

Short-pulse equation: well-posedness and wave breaking

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AIMS Conference on Dynamical Systems, Dresden, May 26, 2010

References:

Yu. Liu, D.P., A. Sakovich, Dynamics of PDE 6, 291-310 (2009)

D.P., A. Sakovich, Communications in PDE 35, 613-629 (2010)

The **short-pulse equation** is a model for propagation of ultra-short pulses [Schäfer, Wayne 2004]:

$$u_{xt} = u + \frac{1}{6} (u^3)_{xx},$$

where all coefficients are normalized thanks to the scaling invariance.

The short-pulse equation

- originates from a scalar Maxwell's equation

$$u_{xx} = u_{tt} + u + (u^3)_{tt}$$

- replaces the nonlinear Schrödinger equation for short wave packets with few cycles on the pulse scale
- features exact solutions for modulated pulses
- enjoys inverse scattering and an infinite set of conserved quantities

Transformation to the sine-Gordon equation

Let $x = x(y, t)$ satisfy

$$\begin{cases} x_y = \cos w, \\ x_t = -\frac{1}{2}w_t^2. \end{cases}$$

Then, $w = w(y, t)$ satisfies the sine-Gordon equation in characteristic coordinates [A. Sakovich, S. Sakovich, J. Phys. A **39**, L361 (2006)]:

$$w_{yt} = \sin(w).$$

Lemma

Let $w(\cdot, t)$, $t \in [0, T]$ is C^1 in the space

$$H_c^s = \left\{ w \in H^s(\mathbb{R}) : \|w\|_{L^\infty} \leq w_c < \frac{\pi}{2} \right\}, \quad s \geq 1.$$

Then, $x(y, t)$ is invertible in y for any $t \in [0, T]$ and $u(x, t) = w_t(y(x, t), t)$ solves the short-pulse equation

$$u_{xt} = u + \frac{1}{6} (u^3)_{xx}, \quad x \in \mathbb{R}, \quad t \in [0, T].$$

Solutions of the short-pulse equation

A kink of the sine–Gordon equation gives a *loop solution* of the short-pulse equation:

$$\begin{cases} u = 2 \operatorname{sech}(y + t), \\ x = y - 2 \tanh(y + t). \end{cases}$$

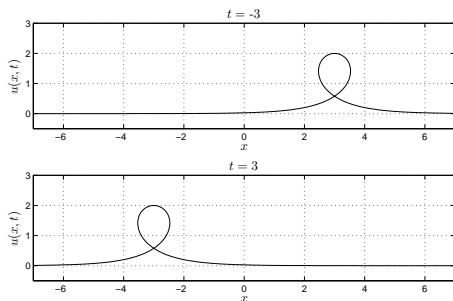


Figure: The loop solution $u(x, t)$ to the short-pulse equation

Solutions of the short-pulse equation

A breather of the sine–Gordon equation gives a *pulse solution* of the short-pulse equation:

$$\begin{cases} u(y, t) = 4mn \frac{m \sin \psi \sinh \phi + n \cos \psi \cosh \phi}{m^2 \sin^2 \psi + n^2 \cosh^2 \phi} = u\left(y - \frac{\pi}{m}, t + \frac{\pi}{m}\right), \\ x(y, t) = y + 2mn \frac{m \sin 2\psi - n \sinh 2\phi}{m^2 \sin^2 \psi + n^2 \cosh^2 \phi} = x\left(y - \frac{\pi}{m}, t + \frac{\pi}{m}\right) + \frac{\pi}{m}, \end{cases}$$

where

$$\phi = m(y + t), \quad \psi = n(y - t), \quad n = \sqrt{1 - m^2},$$

and $m \in \mathbb{R}$ is a free parameter.

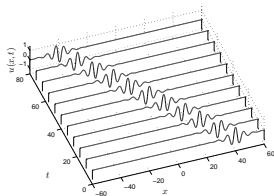


Figure: The pulse solution to the short-pulse equation with $m = 0.25$

The short-pulse equation

$$u_{xt} = u + \frac{1}{6} (u^3)_{xx}, \quad x \in \mathbb{R}, \quad t \in [0, T]$$

and the sine–Gordon equation in characteristic coordinates

$$w_{yt} = \sin(w), \quad y \in \mathbb{R}, \quad t \in [0, T].$$

- Local existence of solutions of the Cauchy problem
- Criteria for global solutions
- Criteria for wave breaking in a finite time
- Orbital and asymptotic stability of pulse solutions

Theorem (Schäfer & Wayne, 2004)

Let $u_0 \in H^s$, $s > 3/2$. There exists a maximal existence time $T = T(u_0) > 0$ and a unique solution to the short-pulse equation

$$u(t) \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$$

that satisfies $u(0) = u_0$ and depends continuously on u_0 .

