Short-pulse equation: well-posedness and wave breaking

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References:
Yu. Liu, D.P., A. Sakovich, Dynamics of PDE 6, 291-310 (2009)
D.P., A. Sakovich, Communications in PDE 35, 613-629 (2010)

Properties of the short-pulse equation

The **short-pulse equation** is a model for propagation of ultra-short pulses with few cycles on the pulse scale [Schäfer, Wayne 2004]:

$$u_{xt} = u + \frac{1}{6} \left(u^3\right)_{xx},$$

where all coefficients are normalized thanks to the scaling invariance.

The short-pulse equation

- originates from a scalar Maxwell's equation

$$u_{xx} = u_{tt} + u + (u^3)_{tt}$$

- replaces the nonlinear Schrödinger equation for short wave packets
- features exact solutions for modulated pulses
- enjoys inverse scattering and an infinite set of conserved quantities

Transformation to the sine-Gordon equation

Let $x = x(y,t)$ satisfy

$$\begin{cases} x_y = \cos w, \\ x_t = -\frac{1}{2} w_t^2. \end{cases}$$

Then, $w = w(y,t)$ satisfies the sine–Gordon equation in characteristic coordinates [A. Sakovich, S. Sakovich, J. Phys. A **39**, L361 (2006)]:

$$w_{yt} = \sin(w).$$

Lemma

Let the mapping $[0,T] \ni t \mapsto w(\cdot,t) \in H^s_c$ *be* C^1 *and*

$$H^s_c = \left\{ w \in H^s(\mathbb{R}) : \quad \|w\|_{L^\infty} \le w_c < \frac{\pi}{2} \right\}, \quad s \ge 1.$$

Then, $x(y,t)$ *is invertible in* y *for any* $t \in [0,T]$ *and* $u(x,t) = w_t(y(x,t),t)$ *solves the short-pulse equation*

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}, \quad x \in \mathbb{R}, \quad t \in [0,T].$$

Solutions of the short-pulse equation

A kink of the sine–Gordon equation gives a *loop solution* of the short-pulse equation:

$$\begin{cases} u = 2\,\mathrm{sech}(y+t), \\ x = y - 2\tanh(y+t). \end{cases}$$

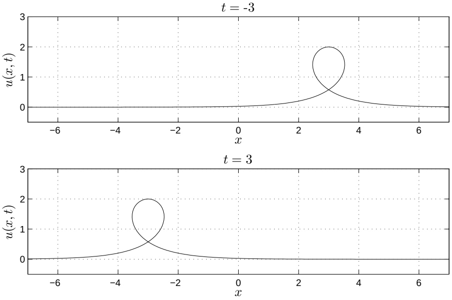

Figure: The loop solution $u(x,t)$ to the short-pulse equation

Solutions of the short-pulse equation

A breather of the sine–Gordon equation gives a *pulse solution* of the short-pulse equation:

$$\begin{cases} u(y,t) = 4mn\dfrac{m\sin\psi\sinh\phi + n\cos\psi\cosh\phi}{m^2\sin^2\psi + n^2\cosh^2\phi} = u\left(y - \frac{\pi}{m}, t + \frac{\pi}{m}\right), \\ x(y,t) = y + 2mn\dfrac{m\sin 2\psi - n\sinh 2\phi}{m^2\sin^2\psi + n^2\cosh^2\phi} = x\left(y - \frac{\pi}{m}, t + \frac{\pi}{m}\right) + \frac{\pi}{m}, \end{cases}$$

where

$$\phi = m(y+t), \quad \psi = n(y-t), \quad n = \sqrt{1-m^2},$$

and $m \in \mathbb{R}$ is a free parameter.

