We obtain precise information on the location of Lax and stability spectra.
- We construct exact solutions describing rogue waves on the background of periodic waves.

A particularly interesting outcome is the explicit relation between instability of periodic waves and full localization of rogue waves.
[F. Baronio, M. Conforti, S. Wabnitz, 2014]

Periodic travelling and standing waves

The derivative nonlinear Schrödinger (DNLS) equation

$$i\psi_t + \psi_{xx} + i(|\psi|^2\psi)_x = 0$$

admits the periodic traveling and standing wave solution

$$\psi(x,t) = e^{4ibt} u(x + 2ct)$$

with two parameters b and c. The envelope $u = u(x)$ satisfies

$$u'' + 2|u|^2 u + 2icu' - 4bu = 0,$$

From here, solutions are usually constructed by separation of variables $u(x) = R(x)e^{i\Theta(x)}$ with

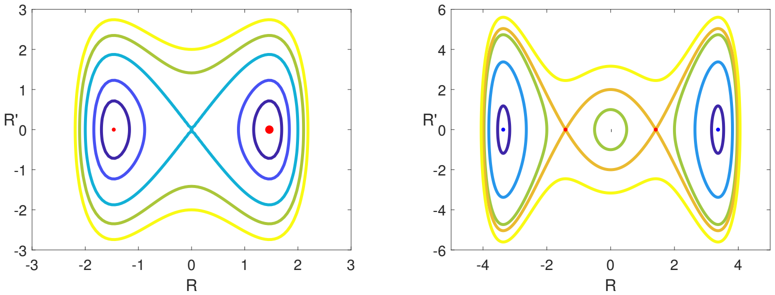
$$\frac{d\Theta}{dx} = -\frac{a}{R^2} - \frac{3}{4}R^2 - c.$$

Periodic waves of "trivial phase" correspond to $a = 0$.
Whatever we do for $a = 0$ can be done for $a \neq 0$.

Families of periodic waves for $a = 0$

$$\left(\frac{dR}{dx}\right)^2 + F(R) = 4d, \quad F(R) = \frac{1}{16}R^6 + \frac{c}{2}R^4 + (c^2 - 4b)R^2.$$

Left: $c^2 < 4b$. Right: $c^2 > 4b$, $c < 0$, and $b > 0$.



Lax spectrum for DNLS

Lax system for the DNLS equation includes the spectral problem **[D.Kaup–A.Newell (1978)]**:

$$\varphi_x = \begin{pmatrix} -i\lambda^2 & \lambda u \\ -\lambda\bar{u} & i\lambda^2 \end{pmatrix} \varphi,$$

where $u(x) = R(x)e^{i\Theta(x)}$ with periodic R and Θ'. Solutions are in the form $\varphi = (p, q)^T$, where $p(x) = P(x)e^{i\Theta(x)/2}$ and $q(x) = Q(x)e^{-i\Theta(x)/2}$.

1. Let $\lambda \in \mathbb{C}\backslash i\mathbb{R}$ be a simple eigenvalue with $\varphi = (p, q)^T$.
   Then, $\bar{\lambda}$ is a simple eigenvalue with $\varphi = (\bar{q}, -\bar{p})^T$.
2. Let $\lambda \in i\mathbb{R}$ be a simple eigenvalue with $\varphi = (p, q)^T$. Then, $q = -i\bar{p}$.

Complex Hamiltonian system

Fix $\lambda = \lambda_1$ with $\varphi = (p_1, q_1)^T$ and $\lambda = \lambda_2$ with $\varphi = (p_2, q_2)^T$ s.t. $\lambda_1 \neq \lambda_2$. Consider the potential u of the Kaup–Newell problem given by either

$$\lambda_1 \in \mathbb{C}\backslash i\mathbb{R}, \quad \lambda_2 = \bar{\lambda}_1 : \qquad \left\{ \begin{array}{l} u = \lambda_1 p_1^2 + \bar{\lambda}_1 \bar{q}_1^2, \\ \bar{u} = \bar{\lambda}_1 \bar{p}_1^2 + \lambda_1 q_1^2 \end{array} \right.$$

or

$$\begin{array}{ll} \lambda_1 = i\beta_1, & \lambda_2 = i\beta_2 \\ q_1 = -i\bar{p}_1, & q_2 = -i\bar{p}_2 \end{array} : \qquad \left\{ \begin{array}{l} u = i\beta_1 p_1^2 + i\beta_2 p_2^2, \\ \bar{u} = -i\beta_1 \bar{p}_1^2 - i\beta_2 \bar{p}_2^2. \end{array} \right.$$

