Short-pulse equation: well-posedness and wave breaking

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References:

Yu. Liu, D.P., A. Sakovich, Dynamics of PDE 6, 291-310 (2009)
D.P., A. Sakovich, Communications in PDE 35, 613-629 (2010)

Properties of the short-pulse equation

The **short-pulse equation** is a model for propagation of ultra-short pulses [Schäfer, Wayne 2004]:

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx},$$

where all coefficients are normalized thanks to the scaling invariance.

The short-pulse equation

- originates from a scalar Maxwell's equation

$$u_{xx} = u_{tt} + u + (u^3)_{tt}$$

- replaces the nonlinear Schrödinger equation for short wave packets with few cycles on the pulse scale
- features exact solutions for modulated pulses
- enjoys inverse scattering and an infinite set of conserved quantities

Transformation to the sine-Gordon equation

Let $x = x(y,t)$ satisfy

$$\begin{cases} x_y = \cos w, \\ x_t = -\frac{1}{2} w_t^2. \end{cases}$$

Then, $w = w(y,t)$ satisfies the sine–Gordon equation in characteristic coordinates [A. Sakovich, S. Sakovich, J. Phys. A **39**, L361 (2006)]:

$$w_{yt} = \sin(w).$$

Lemma

Let $w(\cdot,t)$, $t \in [0,T]$ is C^1 in the space

$$H_c^s = \left\{ w \in H^s(\mathbb{R}) : \quad \|w\|_{L^\infty} \le w_c < \frac{\pi}{2} \right\}, \quad s \ge 1.$$

Then, $x(y,t)$ is invertible in y for any $t \in [0,T]$ and $u(x,t) = w_t(y(x,t),t)$ solves the short-pulse equation

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}, \quad x \in \mathbb{R}, \quad t \in [0,T].$$

Solutions of the short-pulse equation

A kink of the sine–Gordon equation gives a *loop solution* of the short-pulse equation:

$$\begin{cases} u = 2\,\mathrm{sech}(y+t), \\ x = y - 2\tanh(y+t). \end{cases}$$

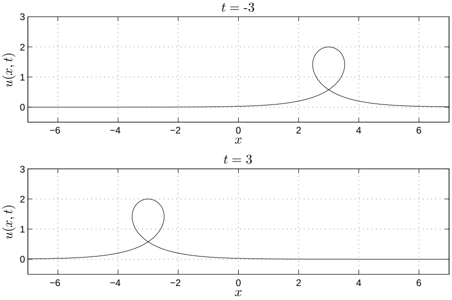

Figure: The loop solution $u(x,t)$ to the short-pulse equation

Solutions of the short-pulse equation

A breather of the sine–Gordon equation gives a *pulse solution* of the short-pulse equation:

$$\begin{cases} u(y,t) = 4mn\dfrac{m\sin\psi\sinh\phi + n\cos\psi\cosh\phi}{m^2\sin^2\psi + n^2\cosh^2\phi} = u\left(y-\frac{\pi}{m}, t+\frac{\pi}{m}\right), \\ x(y,t) = y + 2mn\dfrac{m\sin 2\psi - n\sinh 2\phi}{m^2\sin^2\psi + n^2\cosh^2\phi} = x\left(y-\frac{\pi}{m}, t+\frac{\pi}{m}\right) + \frac{\pi}{m}, \end{cases}$$

where

$$\phi = m(y+t), \quad \psi = n(y-t), \quad n = \sqrt{1-m^2},$$

and $m \in \mathbb{R}$ is a free parameter.

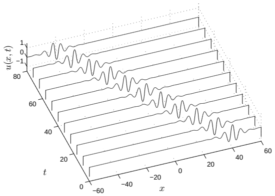

Figure: The pulse solution to the short-pulse equation with $m = 0.25$

The list of problems

The short-pulse equation

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}, \quad x \in \mathbb{R}, \quad t \in [0,T]$$

and the sine–Gordon equation in characteristic coordinates

$$w_{yt} = \sin(w), \quad y \in \mathbb{R}, \quad t \in [0,T].$$

- Local existence of solutions of the Cauchy problem
- Criteria for global solutions
- Criteria for wave breaking in a finite time
- Orbital and asymptotic stability of pulse solutions

Local well-posedness of the short-pulse equation

Theorem (Schäfer & Wayne, 2004)

Let $u_0 \in H^s$, $s > 3/2$. There exists a maximal existence time $T = T(u_0) > 0$ and a unique solution to the short-pulse equation

$$u(t) \in C([0,T), H^s) \cap C^1([0,T), H^{s-1})$$

that satisfies $u(0) = u_0$ and depends continuously on u_0.

