Stability of smooth travelling waves and instability of peaked travelling waves in the Camassa–Holm models

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joint work with Anna Geyer (TU Delft),
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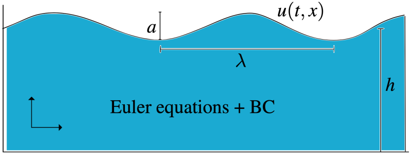
The Camassa-Holm equation

$$u_t - u_{txx} + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx} \qquad \text{(CH)}$$

models the propagation of unidirectional shallow water waves, where $u = u(t,x)$ represents the water surface. [Camassa & Holm, 1993]

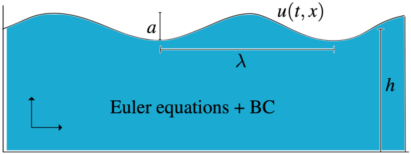


Introduction

It was extended as the Degasperis–Procesi equation

$$u_t - u_{txx} + 4\,u\,u_x = 3\,u_x u_{xx} + u\,u_{xxx} \qquad \text{(DP)}$$

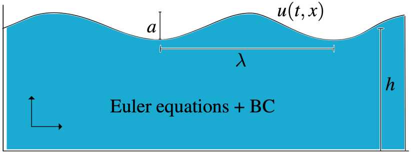
at the same asymptotic accuracy [Degasperis & Procesi, 1999]



It was further extended as the *b*-Camassa–Holm equation

$$u_t - u_{txx} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx} \qquad \text{(b-CH)}$$

by using transformations of integrable KdV equation
[Dullin, Gottwald, & Holm, 2001] [Degasperis, Holm & Hone, 2002]



Introduction

It was further extended as the b-Camassa–Holm equation

$$u_t - u_{txx} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx} \qquad \text{(b-CH)}$$

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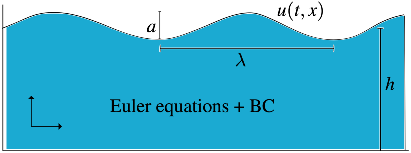
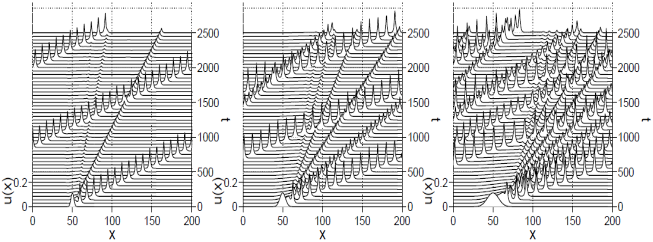


- BBM equation at small amplitudes: $u_t - u_{txx} + (b+1)\, u\, u_x = 0$
- CH and DP cases are integrable for $b = 2$ and $b = 3$.

Similations of the b-family of Camassa-Holm equations

$$u_t - u_{txx} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx}$$

starting with Gaussian initial data $u(0,x)$ [Holm & Staley, 2003]



Peaked solitary waves (*peakons*) are observed for $b > 1$

Solitary waves in b-CH model

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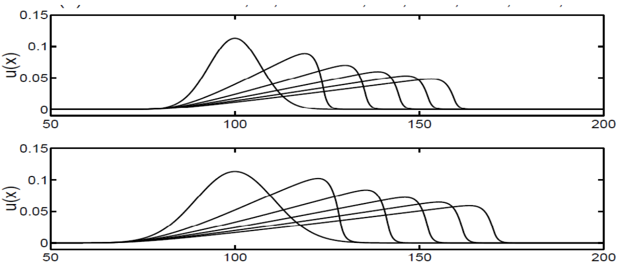

Rarefactive waves are observed for $b \in (-1, 1)$

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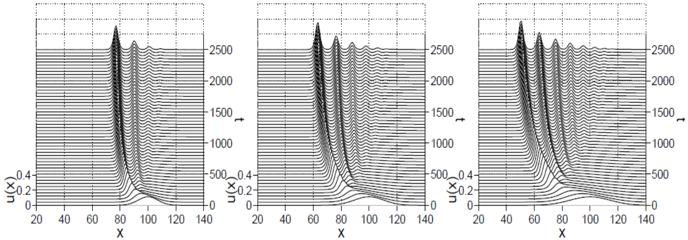

Smooth solitary waves (*leftons*) are observed for $b < -1$

Stability of solitary waves: state of the art

For solitary waves satisfying $u(x) \to 0$ as $|x| \to \infty$

- Orbital stability of peakons in energy space
  $b = 2$: [Constantin & Strauss, 2000] [Constantin & Molinet, 2001]
  $b = 3$: [Lin & Liu, 2009]

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For solitary waves satisfying $u(x) \to k$ as $|x| \to \infty$ with $k > 0$:

- Orbital stability of smooth solitons in energy space
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Similar studies were developed for travelling periodic waves (smooth or peaked) [Lenells, 2004-2006]

Stability of solitary waves: new results

▷ Linear and nonlinear instability of peakons in $H^1 \cap W^{1,\infty}$
$b = 2$: [Natali & P., 2020] [Madiyeva & P., 2021]

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  any $b \in \mathbb{R}$: [Lafortune & P., 2021]

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- Spectral and orbital stability of smooth periodic waves in H^3_{per}
  $b = 2$ [Geyer, Martins, Natali, & P., 2022]

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More work is needed for stability analysis of smooth travelling waves.

