Traveling waves in the Camassa–Holm equations: their stability and instability

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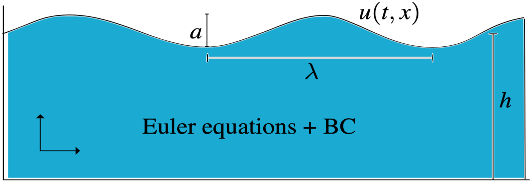
Section 1

Background and motivation

Toy models for fluids

The study of traveling waves in the irrotational motion of an incompressible fluid has a long history.



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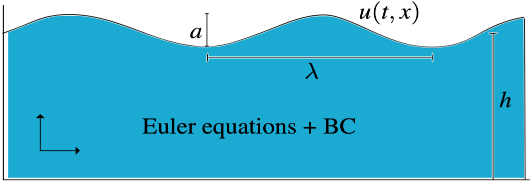

The following evolution equations were used for approximations of such traveling waves in the shallow limit $a \ll h \ll \lambda$.

The Korteweg–de Vries (KdV) equation:

$$u_t + u_x + u_{xxx} + u\, u_x = 0$$

[Boussinesq, 1872]   [Korteweg & de Vries, 1895]

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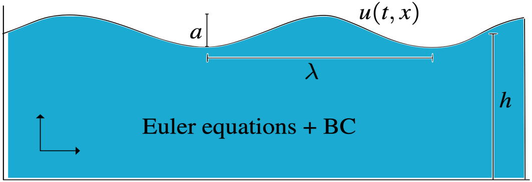

The following evolution equations were used for approximations of such traveling waves in the shallow limit $a \ll h \ll \lambda$.

The Benjamin–Bona–Mahony (BBM) equation

$$u_t + u_x - u_{txx} + u\, u_x = 0$$

[Peregrine, 1966] [Benjamin–Bona–Mahony, 1972]

Toy models for fluids

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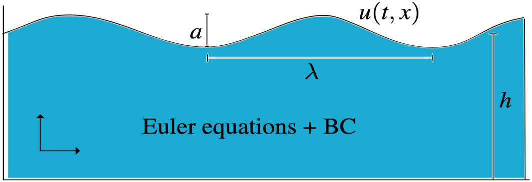


The following evolution equations were used for approximations of such traveling waves in the shallow limit $a \ll h \ll \lambda$.

The Camassa–Holm (CH) equation

$$u_t + u_x - u_{txx} + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx}$$

[Camassa & Holm, 1993] [Johnson, 2000] [Constantin & Lannes, 2009]

CH models

The Camassa-Holm equation

$$u_t + u_x - u_{txx} + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx} \qquad \text{(CH)}$$

was extended as the Degasperis–Procesi equation

$$u_t + u_x - u_{txx} + 4\,u\,u_x = 3\,u_x u_{xx} + u\,u_{xxx} \qquad \text{(DP)}$$

at the same asymptotic accuracy.

[Degasperis & Procesi, 1999] [Constantin & Lannes, 2009]

It is further extended as the b-Camassa–Holm equation

$$u_t + u_x - u_{txx} + (b+1)\,u\,u_x = b\,u_x u_{xx} + u\,u_{xxx} \qquad \text{(b-CH)}$$

by using transformations of integrable KdV equation.

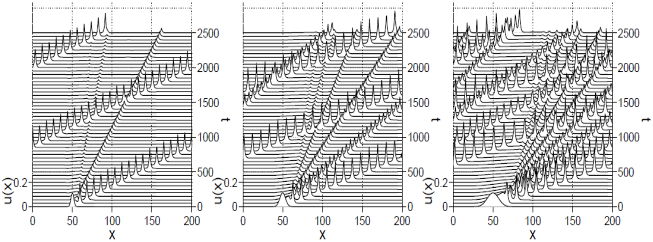
[Dullin, Gottwald, & Holm, 2001] [Degasperis, Holm & Hone, 2002]

Solitary waves in b-CH model

Similations of the b-family of Camassa-Holm equations

$$u_t - u_{txx} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx}$$

starting with Gaussian initial data $u(0,x)$ [Holm & Staley, 2003]