Remarks:

- The proof of Schäfer & Wayne was only developed for $s = 2$.
- There is a constraint on solutions of the short-pulse equation

$$\int_{\mathbb{R}} u(x, t) dx = 0, \quad t > 0,$$

but this constraint was not taken into account.

Conserved quantities of the short-pulse equation

A bi-infinite hierarchy of conserved quantities of the short-pulse equation was found in Brunelli [J.Math.Phys. **46**, 123507 (2005)]:

$$\begin{aligned} & \dots \\ E_{-1} &= \int_{\mathbb{R}} \left(\frac{1}{24} u^4 - \frac{1}{2} (\partial_x^{-1} u)^2 \right) dx, \\ E_0 &= \int_{\mathbb{R}} u^2 dx, \\ E_1 &= \int_{\mathbb{R}} \frac{u_x^2}{1 + \sqrt{1 + u_x^2}} dx, \\ E_2 &= \int_{\mathbb{R}} \frac{u_{xx}^2}{(1 + u_x^2)^{5/2}} dx, \\ & \dots \end{aligned}$$

Theorem (P. & Sakovich, 2010)

Let $u_0 \in H^2$ and the conserved quantities satisfy $2E_1 + E_2 < 1$. Then the short-pulse equation admits a unique solution $u(t) \in C(\mathbb{R}_+, H^2)$ with $u(0) = u_0$.

The values of E_0 , E_1 and E_2 are bounded by $\|u_0\|_{H^2}$ as follows:

$$E_0 = \int_{\mathbb{R}} u^2 dx = \|u_0\|_{L^2}^2,$$

$$E_1 = \int_{\mathbb{R}} \frac{u_x^2}{1 + \sqrt{1 + u_x^2}} dx \leq \frac{1}{2} \|u'_0\|_{L^2}^2,$$

$$E_2 = \int_{\mathbb{R}} \frac{u_{xx}^2}{(1 + u_x^2)^{5/2}} dx \leq \|u''_0\|_{L^2}^2.$$

The existence time $T > 0$ of the local solutions is inverse proportional to the norm $\|u_0\|_{H^2}$ of the initial data. To extend T to ∞ , we need to control the norm $\|u(t)\|_{H^2}$ by a T -independent constant on $[0, T]$.

- Let $\tilde{q}(x, t) = \frac{u_x}{\sqrt{1+u_x^2}}$. Then, we obtain

$$\|\tilde{q}(t)\|_{H^1} \leq \sqrt{2E_1 + E_2} < 1, \quad t \in [0, T].$$

- Thanks to Sobolev's embedding $\|\tilde{q}\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|\tilde{q}\|_{H^1} < 1$, so that $u_x = \frac{\tilde{q}}{\sqrt{1-\tilde{q}^2}}$ satisfies the bound

$$\|u_x(t)\|_{H^1} \leq \frac{\|\tilde{q}\|_{H^1}}{\sqrt{1 - \|\tilde{q}\|_{H^1}^2}}, \quad t \in [0, T]$$

or equivalently

$$\|u(t)\|_{H^2} \leq \left(E_0 + \frac{2E_1 + E_2}{1 - (2E_1 + E_2)} \right)^{1/2}, \quad t \in [0, T].$$

Corollary

Let $u_0 \in H^2$ such that $2\sqrt{2E_1E_2} < 1$. Then the short-pulse equation admits a unique solution $u(t) \in C(\mathbb{R}_+, H^2)$ with $u(0) = u_0$.

Let $\alpha \in \mathbb{R}_+$ be an arbitrary parameter. If $u(x, t)$ is a solution of the short-pulse equation, then $U(X, T)$ is also a solution with

$$X = \alpha x, \quad T = \alpha^{-1}t, \quad U(X, T) = \alpha u(x, t).$$

The scaling invariance yields transformation $\tilde{E}_1 = \alpha E_1$ and $\tilde{E}_2 = \alpha^{-1}E_2$. For a given $u_0 \in H^2$, a family of initial data $U_0 \in H^2$ satisfies

$$\phi(\alpha) = 2\tilde{E}_1 + \tilde{E}_2 = 2\alpha E_1 + \alpha^{-1}E_2 \geq 2\sqrt{2E_1E_2}, \quad \forall \alpha \in \mathbb{R}_+.$$

If $2\sqrt{2E_1E_2} < 1$, there exists α such that $U(X, T)$ is defined for any $T \in \mathbb{R}_+$.