Properties of the Camassa-Holm equation

The local differential equation

$$u_t - u_{txx} + (b+1)\,u\,u_x = b\,u_x u_{xx} + u\,u_{xxx}$$

can be rewritten in the integral form of the perturbed Burgers equation

$$u_t + uu_x + \frac{1}{4}\varphi' * \left(bu^2 + (3-b)u_x^2\right) = 0,$$

where $\varphi := 2(1-\partial_x^2)^{-1}\delta = e^{-|x|}$ is the Green function.

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The model may feature wave breaking:

$$\|u(t,\cdot)\|_{L^\infty} < \infty, \quad \|u_x(t,\cdot)\|_{L^\infty} \to \infty \quad \text{as} \;\; t \to T < \infty$$

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Solutions of the Burgers equation $v_t + vv_x = 0$ with $v(0,x) = f(x)$ feature the same wave breaking:

$$v(t,x) = f(x - tv(t,x)) \quad \Rightarrow \quad v_x = \frac{f'(x-tv)}{1+tf'(x-tv)}.$$

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- locally well-posed in H^s, $s > 3/2$ [Escher & Yin, 2008; Zhou, 2010]
- no continuous dependence in H^s, $s \leq 3/2$
  [Himonas, Grayshan, Holliman (2016)] [Guo, Liu, Molinet, Yin (2018)]
- locally well-posed in $H^1 \cap W^{1,\infty}$.
  [De Lellis, Kappeler, Topalov (2007)] [Linares, Ponce, Sideris (2019)]

Hamiltonian structure of the CH equations

For $b = 2$, the Camassa–Holm equation

$$u_t - u_{txx} + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx}$$

has the first three conserved quantities

$$M(u) = \int u dx, \quad E(u) = \frac{1}{2}\int (u^2 + u_x^2) dx, \quad F(u) = \frac{1}{2}\int (u^3 + u u_x^2)\, dx.$$

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(CH) can be written in Hamiltonian form in two ways:

$$u_t = J F'(u), \quad J = -(1 - \partial_x^2)^{-1}\partial_x$$

and

$$m_t = J_m E'(m), \quad J_m = -\left(m\partial_x + \partial_x m\right),$$

where $m = u - u_{xx}$.

Hamiltonian structure of the b-CH equations

For general $b \neq 1$, the b-Camassa–Holm equation

$$u_t - u_{txx} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx}$$

has three conserved quantities

$$M(m) = \int m dx, \quad E(m) = \int m^{\frac{1}{b}} dx, \quad F(m) = \int \left(\frac{m_x^2}{b^2 m^2} + 1 \right) m^{-\frac{1}{b}} dx,$$

where $m = u - u_{xx}$.

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where $m = u - u_{xx}$.

b-CH can be written in Hamiltonian form:

$$\frac{dm}{dt} = J_m \frac{\delta M}{\delta m},$$

associated with

$$J_m := -\frac{1}{b-1} (bm\partial_x + m_x)(1 - \partial_x^2)^{-1} \partial_x^{-1} (b\partial_x m - m_x).$$

Stability of solitary waves: new results

- **Linear and nonlinear instability of peakons in $H^1 \cap W^{1,\infty}$**
  $b = 2$: [Natali & P., 2020] [Madiyeva & P., 2021]
- **Spectral instability of peakons**
  any $b \in \mathbb{R}$: [Lafortune & P., 2021]
- Spectral and orbital stability of smooth solitary waves in H^3
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  $b = 2$ [Geyer, Martins, Natali, & P., 2022]

Existence of peakons

Peakons exist in the weak form in $H^1 \cap W^{1,\infty}$

$$u(t,x) = ce^{-|x-ct|}.$$

Without loss of generality, we can set $c = 1$. The normalized profile $\varphi(x) = e^{-|x|}$ satisfies the integral equation

$$-\varphi + \frac{1}{2}\varphi^2 + \frac{1}{4}\varphi * \left(b\varphi^2 + (3-b)(\varphi')^2\right) = 0,$$

which follows from integration of

$$u_t + uu_x + \frac{1}{4}\varphi' * \left(bu^2 + (3-b)u_x^2\right) = 0,$$

after the traveling wave reduction $u(t,x) = \varphi(x-t)$.

Orbital stability of peakons: $b = 2$

Theorem (Constantin–Molinet (2001); Lenells (2005))

φ is a unique (up to translation) minimizer of $F(u)$ in H^1 subject to $E(u)$ and $M(u)$.