Peaked solitary waves (*peakons*) are observed for $b > 1$

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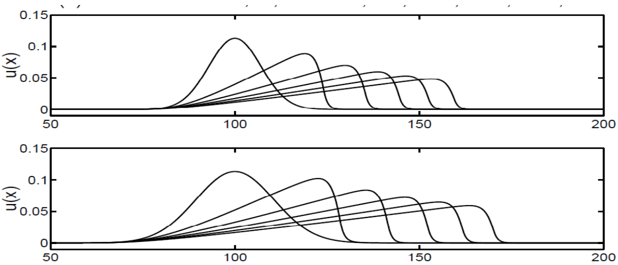

Rarefactive waves are observed for $b \in (-1, 1)$

Solitary waves in b-CH model

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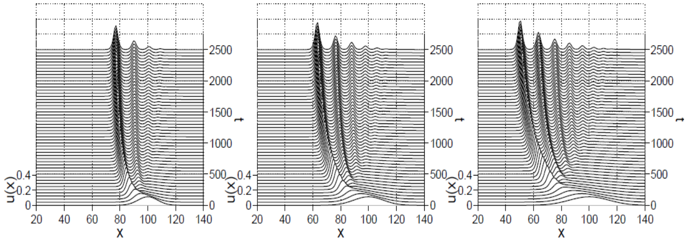

Smooth solitary waves (*leftons*) are observed for $b < -1$

Solitary waves in b-CH model

Similations of the b-family of Camassa-Holm equations

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starting with Gaussian initial data $u(0,x)$ [Holm & Staley, 2003]

Our objectives:

- To study the linear and nonlinear stability of the traveling waves.
- To understand differences in the stability analysis between smooth and peaked profiles of the traveling waves.

Standard approach to orbital stability of nonlinear waves

- Construct an augmented Hamiltonian $\Lambda(u)$, such that the traveling wave solution ϕ is a critical point of Λ: $\underbrace{\Lambda'(\phi) = 0}_{\text{TW-eq}}$
- Compute the spectrum of the linearized operator $\mathcal{L} = \Lambda''(\phi)$ and control the number of negative eigenvalues in $L^2(\mathbb{R})$.
- If $\mathcal{L}$ has only one negative simple eigenvalue and a simple zero eigenvalue, then we need to prove that the traveling wave ϕ is a constrained minimizer of Hamiltonian under fixed momentum, i.e. $\mathcal{L}|_{X_0} \geq 0$, where X_0 is a constrained subspace of L^2
- The traveling wave ϕ is orbitally stable in energy space if local well-posedness has been proven in the energy space.

[Anna Geyer & D. P., *Stability of nonlinear waves in Hamiltonian systems*, AMS Monographs, 2025]

Stability of solitary waves: state-of-the-art *before*

For solitary waves satisfying $u(x) \to 0$ as $|x| \to \infty$

- Orbital stability of peakons in energy space
  $b = 2$: [Constantin & Strauss, 2000] [Constantin & Molinet, 2001]
  $b = 3$: [Lin & Liu, 2009]

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- Orbital stability of leftons in weighted Sobolev spaces
  $b < -1$: [Hone & Lafortune, 2014]

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For smooth solitary waves satisfying $u(x) \to k > 0$ as $|x| \to \infty$:

- Orbital stability of smooth solitons in energy space
  $b = 2$: [Constantin & Strauss, 2002]
  $b = 3$: [Li & Liu & Wu, 2020]

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Similar studies were developed for travelling periodic waves with smooth and peaked profiles: [Lenells, 2004-2006]

Stability of solitary waves: state-of-the-art *after*

- Peakons are linearly and nonlinearly unstable in $H^1 \cap W^{1,\infty}$
  $b = 2$: [Natali & P., 2020] [Madiyeva & P., 2021]

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- Peakons are spectrally unstable in L^2
  any $b \in \mathbb{R}$: [Lafortune & P., 2022a]
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- Smooth solitary waves are orbitally stable in H^3
  $b > 1$: [Lafortune & P., 2022b] [Long & Liu, 2023]

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- Smooth periodic waves are spectrally stable in $L^2_{\rm per}$
  $b = 2$ [Geyer, Martins, Natali, & P., 2022]
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  $b = 3$ [Geyer & P., 2024]

- Smooth solitary waves are linearly transversely stable in 2-dim
  $b = 2$ [Geyer, Liu, & P., 2024]

Section 2

Properties of b-Camassa–Holm equation

Properties of the Camassa-Holm equation on the line

The local differential equation

$$u_t - u_{txx} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx}$$

can be rewritten in the integral form of the perturbed Burgers equation

$$u_t + uu_x + \frac{1}{4}\varphi' * \left(bu^2 + (3-b)u_x^2\right) = 0,$$

where $\varphi := 2(1-\partial_x^2)^{-1}\delta = e^{-|x|}$ is the Green function.