Short-pulse equation in a periodic domain

Let \mathbb{S} be the unit circle and let ∂_x^{-1} be the mean-zero anti-derivative

$$\partial_x^{-1}u = \int_0^x u(x', t)dx' - \int_{\mathbb{S}} \int_0^x u(x', t)dx' dx.$$

The short-pulse equation on a circle is given by

$$\begin{cases} u_t = \frac{1}{2}u^2u_x + \partial_x^{-1}u, \\ u(x, 0) = u_0(x), \end{cases} \quad x \in \mathbb{S}, \quad t \geq 0.$$

Let $u(t) \in C([0, T], H^s(\mathbb{S})) \cap C^1([0, T], H^{s-1}(\mathbb{S}))$ be a local solution such that $u(0) = u_0 \in H^s(\mathbb{S})$.

- The assumption $\int_{\mathbb{S}} u_0(x)dx = 0$ is necessary for existence.
- The following quantities are constant on $[0, T)$:

$$E_0 = \int_{\mathbb{S}} u^2 dx, \quad E_1 = \int_{\mathbb{S}} \sqrt{1 + u_x^2} dx$$

Lemma

Let $u_0 \in H^2(\mathbb{S})$ and $u(t)$ be a local solution of the Cauchy problem. The solution blows up in a finite time $T < \infty$ in the sense $\lim_{t \uparrow T} \|u(\cdot, t)\|_{H^2} = \infty$ if and only if

$$\limsup_{t \uparrow T} \sup_{x \in \mathbb{S}} u(x, t)u_x(x, t) = +\infty.$$

For the inviscid Burgers equation

$$\begin{cases} u_t = \frac{1}{2}u^2u_x, & x \in \mathbb{S}, \quad t \geq 0. \\ u(x, 0) = u_0(x), \end{cases}$$

the problem can be solved by the method of characteristics. The finite-time blow-up occurs for any $u_0(x) \in C^1(\mathbb{S})$ if there is a point $x_0 \in \mathbb{S}$ such that $u_0(x_0)u_0'(x_0) > 0$. The blow-up time is

$$T = \inf_{\xi \in \mathbb{S}} \left\{ \frac{1}{u_0(\xi)u_0'(\xi)} : u_0(\xi)u_0'(\xi) > 0 \right\}.$$

Let $\xi \in \mathbb{S}$, $t \in [0, T)$, and denote

$$x = X(\xi, t), \quad u(x, t) = U(\xi, t), \quad \partial_x^{-1} u(x, t) = G(\xi, t).$$

At characteristics $x = X(\xi, t)$, we obtain

$$\begin{cases} \dot{X}(t) = -\frac{1}{2}U^2, \\ X(0) = \xi, \end{cases} \quad \begin{cases} \dot{U}(t) = G, \\ U(0) = u_0(\xi), \end{cases}$$

- The map $X(\cdot, t) : \mathbb{S} \mapsto \mathbb{R}$ is an increasing diffeomorphism with

$$\partial_\xi X(\xi, t) = \exp\left(\int_0^t u(X(\xi, s), s)u_x(X(\xi, s), s)ds\right) > 0, \quad t \in [0, T), \quad \xi \in \mathbb{S}.$$

- The following quantities are bounded on $[0, T)$:

$$|u(x, t)| \leq \left| \int_{\xi_t}^x u_x(x, t) dx \right| \leq \int_{\mathbb{S}} |u_x(x, t)| dx \leq E_1$$

and

$$|\partial_x^{-1} u(x, t)| \leq \left| \int_{\tilde{\xi}_t}^x u(x, t) dx \right| \leq \int_{\mathbb{S}} |u(x, t)| dx \leq \sqrt{E_0}.$$

Theorem (Liu, P. & Sakovich, 2009)

Let $u_0 \in H^2(\mathbb{S})$ and $\int_{\mathbb{S}} u_0(x) dx = 0$. Assume that there exists $x_0 \in \mathbb{R}$ such that $u_0(x_0)u_0'(x_0) > 0$ and

$$\text{either} \quad |u_0'(x_0)| > \left(\frac{E_1^2}{4E_0^{1/2}} \right)^{1/3},$$

$$|u_0(x_0)||u_0'(x_0)|^2 > E_1 + \left(2E_0^{1/2}|u_0'(x_0)|^3 - \frac{1}{2}E_1^2 \right)^{1/2},$$

$$\text{or} \quad |u_0'(x_0)| \leq \left(\frac{E_1^2}{4E_0^{1/2}} \right)^{1/3}, \quad |u_0(x_0)||u_0'(x_0)|^2 > E_1.$$

Then there exists a finite time $T \in (0, \infty)$ such that the solution $u(t) \in C([0, T), H^2(\mathbb{S}))$ of the Cauchy problem blows up with the property

$$\limsup_{t \uparrow T} \sup_{x \in \mathbb{S}} u(x, t)u_x(x, t) = +\infty, \quad \text{while} \quad \lim_{t \uparrow T} \|u(\cdot, t)\|_{L^\infty} \leq E_1.$$