Theorem (Constantin–Strauss (2000); Lenells (2005))

For every small $\varepsilon > 0$, if the initial data satisfies

$$\|u_0 - \varphi\|_{H^1} < \left(\frac{\varepsilon}{3}\right)^4,$$

then the solution satisfies

$$\|u(t,\cdot) - \varphi(\cdot - \xi(t))\|_{H^1} < \varepsilon, \quad t \in (0,T),$$

where $\xi(t)$ is a point of maximum for $u(t,\cdot)$.

Nonlinear instability of peakons: $b = 2$

Consider solutions of the Cauchy problem:

$$\begin{cases} u_t + uu_x + Q[u] = 0, \\ u|_{t=0} = u_0 \in H^1 \cap W^{1,\infty}, \end{cases} \qquad Q[u] := \frac{1}{4}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right).$$

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Theorem (Natali–P. (2020); Madiyeva–P (2021))

For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H^1 \cap W^{1,\infty}$ satisfying

$$\|u_0 - \varphi\|_{H^1} + \|u_0' - \varphi'\|_{L^\infty} < \delta,$$

s.t. the unique solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ with $T > t_0$ satisfies

$$\|u_x(t_0,\cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty} > 1,$$

where $\xi(t)$ is a point of peak of $u(t,\cdot)$ for $t \in [0,T)$.

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- If $u \in H^1(\mathbb{R})$, then $Q[u] \in C(\mathbb{R})$.
- If $u \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, then $Q[u]$ is Lipschitz continuous.

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If $u(t,\cdot) \in H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{\xi(t)\})$ for $t \in [0,T)$. Then, $\xi(t) \in C^1(0,T)$ and

$$\frac{d\xi}{dt} = u(t,\xi(t)), \quad t \in (0,T).$$

For the peaked traveling wave $u(t,x) = \varphi(x - ct)$, this gives $c = \varphi(0) := \max_{x\in\mathbb{R}} \varphi(x)$.

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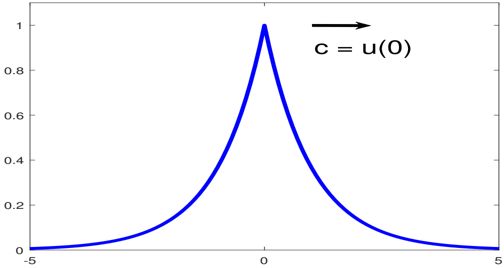
Peaked solitary wave with a single peak:



Decomposition near a single peakon

Consider a decomposition:

$$u(t,x) = \varphi(x - t - a(t)) + v(t, x - t - a(t)), \quad t \in [0, T), \quad x \in \mathbb{R},$$

with the peak at $\xi(t) = t + a(t)$ for $v(t, \cdot) \in H^1(\mathbb{R}) \cap C^1(\mathbb{R} \backslash \{\xi(t)\})$.

Then, $a'(t) = v(t, 0)$ and

$$v_t = (1 - \varphi) v_x + (v|_{x=0} - v)\varphi' + (v|_{x=0} - v) v_x - \varphi' * (\varphi v + \frac{1}{2}\varphi' v_x) - Q[v].$$

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Due to

$$[v(0) - v(x)]\varphi'(x) - \varphi' * \varphi v - \frac{1}{2}\varphi' * \varphi' v_x = \varphi(x) \int_0^x v(y)dy,$$

the evolution of $v(t,x)$ simplifies to

$$v_t = (1-\varphi)v_x + \varphi \int_0^x v(t,y)dy + (v|_{x=0} - v)v_x - Q[v].$$

Nonlinear evolution

For the evolution problem:

$$\begin{cases} v_t = (c - \varphi) v_x + \varphi \int_0^x v(t,y) dy + (v|_{x=0} - v) v_x - Q[v], \quad t \in (0,T), \\ v|_{t=0} = v_0(x), \end{cases}$$

we can look for solutions with the method of characteristic curves:

$$x = X(t,s), \qquad v(t, X(t,s)) = V(t,s).$$

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The characteristic coordinates $X(t,s)$ satisfies

$$\left\{ \begin{array}{l} \frac{dX}{dt} = \varphi(X) - 1 + v(t,X) - v(t,0), \quad t \in (0,T), \\ X|_{t=0} = s. \end{array} \right.$$

Since φ is Lipschitz, there exists the unique characteristic function $X(t,s)$ for each $s \in \mathbb{R}$ if $v(t,\cdot)$ remains in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$
The peak location $X(t,0) = 0$ is invariant in time.