Remarks:

- The proof of Schäfer & Wayne was only developed for $s = 2$.
- There is a constraint on solutions of the short-pulse equation

$$\int_{\mathbb{R}} u(x,t)dx = 0, \quad t > 0,$$

but this constraint was not taken into account.

Conserved quantities of the short-pulse equation

A bi-infinite hierarchy of conserved quantities of the short-pulse equation was found in Brunelli [J.Math.Phys. **46**, 123507 (2005)]:

$$
\begin{aligned}
&\dots \\
E_{-1} &= \int_{\mathbb{R}} \left(\frac{1}{24} u^4 - \frac{1}{2} (\partial_x^{-1} u)^2 \right) dx, \\
E_0 &= \int_{\mathbb{R}} u^2 dx, \\
E_1 &= \int_{\mathbb{R}} \frac{u_x^2}{1 + \sqrt{1 + u_x^2}} dx, \\
E_2 &= \int_{\mathbb{R}} \frac{u_{xx}^2}{(1 + u_x^2)^{5/2}} dx, \\
&\dots
\end{aligned}
$$

Theorem (P. & Sakovich, 2010)

Let $u_0 \in H^2$ and the conserved quantities satisfy $2E_1 + E_2 < 1$. Then the short-pulse equation admits a unique solution $u(t) \in C(\mathbb{R}_+, H^2)$ with $u(0) = u_0$.

The values of E_0, E_1 and E_2 are bounded by $\|u_0\|_{H^2}$ as follows:

$$\begin{aligned} E_0 &= \int_{\mathbb{R}} u^2 dx = \|u_0\|^2_{L^2}, \\ E_1 &= \int_{\mathbb{R}} \frac{u_x^2}{1+\sqrt{1+u_x^2}} dx \leq \frac{1}{2}\|u_0'\|^2_{L^2}, \\ E_2 &= \int_{\mathbb{R}} \frac{u_{xx}^2}{(1+u_x^2)^{5/2}} dx \leq \|u_0''\|^2_{L^2}. \end{aligned}$$

The existence time $T > 0$ of the local solutions is inverse proportional to the norm $\|u_0\|_{H^2}$ of the initial data. To extend T to ∞, we need to control the norm $\|u(t)\|_{H^2}$ by a T-independent constant on $[0, T]$.

Sketch of the proof

- Let $\tilde{q}(x,t) = \frac{u_x}{\sqrt{1+u_x^2}}$. Then, we obtain

$$\|\tilde{q}(t)\|_{H^1} \leq \sqrt{2E_1 + E_2} < 1, \quad t \in [0,T).$$

- Thanks to Sobolev's embedding $\|\tilde{q}\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|\tilde{q}\|_{H^1} < 1$, so that $u_x = \frac{\tilde{q}}{\sqrt{1-\tilde{q}^2}}$ satisfies the bound

$$\|u_x(t)\|_{H^1} \leq \frac{\|\tilde{q}\|_{H^1}}{\sqrt{1-\|\tilde{q}\|_{H^1}^2}}, \quad t \in [0,T)$$

or equivalently

$$\|u(t)\|_{H^2} \leq \left(E_0 + \frac{2E_1 + E_2}{1-(2E_1+E_2)}\right)^{1/2}, \quad t \in [0,T).$$

Sharper condition for global well-posedness

Corollary

Let $u_0 \in H^2$ such that $2\sqrt{2E_1E_2} < 1$. Then the short-pulse equation admits a unique solution $u(t) \in C(\mathbb{R}_+, H^2)$ with $u(0) = u_0$.

Let $\alpha \in \mathbb{R}_+$ be an arbitrary parameter. If $u(x,t)$ is a solution of the short-pulse equation, then $U(X,T)$ is also a solution with

$$X = \alpha x, \quad T = \alpha^{-1} t, \quad U(X,T) = \alpha u(x,t).$$

The scaling invariance yields transformation $\tilde{E}_1 = \alpha E_1$ and $\tilde{E}_2 = \alpha^{-1} E_2$. For a given $u_0 \in H^2$, a family of initial data $U_0 \in H^2$ satisfies

$$\phi(\alpha) = 2\tilde{E}_1 + \tilde{E}_2 = 2\alpha E_1 + \alpha^{-1} E_2 \geq 2\sqrt{2E_1E_2}, \quad \forall \alpha \in \mathbb{R}_+.$$

If $2\sqrt{2E_1E_2} < 1$, there exists α such that $U(X,T)$ is defined for any $T \in \mathbb{R}_+$.

Let $\mathbb{S}$ be the unit circle and let ∂_x^{-1} be the mean-zero anti-derivative

$$\partial_x^{-1} u = \int_0^x u(x', t) dx' - \int_{\mathbb{S}} \int_0^x u(x', t) dx' dx.$$

The short-pulse equation on a circle is given by

$$\begin{cases} u_t = \frac{1}{2} u^2 u_x + \partial_x^{-1} u, \\ u(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{S}, \quad t \geq 0.$$

Let $u(t) \in C([0,T), H^s(\mathbb{S})) \cap C^1([0,T), H^{s-1}(\mathbb{S}))$ be a local solution such that $u(0) = u_0 \in H^s(\mathbb{S})$.