The Kaup–Newell problem becomes a complex Hamiltonian system generated by the Hamiltonian function

$$H = i\lambda_1^2 p_1 q_1 + i\lambda_2^2 p_2 q_2 - \frac{1}{2}(\lambda_1 p_1^2 + \lambda_2 p_2^2)(\lambda_1 q_1^2 + \lambda_2 q_2^2).$$

with additional conserved quantity

$$M = i(p_1 q_1 + p_2 q_2).$$

Both conserved quantities are real for the two cases above.

Travelling wave reduction

Differentiating the constraint between u and eigenfunctions:

$$\begin{aligned} & u = \lambda_1 p_1^2 + \lambda_2 p_2^2, \\ \Rightarrow \quad & \frac{du}{dx} + i|u|^2 u + 2iHu + 2i(\lambda_1^3 p_1^2 + \lambda_2^3 p_2^2) = 0, \\ \Rightarrow \quad & \frac{d^2u}{dx^2} + i\frac{d}{dx}(|u|^2 u) + 2iH\frac{du}{dx} + 4(\lambda_1^5 p_1^2 + \lambda_2^5 p_2^2 + i\lambda_1^4 u p_1 q_1 + i\lambda_2^4 u p_2 q_2) = 0. \end{aligned}$$

The last equation yields the travelling wave reduction of DNLS:

$$\boxed{\frac{d^2u}{dx^2} + i\frac{d}{dx}(|u|^2 u) + 2ic\frac{du}{dx} - 4bu = 0,}$$

where $b := \lambda_1^2 \lambda_2^2 (1 + M)$ and $c := \lambda_1^2 + \lambda_2^2 + H$.

Integrability of the complex Hamiltonian system

The complex Hamiltonian system on (p_1, q_1) and (p_2, q_2) is a compatibility condition of the Lax equation

$$\frac{d}{dx}\Psi = U(\lambda, u)\Psi - \Psi U(\lambda, u),$$

where $U(\lambda, u)$ is the same as in the Kaup–Newell system and

$$\Psi = \left(\begin{array}{cc} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & -\Psi_{11} \end{array} \right)$$

with

$$\Psi_{11} = -i - \frac{\lambda_1^2 p_1 q_1}{\lambda^2 - \lambda_1^2} - \frac{\lambda_2^2 p_2 q_2}{\lambda^2 - \lambda_2^2} = \frac{-i[\lambda^4 - (c + \frac{1}{2}|u|^2)\lambda^2 + b]}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)},$$

$$\Psi_{12} = \lambda \left[\frac{\lambda_1 p_1^2}{\lambda^2 - \lambda_1^2} + \frac{\lambda_2 p_2^2}{\lambda^2 - \lambda_2^2} \right] = \frac{\lambda[\lambda^2 u + \frac{i}{2}(\frac{du}{dx} + i|u|^2 u) - cu}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)}.$$

$\det \Psi$ is constant and has simple poles at $(\pm\lambda_1, \pm\lambda_2)$:

$$\det \Psi = 1 - \frac{2H\lambda^2 - \lambda_1^2\lambda_2^2 M(M+2)}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)} = \frac{P(\lambda)}{(\lambda^2 - \lambda_1^2)^2(\lambda^2 - \lambda_2^2)^2}$$

with

$$\boxed{P(\lambda) = \lambda^8 - 2c\lambda^6 + (a + 2b + c^2)\lambda^4 + (d - c(a+2b))\lambda^2 + b^2.}$$

Recall the stability analysis with $\varphi(x,t) = \chi(x)e^{t\Omega(\lambda)}$ and $\Lambda = 2\Omega(\lambda)$. Then,

$$\boxed{\Omega(\lambda) = \pm 2i\sqrt{P(\lambda)}.}$$

Roots of $P(\lambda)$ are mapped to $\Lambda = 0$ in the stability plane.