Nonlinear evolution

For the evolution problem:

$$\begin{cases} v_t = (c-\varphi)v_x + \varphi \int_0^x v(t,y)dy + (v|_{x=0} - v)v_x - Q[v], & t \in (0,T), \\ v|_{t=0} = v_0(x), \end{cases}$$

we can look for solutions with the method of characteristic curves:

$$x = X(t,s), \qquad v(t, X(t,s)) = V(t,s).$$

From the right side of the peak, $V_0(t) = v(t,0)$, $W_0(t) = v_x(t,0^+)$:

$$\frac{dW_0}{dt} = W_0 + V_0 + V_0^2 - \frac{1}{2}W_0^2 - P[v](0), \quad P[v] := \varphi * \left(v^2 + \frac{1}{2}v_x^2\right).$$

We will show that $W_0(t)$ grows and may diverge in a finite time.

Proof of instability

From orbital stability in H^1 [A. Constant, W. Strauss (2000)]
If $\|v_0\|_{H^1} < (\varepsilon/3)^4$, then

$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

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From the equation on the right side of the peak:

$$\frac{dW_0}{dt} = W_0 + V_0 + V_0^2 - \frac{1}{2}W_0^2 - P[v](0)$$

and since $P[v] > 0$, we have

$$\frac{dW_0}{dt} \leq W_0 + C\varepsilon \quad \Rightarrow \quad W_0(t) \leq [W_0(0) + C\varepsilon]\, e^t$$

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If $W_0(0) = -2C\varepsilon$, then

$$W_0(t) \leq -C\varepsilon e^t,$$

hence $|W_0(t_0)| \geq 1$ for $t_0 := -\log(C\varepsilon)$.

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hence $|W_0(t_0)| \geq 1$ for $t_0 := -\log(C\varepsilon)$.

The initial constraint $\|v_0\|_{L^\infty} + \|v_0'\|_{L^\infty} < \delta$, is satisfied if $\forall \delta > 0$, $\exists \varepsilon > 0$ such that

$$\left(\frac{\varepsilon}{3}\right)^4 + 2C\varepsilon < \delta.$$

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$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|v(t,\cdot)\|_{H^1} < \varepsilon.$$

For the strong instability, we estimate

$$\frac{dW_0}{dt} = W_0 + V_0 + V_0^2 - \frac{1}{2}W_0^2 - P[v](0) \leq W_0 - \frac{1}{2}W_0^2 + C\varepsilon.$$

Proof of instability

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By the ODE comparison theory, $W_0(t) \leq \overline{W}(t)$, where the supersolution satisfies

$$\frac{d\overline{W}}{dt} = \overline{W} - \frac{1}{2}\overline{W}^2 + C\varepsilon$$

with $W_0(0) = \overline{W}(0) = -C\varepsilon$ and $\overline{W}(t) \to -\infty$ as $t \to \overline{T}$.

Illustration of the peakon instability (periodic case)

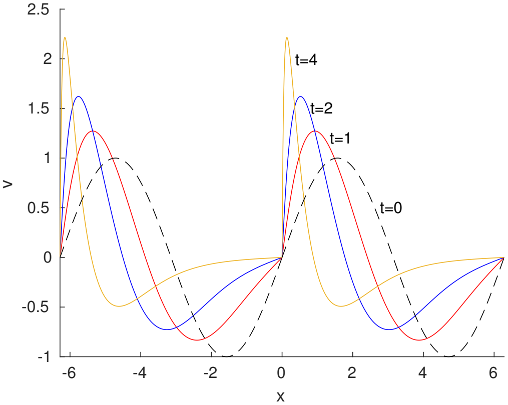

Figure: The plots of perturbation $v(t,x)$ to the peaked wave versus x on $[-2\pi, 2\pi]$ for different values of t in the case $v_0(x) = \sin(x)$.

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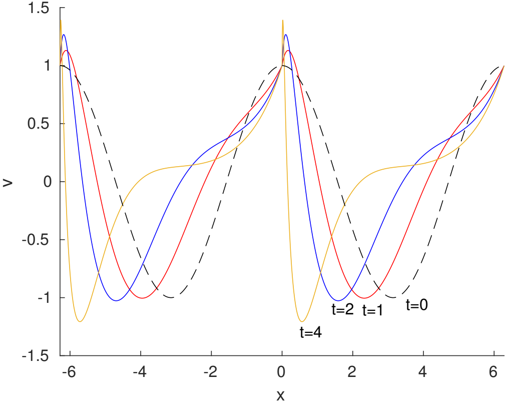

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Linearized evolution: any $b \in \mathbb{R}$

Truncation of the quadratic terms yields the linearized problem for perturbations in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$:

$$\begin{aligned} v_t = & (1-\varphi)v_x + (b-2)(v|_{x=0} - v)\varphi' \\ & + \frac{1}{2}(b-3)\varphi * (\varphi' v) - \frac{1}{2}(2b-3)\varphi' * (\varphi v), \end{aligned}$$

Question: Can we predict instability of peakons for any b from analysis of the linearized operator in $L^2(\mathbb{R})$?

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Question: Can we predict instability of peakons for any b from analysis of the linearized operator in $L^2(\mathbb{R})$?