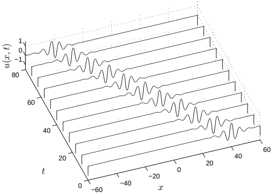

Figure: The pulse solution to the short-pulse equation with $m = 0.25$

The list of problems

The short-pulse equation

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}, \quad x \in \mathbb{R}, \quad t \in [0, T]$$

and the sine–Gordon equation in characteristic coordinates

$$w_{yt} = \sin(w), \quad y \in \mathbb{R}, \quad t \in [0, T].$$

- Local existence of solutions of the Cauchy problem
- Criteria for existence of global solutions
- Criteria for wave breaking in a finite time
- Orbital and asymptotic stability of modulated pulse solutions

Context with other recent works

- A. Stefanov [J. Diff. Eqs. (2010)] considered a family of the generalized short-pulse equations

$$u_{xt} = u + (u^p)_{xx}$$

  and proved global existence and scattering to zero for small initial data if $p \geq 4$.

- Y. Liu, D.P., & A. Sakovich [SIAM J. Math. Anal. (2010)] proved wave breaking for sufficiently large initial data if $p = 2$ but found no proof of global existence for small initial data.

- C. Holliman & A. Himonas [Diff. Int. Eqs. (2010)] proved the lack of continuity with respect to initial data (no local well-posedness) for the Hunter-Saxton equation

$$u_{xt} = (u_x)^2 - (u^2)_{xx}.$$

Remark: The cubic case $p = 2$ is a critical, for which the existence of the modulated pulse solutions implies no scattering to zero for small initial data. Global existence and wave breaking coexist for small and large initial data.

Local well-posedness of the short-pulse equation

Theorem (Schäfer & Wayne, 2004)

Let $u_0 \in H^s$, $s > 3/2$. There exists a maximal existence time $T = T(u_0) > 0$ and a unique solution to the short-pulse equation

$$u(t) \in C([0,T), H^s) \cap C^1([0,T), H^{s-1})$$

that satisfies $u(0) = u_0$ and depends continuously on u_0.

Remarks:

- The proof of Schäfer & Wayne was only developed for $s = 2$.
- There is a constraint on solutions of the short-pulse equation

$$\int_{\mathbb{R}} u(x,t)dx = 0, \quad t > 0,$$

  but this constraint was not taken into account.

Consider the Cauchy problem for the sine-Gordon equation

$$\left\{ \begin{array}{ll} w_{yt} = \sin w, & y \in \mathbb{R}, \quad t > 0 \\ w|_{t=0} = w_0, & y \in \mathbb{R}. \end{array} \right.$$

Note: if $w \in C^1([0,T), H^s(\mathbb{R}))$, $s > \frac{1}{2}$, then

$$\int_{\mathbb{R}} \sin w(y,t) dy = 0, \quad t \in (0,T).$$

The standard method of Picard–Kato would not work because if $w(\cdot,t) \in H^s$, $s > \frac{1}{2}$, then $\sin(w(\cdot,t)) \in H^s$, but $\partial_y^{-1} \sin(w(y,t))dy$ may not be in H^s.

Let $q = \sin(w)$ and rewrite the Cauchy problem in the equivalent form

$$\left\{ \begin{array}{l} q_t = (1 - f(q))\partial_y^{-1} q, \\ q|_{t=0} = q_0, \end{array} \right.$$

where

$$f(q) := 1 - \sqrt{1 - q^2} = \frac{q^2}{1 + \sqrt{1 - q^2}}, \quad \forall |q| \leq 1 : \quad \frac{q^2}{2} \leq f(q) \leq q^2.$$

Consider the initial-value problem

$$\begin{cases} q_t = (1 - f(q))\partial_y^{-1} q, \\ q|_{t=0} = q_0. \end{cases}$$

Now the constraints are

$$\|q(\cdot, t)\|_{L^\infty} < 1, \quad \int_{\mathbb{R}} q(y,t)dy = 0, \quad t > 0.$$

Theorem

Assume that $q_0 \in X_c^s$, $s > \frac{1}{2}$, where

$$X_c^s = \left\{ q \in H^s \cap \dot{H}^{-1}, \ \|q\|_{L^\infty} \le q_c < 1 \right\}.$$

There exist a maximal time $T = T(q_0) > 0$ and a unique solution $q(t) \in C([0,T), X_c^s)$ of the Cauchy problem that satisfies $q(0) = q_0$ and depends continuously on q_0.