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The time evolution consists of two quadratic parts:

$$\boxed{u_t + u u_x} + \frac{1}{4} \boxed{\varphi' * \left(b u^2 + (3-b) u_x^2 \right)} = 0,$$

with Burgers advection $\boxed{u_t + u u_x = 0}$ and convolution smoothing.

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Solutions of the Burgers equation $u_t + uu_x = 0$ with $u(0,x) = f(x)$ admit wave breaking (gradient blowup) for $f \in W^{1,\infty}(\mathbb{R})$:

$$u(t,x) = f(x - tu(t,x)) \quad \Rightarrow \quad u_x = \frac{f'(x-tu)}{1+tf'(x-tu)}.$$

The local differential equation

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where $\varphi := 2(1-\partial_x^2)^{-1}\delta = e^{-|x|}$ is the Green function.

We say that the dynamics leads to the wave breaking if

$$\|u(t,\cdot)\|_{L^\infty} < \infty, \quad \|u_x(t,\cdot)\|_{L^\infty} \to \infty \quad \text{as} \;\; t \to T < \infty$$

Properties of the Camassa-Holm equation on the line

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For $b > 1$, the initial-value problem is

- locally well-posed in H^s, $s > 3/2$ [Escher & Yin, 2008; Zhou, 2010]
- no continuous dependence in H^s, $s \leq 3/2$
  [Himonas, Grayshan, Holliman (2016)] [Guo, Liu, Molinet, Yin (2018)]
- locally well-posed in $H^1 \cap W^{1,\infty}$.
  [De Lellis, Kappeler, Topalov (2007)] [Linares, Ponce, Sideris (2019)]

Hamiltonian structure of the b-CH equations

For $b = 2$, the Camassa–Holm equation

$$u_t - u_{txx} + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx}$$

has the first three conserved quantities

$$M(u) = \int u dx, \quad E(u) = \frac{1}{2}\int (u^2 + u_x^2) dx, \quad F(u) = \frac{1}{2}\int (u^3 + u u_x^2)\, dx.$$

(CH) can be written in Hamiltonian form in three ways:

$$\begin{aligned}
u_t &= JF'(u), & J &= -(1-\partial_x^2)^{-1}\partial_x, \\
m_t &= J_m E'(m), & J_m &= -\left(m\partial_x + \partial_x m\right), \\
m_t &= J_m M'(m), & J_m &= -(2m\partial_x + m_x)(1-\partial_x^2)^{-1}\partial_x^{-1}(2\partial_x m - m_x).
\end{aligned}$$

where $m = u - u_{xx}$.

Hamiltonian structure of the b-CH equations

For $b = 3$, the Degasperis–Procesi equation

$$u_t - u_{txx} + 4\,u\,u_x = 3\,u_x u_{xx} + u\,u_{xxx}$$

has the first three conserved quantities

$$M(u) = \int u dx, \quad E(u) = \frac{1}{2}\int u(1-\partial_x^2)(4-\partial_x^2)^{-1} u dx, \quad F(u) = \frac{1}{6}\int u^3 dx.$$

(DH) can be written in Hamiltonian form in two ways:

$$u_t = JF'(u), \qquad J = -(1-\partial_x^2)^{-1}(4-\partial_x^2)\partial_x,$$

$$m_t = J_m M'(m), \qquad J_m = -\frac{1}{2}(3m\partial_x + m_x)(1-\partial_x^2)^{-1}\partial_x^{-1}(3\partial_x m - m_x).$$

where $m = u - u_{xx}$.

Hamiltonian structure of the b-CH equations

For general $b \neq 1$, the b-Camassa–Holm equation

$$u_t - u_{txx} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx}$$

can be written in Hamiltonian form:

$$m_t = J_m M'(m), \quad J_m := -\frac{1}{b-1}(bm\partial_x + m_x)(1-\partial_x^2)^{-1}\partial_x^{-1}(b\partial_x m - m_x).$$

where $m = u - u_{xx}$. In addition to the conservation of mass $M(m) = \int m dx$, it has two more conserved quantities:

$$E(m) = \int m^{\frac{1}{b}} dx, \quad F(m) = \int \left(\frac{m_x^2}{b^2 m^2} + 1 \right) m^{-\frac{1}{b}} dx,$$

These are Casimir functionals satisfying $J_m E'(m) = 0$, $J_m F'(m) = 0$.