Characterization of periodic waves

On one hand, the periodic waves of the DNLS are related to the polynomial

$$P(\lambda) = \lambda^8 - 2c\lambda^6 + (a + 2b + c^2)\lambda^4 + (d - c(a + 2b))\lambda^2 + b^2.$$

Denote four pairs of roots of $P(\lambda)$ by $\{\pm\lambda_1, \pm\lambda_2, \pm\lambda_3, \pm\lambda_4\}$, where any two roots can be picked for the algebraic method.

On the other hand, the periodic waves are given for $\rho = \frac{1}{2}R^2$:

$$\left(\frac{d\rho}{dx}\right)^2 + Q(\rho) = 0, \; Q(\rho) = \rho^4 + 4c\rho^3 + 2(2c^2 - 8b)\rho^2 + 4(ac - 2d)\rho + a^2$$

Denote four roots of $Q(\rho)$ by $\{u_1, u_2, u_3, u_4\}$.

The two sets of roots are related explicitly by

$$\left\{ \begin{array}{rcl} u_1 & = & -\frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)^2, \\ u_2 & = & -\frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)^2, \end{array} \right. \qquad \left\{ \begin{array}{rcl} u_3 & = & -\frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)^2, \\ u_4 & = & -\frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^2. \end{array} \right.$$

[A. Kamchatnov (1990)]

First family of periodic waves

Four roots of $Q(\rho)$ are real: $u_4 \leq u_3 \leq u_2 \leq u_1$. Then,

$$\rho(x) = u_4 + \frac{(u_1 - u_4)(u_2 - u_4)}{(u_2 - u_4) + (u_1 - u_2)\mathrm{sn}^2(\mu x; k)},$$

with $2\mu = \sqrt{(u_1 - u_3)(u_2 - u_4)}$ and $2\mu k = \sqrt{(u_1 - u_2)(u_3 - u_4)}$.

This family occurs only in two cases:
Two complex quadruplets when $u_4 \leq u_3 \leq 0 \leq u_2 \leq u_1$,

$$\lambda_1 = \bar{\lambda}_2 = \alpha_1 + i\beta_1, \qquad \lambda_3 = \bar{\lambda}_4 = \alpha_2 + i\beta_2.$$

Four pairs of purely imaginary eigenvalues when $0 \leq u_4 \leq u_3 \leq u_2 \leq u_1$,

$$\lambda_1 = i\beta_1, \qquad \lambda_2 = i\beta_2, \qquad \lambda_3 = i\beta_3, \qquad \lambda_4 = i\beta_4.$$

Second family of periodic waves

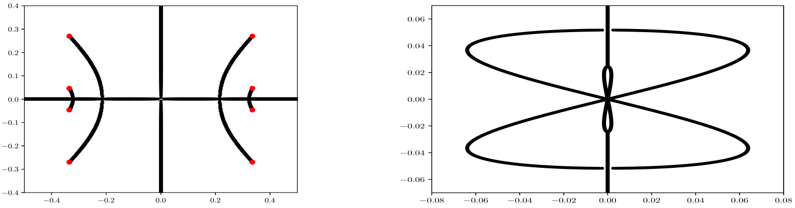
Two roots of $Q(\rho)$ are real $u_2 \leq u_1$ and two roots of $Q(\rho)$ are complex-conjugate $u_{3,4} = \gamma \pm i\eta$. Then,

$$\rho(x) = u_1 + \frac{(u_2 - u_1)(1 - \mathrm{cn}(\mu x; k))}{1 + \delta + (\delta - 1)\mathrm{cn}(\mu x; k)},$$