The linearized operator is

$$L = (1-\varphi)\partial_x - (b-2)\varphi' + K,$$

where $K : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is a compact (Hilbert–Schmidt) operator. Since $\varphi \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, the natural domain of L in $L^2(\mathbb{R})$ is

$$\mathrm{Dom}(L) = \left\{ v \in L^2(\mathbb{R}) : \quad (1-\varphi)v' \in L^2(\mathbb{R}) \right\}.$$

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Question: Can we predict instability of peakons for any b from analysis of the linearized operator in $L^2(\mathbb{R})$?
Since

$$\|(1-\varphi)v'\|_{L^2} \leq \|v'\|_{L^2}$$

$H^1(\mathbb{R})$ is continuously embedded into $\mathrm{Dom}(L)$. However, it is not equivalent to $\mathrm{Dom}(L)$ because $\varphi' \in \mathrm{Dom}(L)$ but $\varphi' \notin H^1(\mathbb{R})$.

Question: How can we get redefine L from $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ to $\mathrm{Dom}(L) \subset L^2(\mathbb{R})$ to study spectral stability of peakons?

Resolution of these questions

It can be checked directly that

$$L\varphi = (2-b)\varphi' \text{ and } L\varphi' = 0.$$

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Starting with $v \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, we write

$$v = v|_{x=0}\varphi + \tilde{v} \quad \text{such that} \;\; \tilde{v}(t,0) = 0$$

and obtain the linearized equation

$$\tilde{v}_t = L\tilde{v} - \frac{3}{2}(b-2)\langle \varphi\varphi', \tilde{v}\rangle \varphi$$

Linear evolution is now well-defined for $\tilde{v} \in \mathrm{Dom}(L) \subset L^2(\mathbb{R})$ for which $\tilde{v}(t,0)$ may not exist.

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In order to reduce the linear evolution to the homogeneous equation, we use the secondary decomposition

$$\tilde{v}(t,x) = \alpha(t)\varphi(x) + \beta(t)\varphi'(x) + w(t,x)$$

and obtain $w_t = Lw$ and

$$\frac{d\alpha}{dt} = (2-b)\beta + \frac{3}{2}(2-b)\langle \phi\phi', w\rangle, \quad \frac{d\beta}{dt} = (2-b)\alpha.$$

For $b \neq 2$, we have instability of peakons in $\mathrm{Dom}(L)$ with $w = 0$. For $b = 2$, we have to analyze the spectrum of L in $L^2(\mathbb{R})$.

Spectrum of a linear operator in a Hilbert space

Let A be a linear operator on a Banach space X with $\mathrm{Dom}(A) \subset X$. The complex plane $\mathbb{C}$ is decomposed into the resolvent set $\rho(A)$ and the spectrum $\sigma(A) = \mathbb{C} \setminus \rho(A)$, the latter consists of the following three disjoint sets:

1. the point spectrum

$$\sigma_{\rm p}(A) = \{\lambda : \quad \mathrm{Ker}(A - \lambda I) \neq \{0\}\},$$

2. the residual spectrum

$$\sigma_{\rm r}(A) = \{\lambda : \quad \mathrm{Ker}(A - \lambda I) = \{0\}, \quad \mathrm{Ran}(A - \lambda I) \neq X\},$$

3. the continuous spectrum

$$\sigma_{\rm c}(A) = \{\lambda : \quad \mathrm{Ker}(A - \lambda I) = \{0\}, \quad \mathrm{Ran}(A - \lambda I) = X, \\ (A - \lambda I)^{-1} : X \to X \text{ is unbounded}\}.$$

Spectrum of a linear operator in a Hilbert space

Theorem (Lafortune–P, 2021)

The spectrum of L with $\mathrm{Dom}(L) \subset L^2(\mathbb{R})$

$$\sigma(L) = \left\{ \lambda \in \mathbb{C} : \;\; |\mathrm{Re}(\lambda)| \leq \left| \frac{5}{2} - b \right| \right\}.$$

Moreover,

- ▷ $\sigma_p(L)$ *is located for* $0 < |\mathrm{Re}(\lambda)| < \frac{5}{2} - b$ *if* $b < \frac{5}{2}$
- ▷ $\sigma_r(L)$ *is located for* $0 < |\mathrm{Re}(\lambda)| < b - \frac{5}{2}$ *if* $b > \frac{5}{2}$
- ▷ $\sigma_c(L)$ *is located for* $\mathrm{Re}(\lambda) = 0$ *and* $\mathrm{Re}(\lambda) = \pm \left| \frac{5}{2} - b \right|$
- ▷ $\lambda = 0$ *is the embedded eigenvalue for every b.*

$\Rightarrow$ the peakon is linearly unstable in $\mathrm{Dom}(L)$ for every $b \neq \frac{5}{2}$.

How do we obtain this result?

Recall that $L = L_0 + K$, where $L_0 := (1-\varphi)\partial_x - (b-2)\varphi'$ with

$$\mathrm{Dom}(L) = \mathrm{Dom}(L_0) = \left\{ v \in L^2(\mathbb{R}) : \quad (1-\varphi)v' \in L^2(\mathbb{R}) \right\}$$

and $K : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is a compact (Hilbert–Schmidt) operator.