[Degasperis, Holm, Hone, 2003]

Section 3

Stability and instability of peakons

Existence of peakons

Peakons exist in the weak form in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$

$$u(t,x) = ce^{-|x-ct|}.$$

Without loss of generality, we can set $c = 1$. The normalized profile $\varphi(x) = e^{-|x|}$ satisfies the integral equation

$$-\varphi + \frac{1}{2}\varphi^2 + \frac{1}{4}\varphi * \left(b\varphi^2 + (3-b)(\varphi')^2\right) = 0,$$

which follows from integration of

$$u_t + uu_x + \frac{1}{4}\varphi' * \left(bu^2 + (3-b)u_x^2\right) = 0,$$

after the traveling wave reduction $u(t,x) = \varphi(x-t)$.

Orbital stability of peakons in H^1: $b = 2$

Theorem (Constantin–Molinet (2001))

φ is a unique (up to translation) minimizer of Hamiltonian $F(u)$ in $H^1(\mathbb{R})$ subject to fixed momentum $E(u)$.

Theorem (Constantin–Strauss (2000))

For every small $\varepsilon > 0$, if the initial data satisfies

$$\|u_0 - \varphi\|_{H^1} < \left(\frac{\varepsilon}{3}\right)^4,$$

then the solution satisfies

$$\|u(t,\cdot) - \varphi(\cdot - \xi(t))\|_{H^1} < \varepsilon, \quad t \in (0,T),$$

where $\xi(t)$ is a point of maximum for $u(t,\cdot)$.

Yet, we claim instability of peakons in $H^1 \cap W^{1,\infty}$: $b = 2$

Consider solutions of the Cauchy problem:

$$\begin{cases} u_t + uu_x + Q[u] = 0, \\ u|_{t=0} = u_0 \in H^1 \cap W^{1,\infty}, \end{cases} \qquad Q[u] := \frac{1}{4}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right).$$

Theorem (Natali–P. (2020))

For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H^1 \cap W^{1,\infty}$ satisfying

$$\|u_0 - \varphi\|_{H^1} + \|u_0' - \varphi'\|_{L^\infty} < \delta,$$

s.t. the unique solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ with $T > t_0$ satisfies

$$\|u_x(t_0,\cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty} > 1,$$

where $\xi(t)$ is a point of peak of $u(t,\cdot)$ for $t \in [0,T)$.

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Consider solutions of the Cauchy problem:

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- If $u \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, then $Q[u]$ is Lipschitz continuous and the method of characteristics can be used to analyze dynamics.
- If there exists a peak at $\xi(t)$ s.t. $u(t,\cdot) \in H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{\xi(t)\})$, then it moves with the local characteristic speed as

$$\frac{d\xi}{dt} = u(t, \xi(t)), \quad t \in (0, T).$$

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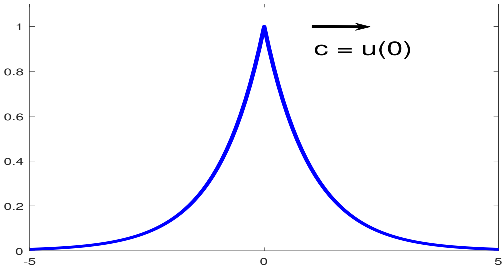
For the peaked traveling wave $u(t,x) = \varphi(x - ct)$,
$\xi'(t) = u(t, \xi(t))$ gives $c = \varphi(0) := \max_{x \in \mathbb{R}} \varphi(x)$.



Evolution of a perturbed peakon

Consider a decomposition near a single peakon:

$$u(t,x) = \varphi(x - t - a(t)) + v(t, x - t - a(t)), \quad t \in [0,T), \quad x \in \mathbb{R},$$

with the peak at $\xi(t) = t + a(t)$ for $v(t,\cdot) \in H^1(\mathbb{R}) \cap C^1(\mathbb{R} \backslash \{\xi(t)\})$.