Duhamel's method

Consider the Cauchy problem for the linearized sine–Gordon equation

$$\begin{cases} Q_t = \partial_y^{-1} Q, \\ Q|_{t=0} = Q_0. \end{cases}$$

Denote

$$L = \partial_y^{-1} \quad \text{and} \quad Q(t) = e^{tL} Q_0.$$

The solution operator e^{tL} is an *isometry* from H^s to H^s for any $s \geq 0$, so that

$$\|Q(t)\|_{H^s} = \|e^{tL} Q_0\|_{H^s} = \|Q_0\|_{H^s}, \quad \forall t \in \mathbb{R}.$$

By Duhamel's principle, we have

$$q(t) = e^{tL} q_0 - \int_0^t e^{(t-t')L} f(q(t'))\, \partial_y^{-1} q \, dt'.$$

Sketch of the proof

Fix $q_c \in (0,1)$, $\delta > 0$ and $\alpha \in (0,1)$ so that the initial data satisfy

$$\|q_0\|_{X^s} \leq \alpha\delta, \quad \|q_0\|_{L^\infty} \leq \alpha q_c$$

We need to show that there exists $T > 0$ such that

- the mapping

$$(Aq)(t) = \int_0^t e^{(t-t')L} f(q(t'))\, \partial_y^{-1} q\, dt' : \quad C([0,T], X_c^s) \mapsto C([0,T], X_c^s)$$

  is Lipschitz continuous and a contraction for sufficiently small $T > 0$.

- The integral equation is well-defined in

$$\|q(t)\|_{X^s} \leq \delta, \quad \|q(t)\|_{L^\infty} \leq q_c, \quad t \in [0,T].$$

Existence, uniqueness, and continuous dependence come from the standard Banach's Fixed-Point Theorem.

More details on the proof

The first estimate is easy:

$$\begin{aligned}\|q(t)\|_{H^s} &\leq \|e^{tL}q_0\|_{H^s} + \int_0^t \|e^{(t-t')L}f(q(t'))p(t')\|_{H^s}dt' \\ &\leq \|q_0\|_{H^s} + C_s\int_0^t \|f(q(t'))\|_{H^s}\|p(t')\|_{H^s}dt'.\end{aligned}$$

The second estimate is more difficult (recall that $L = \partial_y^{-1}$):

$$\|\partial_y^{-1}q(t)\|_{L^2} \leq \|\partial_y^{-1}e^{t\partial_y^{-1}}q_0\|_{L^2} + \int_0^t \|\partial_y^{-1}e^{(t-t')\partial_y^{-1}}f(q(t'))\partial_y^{-1}q(t')\|_{L^2}dt',$$

where we would need to use

$$Le^{(t-t')L}f(q(t'))p(t') = -\int_y^\infty J_0(2\sqrt{(t-t')(y'-y)})f(q(y',t'))p(y',t')dy',$$

as well as Hausdorf–Young's and Hölder's inequalities

$$\|Le^{(t-t')L}f(q(t'))p(t')\|_{L^2} \leq \|J_{t-t'}\|_{L^\infty}\|f(q(t'))p(t')\|_{L^{2/3}} \leq \|f(q(t'))\|_{L^1}\|p(t')\|_{L^2}.$$

Our local well-posedness of the short-pulse equation

Theorem (P., Sakovich, 2010)

Let $u_0 \in H^s \cap \dot{H}^{-1}$*,* $s > 3/2$*. There exists a maximal existence time* $T = T(u_0) > 0$ *and a unique solution to the short-pulse equation*

$$u(t) \in C^1([0,T), H^s \cap \dot{H}^{-1})$$

that satisfies $u(0) = u_0$ *and depends continuously on* u_0*.*

This theorem follows from the local well-posedness of the sine–Gordon equation and the correspondence

$$u = w_t = \frac{q_t}{\sqrt{1-q^2}} = p, \quad u_x = \frac{w_{ty}}{\cos(w)} = \tan(w) = \frac{p_y}{\sqrt{1-q^2}}.$$