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Theorem (Geyer & P (2020))

Let $L : \mathrm{dom}(L) \subset X \to X$ *and* $L_0 : \mathrm{dom}(L_0) \subset X \to X$ *be linear operators on Hilbert space* X *with the same domain such that* $L - L_0 = K$ *is a compact operator in* X. *Assume that the intersections* $\sigma_{\mathrm{p}}(L) \cap \rho(L_0)$ *and* $\sigma_{\mathrm{p}}(L_0) \cap \rho(L)$ *are empty. Then,* $\sigma(L) = \sigma(L_0)$.

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Theorem (Bühler & Salamon (2018))

Let $L : \mathrm{dom}(L) \subset X \to X$ *be a linear operator on Hilbert space* X *and* $L^* : \mathrm{dom}(L^*) \subset X \to X$ *be the adjoint operator. Assume that* $\sigma_{\mathrm{p}}(L)$ *is empty. Then,* $\sigma_{\mathrm{r}}(L) = \sigma_{\mathrm{p}}(L^*)$.

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$L_0 v = \lambda v$ is the first-order equation

$$(1-\varphi)\frac{dv}{dx} + (2-b)\varphi' v = \lambda v$$

with the exact solution

$$v(x) = \begin{cases} v_+ e^{\lambda x}(1-e^{-x})^{2+\lambda-b}, & x > 0, \\ v_- e^{\lambda x}(1-e^{x})^{2-\lambda-b}, & x < 0, \end{cases}$$

If $\mathrm{Re}(\lambda) > 0$, then $v_+ = 0$ and $\mathrm{Re}(\lambda) < \frac{5}{2} - b$.

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$L_0^* v = \lambda v$ is the first-order equation

$$-(1-\varphi)\frac{dv}{dx} + (3-b)\varphi' v = \lambda v$$

with the exact solution

$$v(x) = \begin{cases} v_+ e^{-\lambda x}(1-e^{-x})^{b-3-\lambda}, & x > 0, \\ v_- e^{-\lambda x}(1-e^{x})^{b-3+\lambda}, & x < 0, \end{cases}$$

If $\mathrm{Re}(\lambda) > 0$, then $v_- = 0$ and $\mathrm{Re}(\lambda) < b - \frac{5}{2}$.

CH and DP have different types of peakon instability

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$b = 2$: $\|v(t,\cdot)\|_{L^2(-\infty,0)}$ grows due to point spectrum

$b = 3$: $\|v(t,\cdot)\|_{L^2(0,\infty)}$ grows due to residual spectrum

Stability of solitary waves: new results

- Linear and nonlinear instability of peakons in $H^1 \cap W^{1,\infty}$
  $b = 2$: [Natali & P., 2020] [Madiyeva & P., 2021]

- Spectral instability of peakons
  any $b \in \mathbb{R}$: [Lafortune & P., 2021]

- **Spectral and orbital stability of smooth solitary waves in H^3**
  $b > 1$: [Lafortune & P., 2022]

- **Spectral and orbital stability of smooth periodic waves in $H^3_{\rm per}$**
  $b = 2$ [Geyer, Martins, Natali, & P., 2022]

Standard approach to orbital stability

- Construct an augmented Hamiltonian $\Lambda(u)$, such that the traveling wave solution ϕ is a critical point of Λ: $\underbrace{\Lambda'(\phi) = 0}_{\text{TW-eq}}$

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- If $\mathcal{L}$ has only one negative simple eigenvalue and a simple zero eigenvalue, then we need to prove that the traveling wave ϕ is a constrained minimizer of Hamiltonian under fixed momentum, i.e. $\mathcal{L}|_{X_0} \geq 0$, where X_0 is a constrained subspace of L^2

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- The traveling wave ϕ is orbitally stable in energy space.

Existence of smooth solitary waves

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Further integration gives

$$\frac{1}{2}(b-1)[(\phi')^2 - \phi^2] + \frac{a}{(c - \phi)^{b-1}} = g, \quad g \in \mathbb{R}.$$

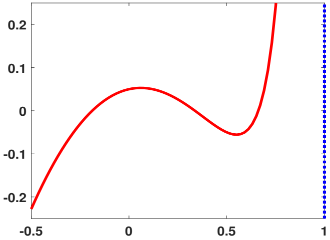
Smooth waves with $c > 0$ exist if $\phi < c$.

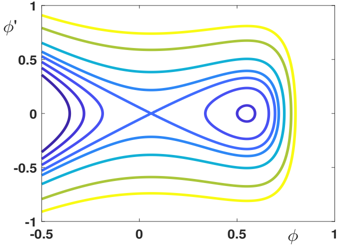
Existence of smooth solitary waves from the phase portrait

Newton's particle with mass $m = b - 1$ and potential energy $U(\phi)$

$$\frac{1}{2}(b-1)(\phi')^2 + U(\phi) = g, \quad U(\phi) = -\frac{1}{2}(b-1)\phi^2 + \frac{a}{(c-\phi)^{b-1}}.$$

For $b > 1$ and $a \in (0, a_0)$ two critical points of $U(\phi)$ exists with ordering $0 < \phi_1 < \phi_2 < c$.