Then, $\xi'(t) = u(t, \xi(t))$ yields $a'(t) = v(t,0)$ and the perturbation $v(t,\cdot)$ satisfies

$$\boxed{v_t = (1-\varphi)v_x + \varphi \int_0^x v(t,y)dy} + \boxed{(v|_{x=0} - v)v_x - Q[v].}$$

Translational invariance on the line is broken by the peak located at $\xi(t) = t + a(t)$.

Nonlinear evolution

For the evolution problem:

$$\begin{cases} v_t = (c-\varphi)v_x + \varphi \int_0^x v(t,y)dy + (v|_{x=0} - v)v_x - Q[v], \quad t \in (0,T), \\ v|_{t=0} = v_0(x), \end{cases}$$

we can look for solutions with the method of characteristic curves:

$$x = X(t,s), \qquad v(t, X(t,s)) = V(t,s).$$

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$$x = X(t,s), \qquad v(t, X(t,s)) = V(t,s).$$

The characteristic coordinates $X(t,s)$ satisfies

$$\begin{cases} \frac{dX}{dt} = \varphi(X) - 1 + v(t,X) - v(t,0), \quad t \in (0,T), \\ X|_{t=0} = s. \end{cases}$$

Since φ is Lipschitz, there exists the unique characteristic function $X(t,s)$ for each $s \in \mathbb{R}$ if $v(t,\cdot)$ remains in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$
The peak location $X(t,0) = 0$ is invariant in time.

Nonlinear evolution

For the evolution problem:

$$\begin{cases} v_t = (c-\varphi)v_x + \varphi \int_0^x v(t,y)dy + (v|_{x=0} - v)v_x - Q[v], \quad t \in (0,T), \\ v|_{t=0} = v_0(x), \end{cases}$$

we can look for solutions with the method of characteristic curves:

$$x = X(t,s), \qquad v(t, X(t,s)) = V(t,s).$$

From the right side of the peak, $V_0(t) = v(t,0)$, $W_0(t) = v_x(t,0^+)$:

$$\frac{dW_0}{dt} = W_0 + V_0 + V_0^2 - \frac{1}{2}W_0^2 - P[v](0), \quad P[v] := \varphi * \left(v^2 + \frac{1}{2}v_x^2\right).$$

The proof is achieved if we show that $W_0(t)$ grows and may diverge in a finite time.

Proof of the nonlinear instability

From the orbital stability in $H^1(\mathbb{R})$ [A. Constantin, W. Strauss (2000)]
If $\|v_0\|_{H^1} < (\varepsilon/3)^4$, then

$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

Proof of the nonlinear instability

From the orbital stability in $H^1(\mathbb{R})$ [A. Constantin, W. Strauss (2000)]
If $\|v_0\|_{H^1} < (\varepsilon/3)^4$, then

$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

To show instability, we use eq. on the right side of the peak:

$$\frac{dW_0}{dt} = W_0 + V_0 + V_0^2 - \frac{1}{2}W_0^2 - P[v](0)$$

and since $P[v] > 0$, we have

$$\frac{dW_0}{dt} \leq W_0 + C\varepsilon \quad \Rightarrow \quad W_0(t) \leq [W_0(0) + C\varepsilon]\, e^t$$

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$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

If $W_0(0) = -2C\varepsilon$, then

$$W_0(t) \leq -C\varepsilon e^t,$$

hence $|W_0(t_0)| \geq 1$ for $t_0 := -\log(C\varepsilon)$.

Proof of the nonlinear instability

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$$W_0(t) \leq -C\varepsilon e^t,$$

hence $|W_0(t_0)| \geq 1$ for $t_0 := -\log(C\varepsilon)$.

The initial constraint $\|v_0\|_{L^\infty} + \|v_0'\|_{L^\infty} < \delta$, is satisfied
if $\forall \delta > 0$, $\exists \varepsilon > 0$ such that

$$\left(\frac{\varepsilon}{3}\right)^4 + 2C\varepsilon < \delta.$$

Proof of the nonlinear instability

From the orbital stability in $H^1(\mathbb{R})$ [A. Constantin, W. Strauss (2000)]
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$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

To show the finite-time wave breaking, we estimate

$$\frac{dW_0}{dt} = W_0 + V_0 + V_0^2 - \frac{1}{2}W_0^2 - P[v](0) \leq W_0 - \frac{1}{2}W_0^2 + C\varepsilon.$$

Proof of the nonlinear instability

From the orbital stability in $H^1(\mathbb{R})$ [A. Constantin, W. Strauss (2000)]
If $\|v_0\|_{H^1} < (\varepsilon/3)^4$, then

$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

By the ODE comparison theory, $W_0(t) \leq \overline{W}(t)$, where the supersolution satisfies

$$\frac{d\overline{W}}{dt} = \overline{W} - \frac{1}{2}\overline{W}^2 + C\varepsilon$$

with $W_0(0) = \overline{W}(0) = -C\varepsilon$ and $\overline{W}(t) \to -\infty$ as $t \to \overline{T}$.