Conserved quantities of the short-pulse equation

A bi-infinite hierarchy of conserved quantities of the short-pulse equation was found in Brunelli [J.Math.Phys. **46**, 123507 (2005)]:

$$
\begin{aligned}
&\dots \\
E_{-1} &= \int_{\mathbb{R}} \left(\frac{1}{24} u^4 - \frac{1}{2} (\partial_x^{-1} u)^2 \right) dx, \\
E_0 &= \int_{\mathbb{R}} u^2 dx, \\
E_1 &= \int_{\mathbb{R}} \frac{u_x^2}{1 + \sqrt{1 + u_x^2}} dx, \\
E_2 &= \int_{\mathbb{R}} \frac{u_{xx}^2}{(1 + u_x^2)^{5/2}} dx, \\
&\dots
\end{aligned}
$$

Balance equations for the conserved quantities:

$$\begin{aligned}
\partial_t \left(u^2\right) &= \partial_x \left(v^2 + \frac{1}{4}u^4\right), \\
\partial_t \left(\sqrt{1+u_x^2} - 1\right) &= \frac{1}{2}\partial_x \left(u^2\sqrt{1+u_x^2}\right), \\
\partial_t \left(\frac{u_{xx}^2}{\sqrt{(1+u_x^2)^5}}\right) &= \partial_x \left(\frac{2u_x^2}{\sqrt{1+u_x^2}} - \frac{u^2 u_{xx}^2}{2\sqrt{(1+u_x^2)^5}}\right),
\end{aligned}$$

where $v = \partial_x^{-1} u = u_t - \frac{1}{2}u^2 u_x$ and $u(t) \in C^1([0,T), H^2)$.

Thanks to the relation to the sine–Gordon equation, we obtain

$$\frac{1}{2}uu_{xx} - u_x^2 = \frac{u_{xt}}{u} - 1 = \tan^2(w) = \frac{q^2}{1-q^2},$$

so that $uu_{xx} \to 0$ as $|x| \to \infty$ if $q(t) \in C([0,T), X_c^s)$, $s > \frac{1}{2}$.

Theorem (P. & Sakovich, 2010)

Let $u_0 \in H^2$ *and the conserved quantities satisfy* $2E_1 + E_2 < 1$. *Then the short-pulse equation admits a unique solution* $u(t) \in C(\mathbb{R}_+, H^2)$ *with* $u(0) = u_0$.

The values of E_0, E_1 and E_2 are bounded by $\|u_0\|_{H^2}$ as follows:

$$\begin{aligned} E_0 &= \int_{\mathbb{R}} u^2 dx = \|u_0\|_{L^2}^2, \\ E_1 &= \int_{\mathbb{R}} \frac{u_x^2}{1+\sqrt{1+u_x^2}} dx \leq \frac{1}{2}\|u_0'\|_{L^2}^2, \\ E_2 &= \int_{\mathbb{R}} \frac{u_{xx}^2}{(1+u_x^2)^{5/2}} dx \leq \|u_0''\|_{L^2}^2. \end{aligned}$$

The existence time $T > 0$ of the local solutions is inverse proportional to the norm $\|u_0\|_{H^2}$ of the initial data. To extend T to ∞, we need to control the norm $\|u(t)\|_{H^2}$ by a T-independent constant on $[0, T]$.