- The assumption $\int_{\mathbb{S}} u_0(x) dx = 0$ is necessary for existence.
- The following quantities are constant on $[0,T)$:

$$E_0 = \int_{\mathbb{S}} u^2 dx, \quad E_1 = \int_{\mathbb{S}} \sqrt{1 + u_x^2} dx$$

Finite-time blow-up scenario

Lemma

Let $u_0 \in H^2(\mathbb{S})$ and $u(t)$ be a local solution of the Cauchy problem. The solution blows up in a finite time $T < \infty$ in the sense $\lim_{t\uparrow T} \|u(\cdot,t)\|_{H^2} = \infty$ if and only if

$$\lim_{t\uparrow T} \sup_{x\in\mathbb{S}} u(x,t)u_x(x,t) = +\infty.$$

For the inviscid Burgers equation

$$\begin{cases} u_t = \frac{1}{2}u^2 u_x, \\ u(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{S}, \quad t \geq 0.$$

the problem can be solved by the method of characteristics. The finite-time blow-up occurs for any $u_0(x) \in C^1(\mathbb{S})$ if there is a point $x_0 \in \mathbb{S}$ such that $u_0(x_0)u_0'(x_0) > 0$. The blow-up time is

$$T = \inf_{\xi\in\mathbb{S}} \left\{ \frac{1}{u_0(\xi)u_0'(\xi)} : \quad u_0(\xi)u_0'(\xi) > 0 \right\}.$$

Method of characteristics

Let $\xi \in \mathbb{S}$, $t \in [0,T)$, and denote

$$x = X(\xi,t), \quad u(x,t) = U(\xi,t), \quad \partial_x^{-1} u(x,t) = G(\xi,t).$$

At characteristics $x = X(\xi,t)$, we obtain

$$\begin{cases} \dot{X}(t) = -\frac{1}{2}U^2, \\ X(0) = \xi, \end{cases} \qquad \begin{cases} \dot{U}(t) = G, \\ U(0) = u_0(\xi), \end{cases}$$

- The map $X(\cdot,t) : \mathbb{S} \mapsto \mathbb{R}$ is an increasing diffeomorphism with

$$\partial_\xi X(\xi,t) = \exp\left(\int_0^t u(X(\xi,s),s) u_x(X(\xi,s),s) ds\right) > 0, \;\; t \in [0,T), \;\; \xi \in \mathbb{S}.$$

- The following quantities are bounded on $[0,T)$:

$$|u(x,t)| \leq \left| \int_{\xi_t}^x u_x(x,t)\, dx \right| \leq \int_{\mathbb{S}} |u_x(x,t)| dx \leq E_1$$

and

$$|\partial_x^{-1} u(x,t)| \leq \left| \int_{\tilde{\xi}_t}^x u(x,t)\, dx \right| \leq \int_{\mathbb{S}} |u(x,t)| dx \leq \sqrt{E_0}.$$

Sufficient condition for wave breaking

Theorem (Liu, P. & Sakovich, 2009)

Let $u_0 \in H^2(\mathbb{S})$ *and* $\int_{\mathbb{S}} u_0(x)\,dx = 0$*. Assume that there exists* $x_0 \in \mathbb{R}$ *such that* $u_0(x_0)u_0'(x_0) > 0$ *and*

$$\text{either} \qquad |u_0'(x_0)| > \left(\frac{E_1^2}{4E_0^{1/2}}\right)^{1/3},$$

$$|u_0(x_0)||u_0'(x_0)|^2 > E_1 + \left(2E_0^{1/2}|u_0'(x_0)|^3 - \frac{1}{2}E_1^2\right)^{1/2},$$

$$\text{or} \qquad |u_0'(x_0)| \leq \left(\frac{E_1^2}{4E_0^{1/2}}\right)^{1/3}, \quad |u_0(x_0)||u_0'(x_0)|^2 > E_1.$$

Then there exists a finite time $T \in (0,\infty)$ *such that the solution* $u(t) \in C([0,T), H^2(\mathbb{S}))$ *of the Cauchy problem blows up with the property*

$$\limsup_{t\uparrow T} \sup_{x\in\mathbb{S}} u(x,t)u_x(x,t) = +\infty, \quad \textit{while} \quad \lim_{t\uparrow T} \|u(\cdot,t)\|_{L^\infty} \leq E_1.$$