Illustration of the peakon instability (periodic case)

For the linearized equation $\boxed{v_t = (1-\varphi)v_x + \varphi \int_0^x v(t,y)dy}$, we can obtain exact solutions and illustrate the peakon instability.

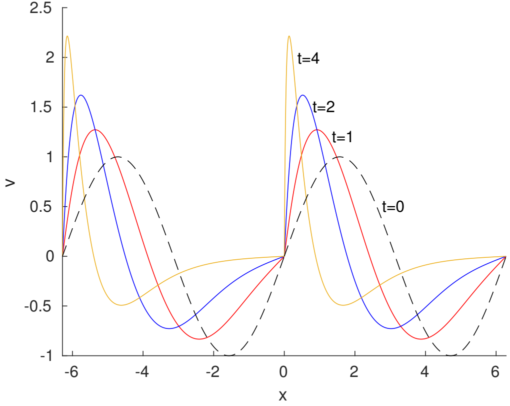

Figure: $v(t,x)$ versus x for $t = 0, 1, 2, 4$ in the case $v_0(x) = \sin(x)$.

Section 4

Spectral instability of peakons for any $b > 1$

Linear evolution equation for a perturbed peakon

The linearized equation is well-posed in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$:

$$\begin{aligned} v_t = & (1-\varphi)v_x + (b-2)(v|_{x=0} - v)\varphi' \\ & + \frac{1}{2}(b-3)\varphi * (\varphi' v) - \frac{1}{2}(2b-3)\varphi' * (\varphi v), \end{aligned}$$

Linear evolution equation for a perturbed peakon

The linearized equation is well-posed in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$:

$$
\begin{aligned}
v_t = {} & (1-\varphi)v_x + (b-2)(v|_{x=0} - v)\varphi' \\
& + \frac{1}{2}(b-3)\varphi * (\varphi' v) - \frac{1}{2}(2b-3)\varphi' * (\varphi v),
\end{aligned}
$$

Question: Can we show the linear instability from analysis of the linearized operator in $L^2(\mathbb{R})$?

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The linearized operator is

$$L = (1-\varphi)\partial_x - (b-2)\varphi' + K,$$

where $K : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is a compact (Hilbert–Schmidt) operator. Since $\varphi \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, the natural domain of L in $L^2(\mathbb{R})$ is

$$\text{Dom}(L) = \left\{ v \in L^2(\mathbb{R}) : \quad (1-\varphi)v' \in L^2(\mathbb{R}) \right\}.$$

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$$\mathrm{Dom}(L) = \left\{ v \in L^2(\mathbb{R}) : \quad (1-\varphi)v' \in L^2(\mathbb{R}) \right\}.$$

$H^1(\mathbb{R})$ is continuously embedded into $\mathrm{Dom}(L)$. However, it is not equivalent to $\mathrm{Dom}(L)$ because $\varphi' \in \mathrm{Dom}(L)$ but $\varphi' \notin H^1(\mathbb{R})$.

Spectrum of a linear operator

Let A be a linear operator on a Banach space X with $\mathrm{Dom}(A) \subset X$. The complex plane $\mathbb{C}$ is decomposed into the resolvent set $\rho(A)$ and the spectrum $\sigma(A) = \mathbb{C} \setminus \rho(A)$, the latter consists of the following three disjoint sets:

1. the point spectrum

$$\sigma_{\mathrm{p}}(A) = \{\lambda : \quad \mathrm{Ker}(A - \lambda I) \neq \{0\}\},$$

2. the residual spectrum

$$\sigma_{\mathrm{r}}(A) = \{\lambda : \quad \mathrm{Ker}(A - \lambda I) = \{0\}, \quad \mathrm{Ran}(A - \lambda I) \neq X\},$$