Properties of smooth solitary waves

For fixed $b > 1$ and $c > 0$, the family of solitary waves have one parameter, which can be chosen as $k \in (0, k_0)$ such that

$$\phi(x) \to k \quad \text{as} \quad |x| \to \infty \;\; \text{exponentially},$$

where $k_0 := (b+1)^{-1} c$. Moreover, $0 < \phi < c$ and

$$\mu = \phi - \phi'' = k \frac{(c-k)^b}{(c-\phi)^b}$$

satisfies $0 < \mu < \infty$.

Note that $u(t,x) = k + v(t, x - kt)$ brings the b-CH equation to

$$v_t - v_{txx} + (b+1) v v_x = k v_x + b v_x v_{xx} + v v_{xxx},$$

for which solitary waves with $v(t,x) \to 0$ as $|x| \to \infty$ were considered in the literature for $b = 2$ and $b = 3$.

Hamiltonian structure of the b-CH equations

Recall that the b-Camassa–Holm equation with $b \neq 1$

$$u_t - u_{txx} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx}$$

has three conserved quantities

$$M(m) = \int m dx, \quad E(m) = \int m^{\frac{1}{b}} dx, \quad F(m) = \int \left(\frac{m_x^2}{b^2 m^2} + 1 \right) m^{-\frac{1}{b}} dx,$$

where $m = u - u_{xx}$.

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where $m = u - u_{xx}$.

The conserved quantities can be redefined as

$$\hat{E}(m) = \int_{\mathbb{R}} \left[m^{\frac{1}{b}} - k^{\frac{1}{b}} \right] dx, \quad \hat{F}(m) = \int_{\mathbb{R}} \left[\left(\frac{m_x^2}{b^2 m^2} + 1 \right) m^{-\frac{1}{b}} - k^{-\frac{1}{b}} \right] dx$$

in the set of functions with fixed $k > 0$:

$$X_k = \left\{ m - k \in H^1(\mathbb{R}) : \quad m(x) > 0, \;\; x \in \mathbb{R} \right\}.$$

Stability of smooth solitary waves

Let $m(t,x) = \mu(x-ct)$ with $\mu \in X_k$. We say that the travelling wave is orbitally stable in X_k if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $m_0 \in X_k$ satisfying $\|m_0 - \mu\|_{H^1} < \delta$, there exists a unique solution $m \in C^0(\mathbb{R}, X_k)$ of the b-CH equation satisfying

$$\inf_{x_0 \in \mathbb{R}} \|m(t,\cdot) - \mu(\cdot - x_0)\|_{H^1} < \varepsilon, \quad t \in \mathbb{R}.$$

Theorem (Lafortune–P, 2022)

For fixed $b > 1$, $c > 0$, and $k \in (0, k_0)$, there exists a unique solitary wave $m(t,x) = \mu(x-ct)$ of the b-CH equation, which is orbitally stable in X_k if the mapping

$$k \mapsto Q(\phi) := \int_{\mathbb{R}} \left[b\left(\frac{c-k}{c-\phi}\right) - \left(\frac{c-k}{c-\phi}\right)^b - b + 1 \right] dx$$

is strictly increasing.

How do we obtain this result?

1. We verify that the solitary wave $\mu \in X_k$ is a critical point of the augmented Hamiltonian

$$\Lambda_{\omega_1,\omega_2}(m) := \hat{M}(m) - \omega_1 \hat{E}(m) - \omega_2 \hat{F}(m),$$

for some (ω_1, ω_2) that depend on (b, c, k).

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2. Then, we expand

$$\Lambda_{\omega_1,\omega_2}(\mu + \tilde{m}) - \Lambda_{\omega_1,\omega_2}(\mu) = \langle \mathcal{L}\tilde{m}, \tilde{m} \rangle + \|\tilde{m}\|^3_{H^1}$$

for every small $\tilde{m} \in H^1(\mathbb{R})$ where $\mathcal{L}$ is the Sturm–Liouville operator in $L^2(\mathbb{R})$ with the dense domain $H^2(\mathbb{R})$. Since $\mathcal{L}\mu' = 0$ and $\mu'(x)$ has only one zero on $\mathbb{R}$, $\mathcal{L}$ admits exactly one simple negative eigenvalue and a simple zero eigenvalue.

How do we obtain this result?