Sketch of the proof

- Let $\tilde{q}(x,t) = \frac{u_x}{\sqrt{1+u_x^2}}$. Then, we obtain

$$\|\tilde{q}(t)\|_{H^1} \leq \sqrt{2E_1 + E_2} < 1, \quad t \in [0,T).$$

- Thanks to Sobolev's embedding $\|\tilde{q}\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|\tilde{q}\|_{H^1} < 1$, so that $u_x = \frac{\tilde{q}}{\sqrt{1-\tilde{q}^2}}$ satisfies the bound

$$\|u_x(t)\|_{H^1} \leq \frac{\|\tilde{q}\|_{H^1}}{\sqrt{1-\|\tilde{q}\|_{H^1}^2}}, \quad t \in [0,T)$$

or equivalently

$$\|u(t)\|_{H^2} \leq \left(E_0 + \frac{2E_1 + E_2}{1-(2E_1+E_2)}\right)^{1/2}, \quad t \in [0,T).$$

Sharper condition for global well-posedness

Corollary

Let $u_0 \in H^2$ such that $2\sqrt{2E_1E_2} < 1$. Then the short-pulse equation admits a unique solution $u(t) \in C(\mathbb{R}_+, H^2)$ with $u(0) = u_0$.

Let $\alpha \in \mathbb{R}_+$ be an arbitrary parameter. If $u(x,t)$ is a solution of the short-pulse equation, then $U(X,T)$ is also a solution with

$$X = \alpha x, \quad T = \alpha^{-1} t, \quad U(X,T) = \alpha u(x,t).$$

The scaling invariance yields transformation $\tilde{E}_1 = \alpha E_1$ and $\tilde{E}_2 = \alpha^{-1} E_2$. For a given $u_0 \in H^2$, a family of initial data $U_0 \in H^2$ satisfies

$$\phi(\alpha) = 2\tilde{E}_1 + \tilde{E}_2 = 2\alpha E_1 + \alpha^{-1} E_2 \geq 2\sqrt{2E_1E_2}, \quad \forall \alpha \in \mathbb{R}_+.$$

If $2\sqrt{2E_1E_2} < 1$, there exists α such that $U(X,T)$ is defined for any $T \in \mathbb{R}_+$.

Let $\mathbb{S}$ be the unit circle and let ∂_x^{-1} be the mean-zero anti-derivative

$$\partial_x^{-1} u = \int_0^x u(x', t)dx' - \int_{\mathbb{S}} \int_0^x u(x', t)dx'dx.$$

The short-pulse equation on a circle is given by

$$\begin{cases} u_t = \frac{1}{2}u^2 u_x + \partial_x^{-1} u, \\ u(x, 0) = u_0(x), \end{cases} \quad x \in \mathbb{S}, \quad t \geq 0.$$

Let $u(t) \in C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{S}))$ be a local solution such that $u(0) = u_0 \in H^s(\mathbb{S})$.

- The assumption $\int_{\mathbb{S}} u_0(x)dx = 0$ is necessary for existence.
- The following quantities are constant on $[0, T)$:

$$E_0 = \int_{\mathbb{S}} u^2 dx, \quad E_1 = \int_{\mathbb{S}} \sqrt{1 + u_x^2} dx$$

Finite-time blow-up scenario

Lemma

Let $u_0 \in H^2(\mathbb{S})$ and $u(t)$ be a local solution of the Cauchy problem. The solution blows up in a finite time $T < \infty$ in the sense $\lim_{t\uparrow T} \|u(\cdot,t)\|_{H^2} = \infty$ if and only if

$$\lim_{t\uparrow T} \sup_{x\in\mathbb{S}} u(x,t)u_x(x,t) = +\infty.$$

For the inviscid Burgers equation

$$\begin{cases} u_t = \frac{1}{2}u^2 u_x, \\ u(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{S}, \quad t \geq 0.$$

the problem can be solved by the method of characteristics. The finite-time blow-up occurs for any $u_0(x) \in C^1(\mathbb{S})$ if there is a point $x_0 \in \mathbb{S}$ such that $u_0(x_0)u_0'(x_0) > 0$. The blow-up time is

$$T = \inf_{\xi\in\mathbb{S}} \left\{ \frac{1}{u_0(\xi)u_0'(\xi)} : \quad u_0(\xi)u_0'(\xi) > 0 \right\}.$$