3. the continuous spectrum

$$\sigma_{\mathrm{c}}(A) = \{\lambda : \quad \mathrm{Ker}(A - \lambda I) = \{0\}, \quad \mathrm{Ran}(A - \lambda I) = X, \\ (A - \lambda I)^{-1} : X \to X \text{ is unbounded}\}.$$

Spectrum of a linear operator

Theorem (Lafortune–P, SIMA 54 (2022) 4572–4590)

The spectrum of L with $\mathrm{Dom}(L) \subset L^2(\mathbb{R})$

$$\sigma(L) = \left\{ \lambda \in \mathbb{C} : \;\; |\mathrm{Re}(\lambda)| \leq \left| \frac{5}{2} - b \right| \right\}.$$

Moreover,

- $\sigma_p(L)$ *is located for* $0 < |\mathrm{Re}(\lambda)| < \frac{5}{2} - b$ *if* $b < \frac{5}{2}$
- $\sigma_r(L)$ *is located for* $0 < |\mathrm{Re}(\lambda)| < b - \frac{5}{2}$ *if* $b > \frac{5}{2}$
- $\sigma_c(L)$ *is located for* $\mathrm{Re}(\lambda) = 0$ *and* $\mathrm{Re}(\lambda) = \pm \left| \frac{5}{2} - b \right|$
- $\lambda = 0$ *is the embedded eigenvalue for every b.*

$\Rightarrow$ the peakon is linearly unstable in $\mathrm{Dom}(L)$ for every $b \neq \frac{5}{2}$.

Spectrum of a linear operator

Theorem (Lafortune–P, SIMA 54 (2022) 4572–4590)

The spectrum of L *with* $\mathrm{Dom}(L) \subset L^2(\mathbb{R})$

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$b = 2$: $\|v(t,\cdot)\|_{L^2}$ grows due to point spectrum

$b = 3$: $\|v(t,\cdot)\|_{L^2}$ grows due to residual spectrum

Spectrum of a linear operator

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- ▷ $\lambda = 0$ *is the embedded eigenvalue for every b.*

Instability in the vertical strip holds for peaked waves in the reduced Ostrovsky equation $u_t + uu_x = \partial_x^{-1} u$ [Geyer & P. (2020)] and for Euler flows [Shvidkoy & Latushkin (2003)]

Proofs of spectral instability

Recall that $L = L_0 + K$, where $L_0 := (1-\varphi)\partial_x - (b-2)\varphi'$ with

$$\mathrm{Dom}(L) = \mathrm{Dom}(L_0) = \left\{ v \in L^2(\mathbb{R}) : \quad (1-\varphi)v' \in L^2(\mathbb{R}) \right\}$$

and $K : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is a compact (Hilbert–Schmidt) operator.

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Theorem (Geyer & P (2020))

If $\sigma_{\rm p}(L) \cap \rho(L_0)$ and $\sigma_{\rm p}(L_0) \cap \rho(L)$ are empty, then $\sigma(L) = \sigma(L_0)$.

Theorem (Bühler & Salamon (2018))

If $\sigma_{\rm p}(L)$ is empty, then $\sigma_{\rm r}(L) = \sigma_{\rm p}(L^*)$.

Proofs of spectral instability

Recall that $L = L_0 + K$, where $L_0 := (1-\varphi)\partial_x - (b-2)\varphi'$ with

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and $K : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is a compact (Hilbert–Schmidt) operator.

Truncated equation $L_0 v = \lambda v$ is the first-order equation

$$(1-\varphi)\frac{dv}{dx} + (2-b)\varphi' v = \lambda v$$

with the exact solution

$$v(x) = \begin{cases} v_+ e^{\lambda x}(1-e^{-x})^{2+\lambda-b}, & x > 0, \\ v_- e^{\lambda x}(1-e^{x})^{2-\lambda-b}, & x < 0, \end{cases}$$

The solution $v \in L^2(\mathbb{R})$ if $v_+ = 0$ and $0 < \mathrm{Re}(\lambda) < \frac{5}{2} - b$.