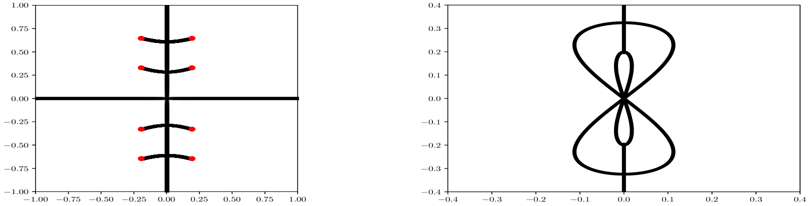
with δ, μ, and k are given in terms of u_1, u_2, γ, and η.

This family occurs only in one case:
One complex quadruplet and two pairs of purely imaginary eigenvalues when $0 \leq u_2 \leq u_1$.

Two complex quadruplets

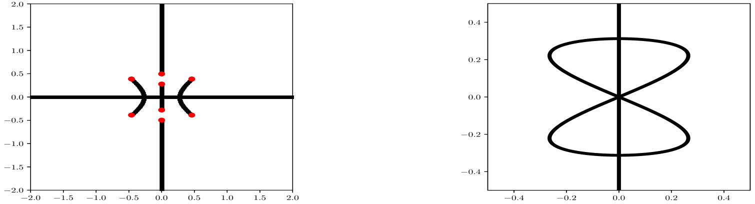


(a) $u_1 = 0.2$, $u_2 = 0.1$, $u_3 = 0$, $u_4 = -0.9$.

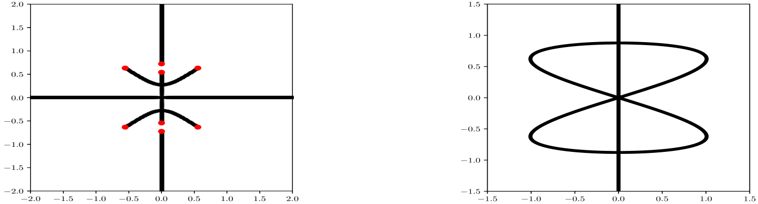


(b) $u_1 = 1.9$, $u_2 = 0.2$, $u_3 = 0$, $u_4 = -0.3$.

One complex quadruplet and two pairs of imaginary

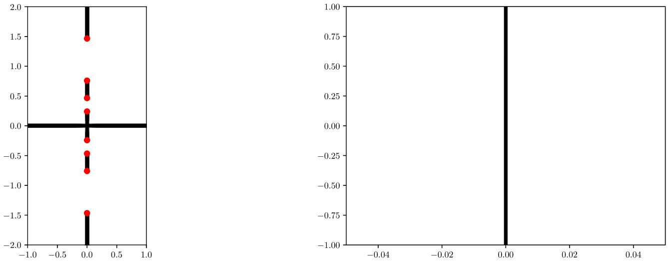


(a) $u_1 = 1.2$, $u_2 = 0$, $u_3 = -0.4 - 0.2i$, $u_4 = -0.4 + 0.2i$.



(b) $u_1 = 3.2$, $u_2 = 0$, $u_3 = -0.6 + 0.2i$, $u_4 = -0.6 - 0.2i$.

Four pairs of imaginary eigenvalues



Remarkable:

- Stability is observed only for $c^2 > 4b$, $c < 0$, and $b > 0$ for periodic waves $\psi(x,t) = e^{4ibt}u(x+2ct)$
- Two different families of periodic waves (positive and sign-indefinite) share the same Lax spectrum and the same stability.

Rogue waves on the periodic wave background

Does there exist a rogue wave on the periodic wave background such that

$$\sup_{x\in\mathbb{R}} \left| \psi(x,t) - e^{4ibt+i\alpha_\pm} u(x+2ct+x_\pm) \right| \to 0 \quad \text{as} \quad t \to \pm\infty$$

for some $\alpha_\pm$ and $x_\pm$? This rogue wave would *appear from nowhere and disappear without trace*.