3. We add the constraint of a conserved quantity

$$b\hat{E}(m) - k^{\frac{1}{b}-1}\hat{M}(m)$$

which restricts perturbations $\tilde{m}$ to the class

$$\langle \mu^{\frac{1}{b}-1} - k^{\frac{1}{b}-1}, \tilde{m} \rangle = 0.$$

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4. To prove that $\mathcal{L}|_{\{v_0\}^\perp} \geq 0$, we need to show that $\langle \mathcal{L}^{-1} v_0, v_0 \rangle < 0$, where $v_0 := \mu^{\frac{1}{b}-1} - k^{\frac{1}{b}-1}$. This is true if and only if the mapping

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Verification of the stability criterion

Smooth solitary waves are solutions $\phi(x) \to k$ as $|x| \to \infty$ of

$$(c - \phi)(\phi - \phi'') + \frac{1}{2}(b - 1)(\phi'^2 - \phi^2) = ck - \frac{1}{2}(b + 1)k^2.$$

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Transformations

$$\phi(x) = \psi(z), \qquad z = \int_0^x [c-\phi(x)]^{-\frac{b-1}{2}} dx$$

and

$$\psi(z) = k + (c-k)\varphi(\zeta), \quad \zeta = \sqrt{c-k(b+1)}(c-k)^{\frac{b-2}{2}} z$$

bring the second-order equation to the semi-linear term

$$-\varphi''(\zeta) + \varphi(1-\varphi)^{b-2}\left[1-(2\gamma)^{-1}(b+1)\varphi\right] = 0, \quad \gamma := \frac{c-k(b+1)}{c-k},$$

where $\gamma \in (0,1)$ replaces $k \in (0,(b+1)^{-1}c)$.

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Since

$$\frac{d\gamma}{dk} = \frac{-bc}{(c-k)^2} < 0,$$

the stability criterion is $\frac{d}{d\gamma}Q(\phi) < 0$.

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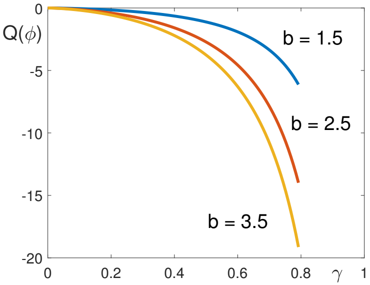
For general $b > 1$, we have proven it asymptotically as $\gamma \to 0$ and $\gamma \to 1$ and confirmed numerically:



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For $b = 2$ (CH), this can be proven for every $\gamma \in (0,1)$:

$$-\varphi''(\zeta) + \varphi\left[1 - \frac{3}{2\gamma}\varphi\right] = 0, \quad \Rightarrow \quad \varphi(\zeta) = \gamma \mathrm{sech}^2(\frac{1}{2}\zeta)$$

so that

$$\frac{d}{d\gamma}Q(\phi) = -\frac{3}{2}\gamma^{1/2}\int_{\mathbb{R}} \frac{\mathrm{sech}^4(\frac{1}{2}\zeta)}{(1-\gamma\mathrm{sech}^2(\frac{1}{2}\zeta))^{5/2}} d\zeta < 0.$$

This result complements the proof in [Constantin & Strauss, 2002]

Verification of the stability criterion

Smooth solitary waves are solutions $\phi(x) \to k$ as $|x| \to \infty$ of

$$(c-\phi)(\phi-\phi'') + \frac{1}{2}(b-1)(\phi'^2-\phi^2) = ck - \frac{1}{2}(b+1)k^2.$$

For $b = 3$ (DP), this can also be proven for every $\gamma \in (0,1)$:

$$-\varphi''(\zeta) + \varphi(1-\varphi)\left[1 - \frac{2}{\gamma}\varphi\right] = 0,$$

with the exact solution

$$\varphi(\zeta) = \frac{3\gamma}{2+\gamma+\sqrt{(1-\gamma)(4-\gamma)}\cosh(\zeta)}.$$

This result complements the proof in [Li & Liu & Wu, 2020]

Summary

We have considered the b-Camassa–Holm equation

$$u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}$$

which models unidirectional small-amplitude shallow water waves.

Summary

We have considered the b-Camassa–Holm equation

$$u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}$$

which models unidirectional small-amplitude shallow water waves.

- Peaked traveling waves are unstable in $H^1 \cap W^{1,\infty}$
  - LWP only holds in $H^1 \cap W^{1,\infty}$.
  - Perturbations are bounded in H^1 (at least for $b = 2$).
  - Perturbations grow in $W^{1,\infty}$ norm.
  - Spectral instability holds for every b.

- Smooth traveling waves are stable in H^3 for $b > 1$
  - LWP and GWP hold for perturbations with $m = u - u'' > 0$
  - Hamiltonian formulation exists for every b
  - TW is constrained minimizer of the augmented Hamiltonian.

Summary

We have considered the b-Camassa–Holm equation

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Further directions:

- Stability of smooth traveling solitary waves for $b \leq 1$.
- Stability of smooth traveling periodic waves for $b \neq 2$.
- Robustness of peaked traveling waves in spite their instability.
- Universality of instability of peaked traveling waves.
- Proof of instability of cusped travelling waves.