Method of characteristics

Let $\xi \in \mathbb{S}$, $t \in [0,T)$, and denote

$$x = X(\xi,t), \quad u(x,t) = U(\xi,t), \quad \partial_x^{-1} u(x,t) = G(\xi,t).$$

At characteristics $x = X(\xi,t)$, we obtain

$$\left\{ \begin{array}{l} \dot{X}(t) = -\frac{1}{2}U^2, \\ X(0) = \xi, \end{array} \right. \qquad \left\{ \begin{array}{l} \dot{U}(t) = G, \\ U(0) = u_0(\xi), \end{array} \right.$$

- The map $X(\cdot,t) : \mathbb{S} \mapsto \mathbb{R}$ is an increasing diffeomorphism with

  $$\partial_\xi X(\xi,t) = \exp\left(\int_0^t u(X(\xi,s),s) u_x(X(\xi,s),s) ds \right) > 0, \;\; t \in [0,T), \;\; \xi \in \mathbb{S}.$$

- The following quantities are bounded on $[0,T)$:

  $$|u(x,t)| \leq \left| \int_{\xi_t}^x u_x(x,t)\, dx \right| \leq \int_{\mathbb{S}} |u_x(x,t)| dx \leq E_1$$

  and

  $$|\partial_x^{-1} u(x,t)| \leq \left| \int_{\tilde{\xi}_t}^x u(x,t)\, dx \right| \leq \int_{\mathbb{S}} |u(x,t)| dx \leq \sqrt{E_0}.$$

Sufficient condition for wave breaking

Theorem (Liu, P. & Sakovich, 2009)

Let $u_0 \in H^2(\mathbb{S})$ *and* $\int_{\mathbb{S}} u_0(x)\,dx = 0$. *Assume that there exists* $x_0 \in \mathbb{R}$ *such that* $u_0(x_0)u_0'(x_0) > 0$ *and*

$$\text{either} \qquad |u_0'(x_0)| > \left(\frac{E_1^2}{4E_0^{1/2}}\right)^{1/3},$$

$$|u_0(x_0)||u_0'(x_0)|^2 > E_1 + \left(2E_0^{1/2}|u_0'(x_0)|^3 - \frac{1}{2}E_1^2\right)^{1/2},$$

$$\text{or} \qquad |u_0'(x_0)| \le \left(\frac{E_1^2}{4E_0^{1/2}}\right)^{1/3}, \quad |u_0(x_0)||u_0'(x_0)|^2 > E_1.$$

Then there exists a finite time $T \in (0,\infty)$ *such that the solution* $u(t) \in C([0,T), H^2(\mathbb{S}))$ *of the Cauchy problem blows up with the property*

$$\limsup_{t\uparrow T} \sup_{x\in\mathbb{S}} u(x,t)u_x(x,t) = +\infty, \quad \textit{while} \quad \lim_{t\uparrow T} \|u(\cdot,t)\|_{L^\infty} \le E_1.$$

Sketch of the proof

Let $V(\xi,t) = u_x(X(\xi,t),t)$ and $W(\xi,t) = U(\xi,t)V(\xi,t)$. Then

$$\begin{cases} \dot{V} &= VW + U, \\ \dot{W} &= W^2 + VG + U^2. \end{cases}$$

Under the conditions of the theorem, there exists $\xi_0 \in \mathbb{S}$ such that $V(\xi_0,t)$ and $W(\xi_0,t)$ satisfy the apriori estimates

$$\begin{cases} \dot{V} &\geq VW - E_1, \\ \dot{W} &\geq W^2 - V\sqrt{E_0}. \end{cases}$$

We show that $V(\xi_0,t)$ and $W(\xi_0,t)$ go to infinity in a finite time.