Let $V(\xi, t) = u_x(X(\xi, t), t)$ and $W(\xi, t) = U(\xi, t)V(\xi, t)$. Then

$$\begin{cases} \dot{V} &= VW + U, \\ \dot{W} &= W^2 + VG + U^2. \end{cases}$$

Under the conditions of the theorem, there exists $\xi_0 \in \mathbb{S}$ such that $V(\xi_0, t)$ and $W(\xi_0, t)$ satisfy the a priori estimates

$$\begin{cases} \dot{V} &\geq VW - E_1, \\ \dot{W} &\geq W^2 - V\sqrt{E_0}. \end{cases}$$

We show that $V(\xi_0, t)$ and $W(\xi_0, t)$ go to infinity in a finite time.

Criteria of well-posedness and wave breaking

Consider Gaussian initial data

$$u_0(x) = a(1 - 2bx^2)e^{-bx^2}, \quad x \in \mathbb{R},$$

where (a, b) are arbitrary and $\int_{\mathbb{R}} u_0(x) dx = 0$ is satisfied.

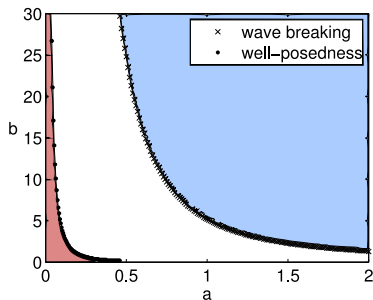


Figure: Global solutions exist below the lower curve and the wave breaking occurs above the upper curve.

Using the pseudospectral method, we solve

$$\frac{\partial}{\partial t} \hat{u}_k = -\frac{i}{k} \hat{u}_k + \frac{ik}{6} \mathcal{F} \left[(\mathcal{F}^{-1} \hat{u})^3 \right]_k, \quad k \neq 0, \quad t > 0.$$

Consider the 1-periodic initial data

$$u_0(x) = a \cos(2\pi x)$$

- Criterion for wave breaking: $a > 1.053$.
- Criterion for global solutions: $a < 0.0354$.

Evolution of the cosine initial data

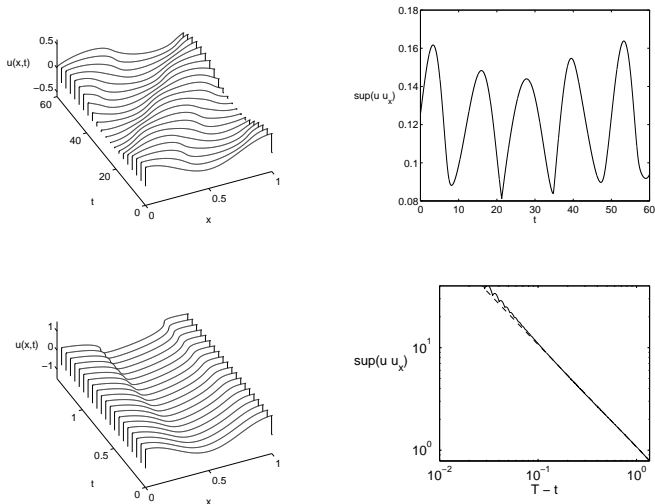


Figure: Solution surface $u(x, t)$ (left) and the supremum norm $W(t)$ (right) for $a = 0.2$ (top) and $a = 0.5$ (bottom). The dashed curve on the bottom right picture shows the linear regression with $C = 1.072$, $T = 1.356$.

We compute the best power fit for

$$W(t) := \sup_{x \in \mathbb{S}} u(x, t) u_x(x, t)$$

according to the blow-up law

$$W(t) \simeq \frac{C}{T-t} \quad \text{for } 0 < T-t \ll 1.$$

Note that the inviscid Burgers equation has the exact blow-up law

$$W(t) = \frac{1}{T-t}.$$

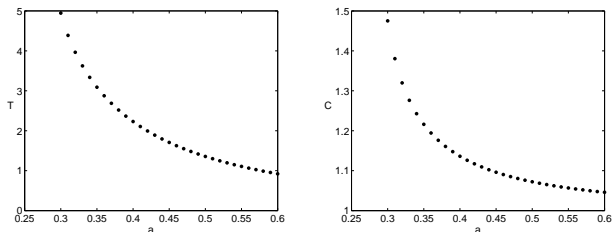


Figure: Time of wave breaking T versus a (left). Constant C of the linear regression versus a (right).

- We found sufficient conditions for global well-posedness of the short-pulse equation for small initial data.
- We found sufficient conditions for wave breaking of the short-pulse equation for large initial data.
- We illustrated both global existence and wave breaking numerically.
- Numerical results suggest orbital stability of the exact modulated pulses of the short-pulse equation.