Sketch of the proof

Let $V(\xi,t) = u_x(X(\xi,t),t)$ and $W(\xi,t) = U(\xi,t)V(\xi,t)$. Then

$$\left\{ \begin{array}{rcl} \dot{V} & = & VW + U, \\ \dot{W} & = & W^2 + VG + U^2. \end{array} \right.$$

Under the conditions of the theorem, there exists $\xi_0 \in \mathbb{S}$ such that $V(\xi_0,t)$ and $W(\xi_0,t)$ satisfy the apriori estimates

$$\left\{ \begin{array}{rcl} \dot{V} & \geq & VW - E_1, \\ \dot{W} & \geq & W^2 - V\sqrt{E_0}. \end{array} \right.$$

We show that $V(\xi_0,t)$ and $W(\xi_0,t)$ go to infinity in a finite time.

Consider Gaussian initial data

$$u_0(x) = a(1 - 2bx^2)e^{-bx^2}, \quad x \in \mathbb{R},$$

where (a, b) are arbitrary and $\int_{\mathbb{R}} u_0(x)dx = 0$ is satisfied.

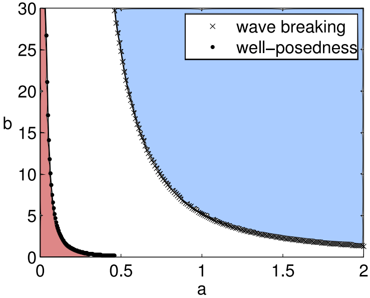

Figure: Global solutions exist below the lower curve and the wave breaking occurs above the upper curve.

Numerical simulation

Using the pseudospectral method, we solve

$$\frac{\partial}{\partial t}\hat{u}_k = -\frac{i}{k}\hat{u}_k + \frac{ik}{6}\mathcal{F}\left[\left(\mathcal{F}^{-1}\hat{u}\right)^3\right]_k, \quad k \neq 0, \quad t > 0.$$

Consider the 1-periodic initial data

$$u_0(x) = a\cos(2\pi x)$$

- Criterion for wave breaking: $a > 1.053$.
- Criterion for global solutions: $a < 0.0354$.

Evolution of the cosine initial data

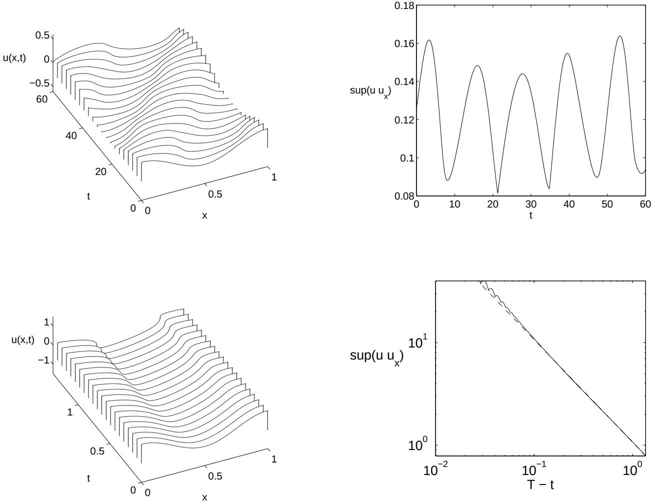

Figure: Solution surface $u(x,t)$ (left) and the supremum norm $W(t)$ (right) for $a = 0.2$ (top) and $a = 0.5$ (bottom). The dashed curve on the bottom right picture shows the linear regression with $C = 1.072$, $T = 1.356$.

Power fit

We compute the best power fit for

$$W(t) := \sup_{x \in \mathbb{S}} u(x,t) u_x(x,t)$$

according to the blow-up law

$$W(t) \simeq \frac{C}{T-t} \quad \text{for} \quad 0 < T - t \ll 1.$$

Note that the inviscid Burgers equation has the exact blow-up law $W(t) = \frac{1}{T-t}$.

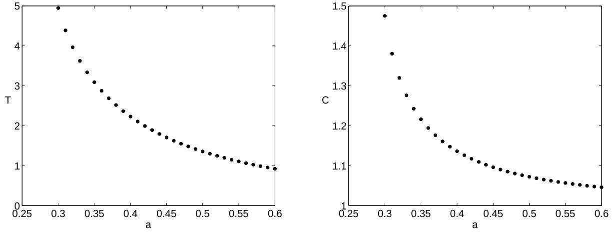

Figure: Time of wave breaking T versus a (left). Constant C of the linear regression versus a (right).

Summary of our results

- We found sufficient conditions for global well-posedness of the short-pulse equation for small initial data.
- We found sufficient conditions for wave breaking of the short-pulse equation for large initial data.
- We illustrated both global existence and wave breaking numerically.
- Numerical results suggest orbital stability of the exact modulated pulses of the short-pulse equation.