Consider Gaussian initial data

$$u_0(x) = a(1 - 2bx^2)e^{-bx^2}, \quad x \in \mathbb{R},$$

where (a, b) are arbitrary and $\int_{\mathbb{R}} u_0(x)dx = 0$ is satisfied.

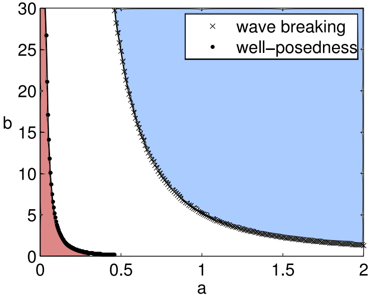

Figure: Global solutions exist below the lower curve and the wave breaking occurs above the upper curve.

Numerical simulation

Using the pseudospectral method, we solve

$$\frac{\partial}{\partial t}\hat{u}_k = -\frac{i}{k}\hat{u}_k + \frac{ik}{6}\mathcal{F}\left[\left(\mathcal{F}^{-1}\hat{u}\right)^3\right]_k, \quad k \neq 0, \quad t > 0.$$

Consider the 1-periodic initial data

$$u_0(x) = a\cos(2\pi x)$$

- Criterion for wave breaking: $a > 1.053$.
- Criterion for global solutions: $a < 0.0354$.

Evolution of the cosine initial data

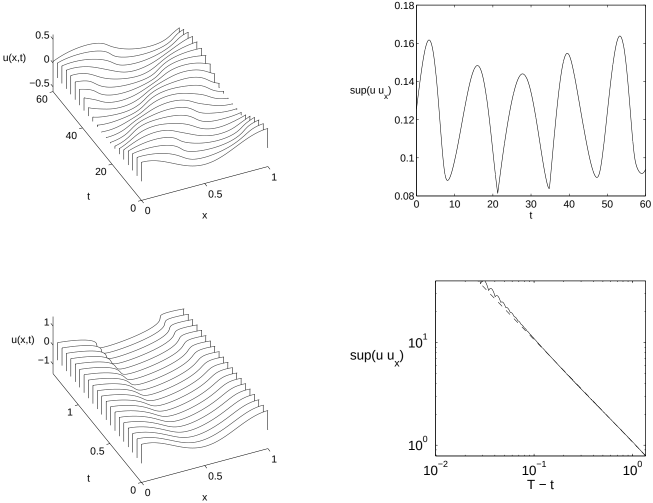

Figure: Solution surface $u(x,t)$ (left) and the supremum norm $W(t)$ (right) for $a = 0.2$ (top) and $a = 0.5$ (bottom). The dashed curve on the bottom right picture shows the linear regression with $C = 1.072$, $T = 1.356$.

Power fit

We compute the best power fit for

$$W(t) := \sup_{x \in \mathbb{S}} u(x,t) u_x(x,t)$$

according to the blow-up law

$$W(t) \simeq \frac{C}{T-t} \quad \text{for} \quad 0 < T - t \ll 1.$$

Note that the inviscid Burgers equation has the exact blow-up law $W(t) = \frac{1}{T-t}$.

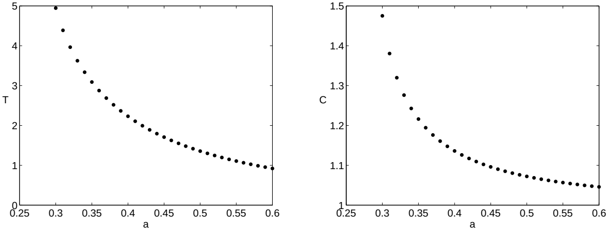

Figure: Time of wave breaking T versus a (left). Constant C of the linear regression versus a (right).

Summary of our results

- We found sufficient conditions for global well-posedness of the short-pulse equation for small initial data.
- We found sufficient conditions for wave breaking of the short-pulse equation for large initial data.
- We illustrated both global existence and wave breaking numerically.
- Numerical results suggest orbital stability of the exact modulated pulses of the short-pulse equation.