Rogue waves are generated by Darboux transformations in the form $\hat{\psi}(x,t) = \hat{u}(x+2ct,t)e^{4ibt}$ with

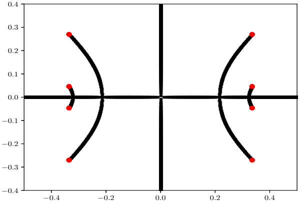
$$\lambda_1 \in \mathbb{C}\backslash i\mathbb{R}: \qquad \hat{u} = \left(\frac{\bar{\lambda}_1 |p_1|^2 + \lambda_1 |q_1|^2}{\lambda_1 |p_1|^2 + \bar{\lambda}_1 |q_1|^2} \right)^2 \left[u - \frac{2i(\lambda_1^2 - \bar{\lambda}_1^2) p_1 \bar{q}_1}{\bar{\lambda}_1 |p_1|^2 + \lambda_1 |q_1|^2} \right].$$

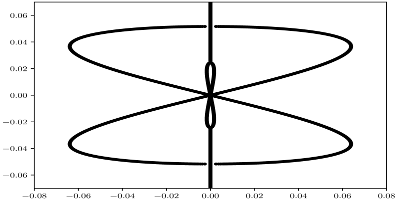
and

$$\lambda_1 = i\beta_1, \quad q_1 = -i\bar{p}_1 : \qquad \hat{u} = -\frac{\bar{p}_1^2}{p_1^2} \left[u + 2i\beta_1 \frac{p_1}{\bar{p}_1} \right] e^{-8ibt}$$

[K. Imai (1999); H. Steudel (2003)]

Complex quadruplets





Let $\phi = (p_1, q_1)^T$ be the periodic eigenvector for the eigenvalue λ_1 with $\Lambda = 0$ (a root in the algebraic method). Darboux transformation

$$\lambda_1 \in \mathbb{C} \backslash i\mathbb{R} : \qquad \hat{u} = \left(\frac{\bar{\lambda}_1 |p_1|^2 + \lambda_1 |q_1|^2}{\lambda_1 |p_1|^2 + \bar{\lambda}_1 |q_1|^2} \right)^2 \left[u - \frac{2i(\lambda_1^2 - \bar{\lambda}_1^2) p_1 \bar{q}_1}{\bar{\lambda}_1 |p_1|^2 + \lambda_1 |q_1|^2} \right]$$

generates the same periodic wave translated in x.

The second, linearly independent solution $\phi = (\hat{p}_1, \hat{q}_1)$ for the same eigenvalue λ_1 can be presented in the form

$$\hat{p}_1 = p_1 \chi_1 - \frac{\bar{q}_1}{|p_1|^2 + |q_1|^2}, \quad \hat{q}_1 = q_1 \chi_1 + \frac{\bar{p}_1}{|p_1|^2 + |q_1|^2},$$

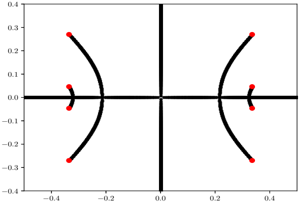
where

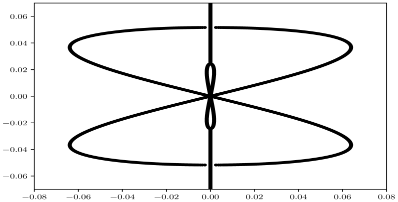
$$\frac{\partial \chi_1}{\partial x} = \frac{2i(\lambda_1^2 - \bar{\lambda}_1^2)\bar{p}_1 \bar{q}_1 + (\lambda_1 - \bar{\lambda}_1)(u\bar{p}_1^2 + \bar{u}\bar{q}_1^2)}{(|p_1|^2 + |q_1|^2)^2}$$

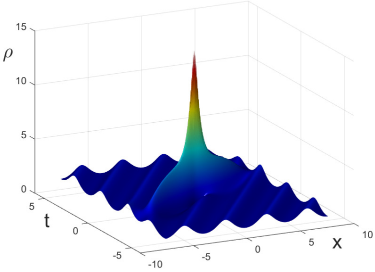
and

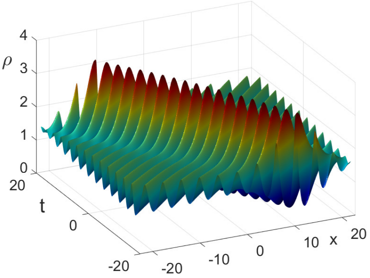
$$\frac{\partial \chi_1}{\partial t} = 2\lambda_1^2(\lambda_1^2 - \bar{\lambda}_1^2).$$

If $\lambda_1 \in \mathbb{C} \backslash i\mathbb{R}$, then $|\chi_1(x,t)| \to \infty$ as $|x| + |t| \to \infty$ so that the Darboux transformation defines rogue wave localized in (x, t).



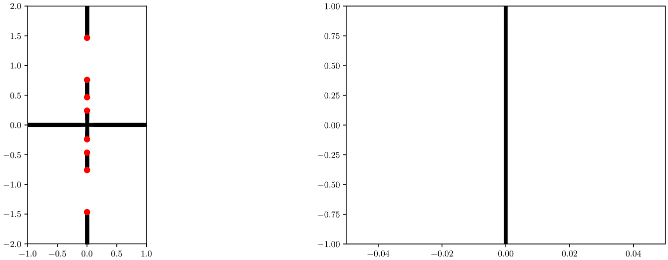






Magnification factors of rogue waves can be computed explicitly.

Purely imaginary eigenvalues



Let $\phi = (p_1, q_1)^T$ be the periodic eigenvector for the eigenvalue λ_1 with $\Lambda = 0$ (a root in the algebraic method). Darboux transformation

$$\lambda_1 = i\beta_1, \quad q_1 = -i\bar{p}_1 : \qquad \hat{u} = -\frac{\bar{p}_1^2}{p_1^2}\left[u + 2i\beta_1 \frac{p_1}{\bar{p}_1}\right] e^{-8ibt}$$

generates the periodic wave of the same class.

The second, linearly independent solution $\phi = (\hat{p}_1, \hat{q}_1)$ for the same eigenvalue λ_1 can be presented in the form

$$\hat{p}_1 = p_1 \chi_1 - \frac{1}{2q_1}, \qquad \hat{q}_1 = q_1 \chi_1 + \frac{1}{2p_1},$$

where

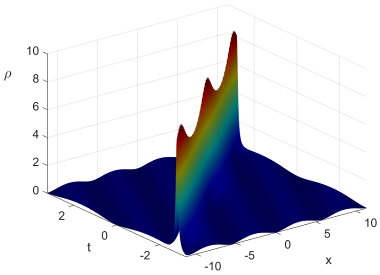
$$\frac{\partial \chi_1}{\partial x} = \frac{i\beta_1}{2|p_1|^4} \left(u\bar{p}_1^2 - \bar{u} p_1^2 \right) \quad \text{and} \quad \frac{\partial \chi_1}{\partial t} = 2\beta_1^2.$$

If $\lambda_1 \in i\mathbb{R}$, then $|\chi_1(x,t)|$ is bounded as $|x| + |t| \to \infty$ along the direction:

$$k_1(x + 2ct) + 2\beta_1^2 t = 0$$

where

$$k_1 = \frac{\nu \beta_1^4}{4K(k)} \int_0^{2K(k)\nu^{-1}} \frac{(a - 2(c + 2\beta_1^2)\rho - \rho^2)}{(b + c\beta_1^2 + \beta_1^4 + \beta_1^2 \rho)^2} dx$$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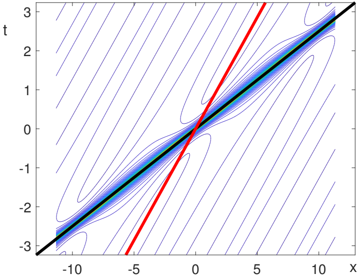
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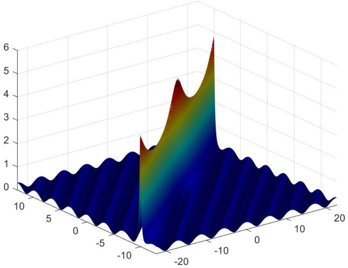
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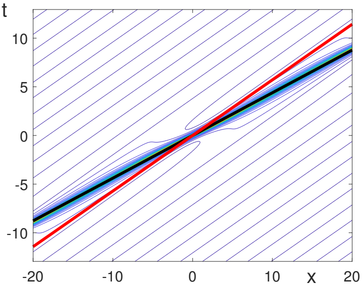
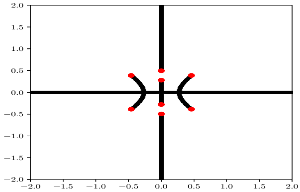
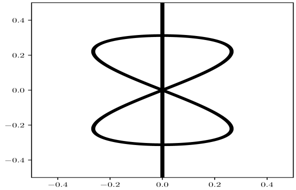
4
3
2
1
0
10
5
0
-5
-10
-20
-10
0
10
20

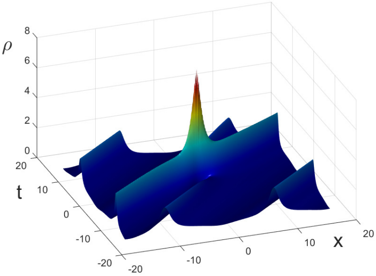

t
10
5
0
-5
-10
-20
-10
0
10
x
20

One complex quadruplet and two pairs of imaginary

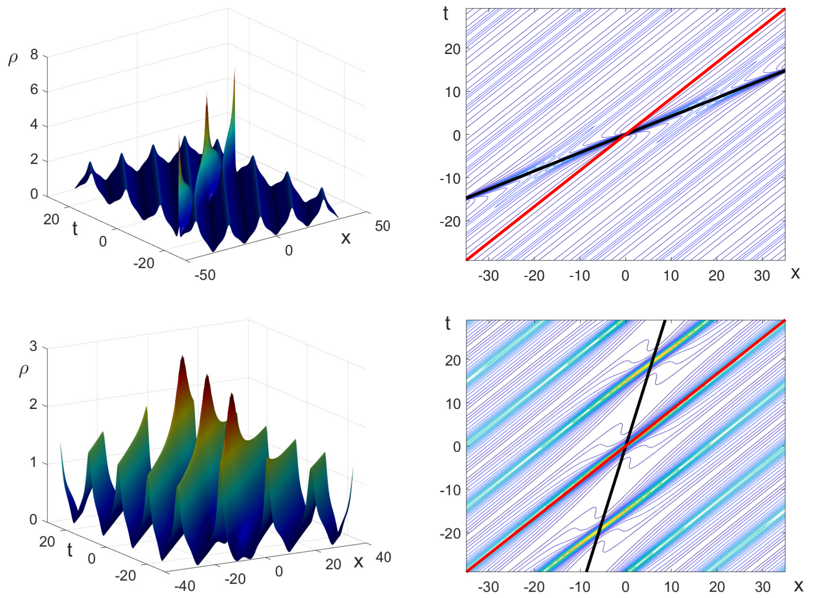




Here is the rogue wave for the complex quadruplet:



Here are two algebraic solitons for two pairs of purely imaginary eigenvalues:



Summary

- Periodic waves of the DNLS equation are constructed by using integrable Hamiltonian systems
- This allows us to characterize the periodic waves in terms of eigenvalues of the Lax equations associated with the periodic eigenfunctions
- We obtain the precise location of Lax and stability spectra, with assistance of the numerical package based on the Hill's method.
- We further obtain exact solutions describing localized structures on the background of periodic waves (either rogue waves or propagating algebraic solitons)
- Localization of rogue waves is related to the modulational instability of the background periodic wave.