Section 5

Stability of smooth solitary waves

Existence of smooth solitary waves: $b > 1$

Smooth traveling waves of the form $u(x,t) = \phi(x - ct)$ satisfy

$$-(c - \phi)(\phi''' - \phi') + b\phi'(\phi'' - \phi) = 0.$$

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Smooth traveling waves of the form $u(x,t) = \phi(x - ct)$ satisfy

$$-(c-\phi)(\phi''' - \phi') + b\phi'(\phi'' - \phi) = 0.$$

After multiplication by $(c-\phi)^{b-1}$, the equation can be integrated into

$$-(c-\phi)^b(\phi'' - \phi) = a, \quad a \in \mathbb{R}.$$

Further integration gives

$$\frac{1}{2}(b-1)[(\phi')^2 - \phi^2] + \frac{a}{(c-\phi)^{b-1}} = g, \quad g \in \mathbb{R}.$$

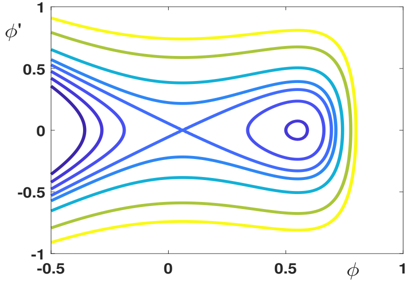
Smooth waves with $c > 0$ exist if $\phi < c$.

Existence of smooth solitary waves: $b > 1$

Newton's particle with mass $m = b - 1$ and potential energy $U(\phi)$

$$\frac{1}{2}(b-1)(\phi')^2 + U(\phi) = g, \quad U(\phi) = -\frac{1}{2}(b-1)\phi^2 + \frac{a}{(c-\phi)^{b-1}}.$$

There exists $a_0 > 0$ such that for every $a \in (0, a_0)$ two critical points of $U(\phi)$ exists with ordering $0 < \phi_1 < \phi_2 < c$.



Properties of smooth solitary waves: $b > 1$

For every $c > 0$, the family of solitary waves has one additional parameter, which can be chosen as $k \in (0, k_0)$ such that

$$\phi(x) \to k \quad \text{as} \quad |x| \to \infty \text{ exponentially.}$$

Moreover, $0 < \phi < c$ and

$$\mu = \phi - \phi'' = k \frac{(c-k)^b}{(c-\phi)^b} > 0.$$

Hamiltonian structure of the b-CH equations

Recall that the b-Camassa–Holm equation with $b \neq 1$

$$u_t - u_{txx} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx}$$

has conserved quantities

$$E(m) = \int m^{\frac{1}{b}} dx, \quad F(m) = \int \left(\frac{m_x^2}{b^2 m^2} + 1 \right) m^{-\frac{1}{b}} dx,$$

where $m = u - u_{xx}$.

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where $m = u - u_{xx}$.

The conserved quantities can be redefined as

$$\hat{E}(m) = \int_{\mathbb{R}} \left[m^{\frac{1}{b}} - k^{\frac{1}{b}} \right] dx, \quad \hat{F}(m) = \int_{\mathbb{R}} \left[\left(\frac{m_x^2}{b^2 m^2} + 1 \right) m^{-\frac{1}{b}} - k^{-\frac{1}{b}} \right] dx$$

in the set of functions with fixed $k > 0$:

$$X_k = \left\{ m - k \in H^1(\mathbb{R}) : \quad m(x) > 0, \;\; x \in \mathbb{R} \right\}.$$

Stability of smooth solitary waves: $b > 1$

Let $m(t,x) = \mu(x-ct)$ with $\mu \in X_k$. We say that the travelling wave is orbitally stable in X_k if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $m_0 \in X_k$ satisfying $\|m_0 - \mu\|_{H^1} < \delta$, there exists a unique solution $m \in C^0(\mathbb{R}, X_k)$ of the b-CH equation satisfying

$$\inf_{x_0 \in \mathbb{R}} \|m(t,\cdot) - \mu(\cdot - x_0)\|_{H^1} < \varepsilon, \quad t \in \mathbb{R}.$$

Theorem (Lafortune–P, Physica D 440 (2022) 133477)

For every $c > 0$ and $k \in (0, k_0)$, there exists a unique solitary wave $m(t,x) = \mu(x-ct)$ of the b-CH equation, which is orbitally stable in X_k if the mapping

$$k \mapsto Q(\phi) := \int_{\mathbb{R}} \left[b\left(\frac{c-k}{c-\phi}\right) - \left(\frac{c-k}{c-\phi}\right)^b - b + 1 \right] dx$$

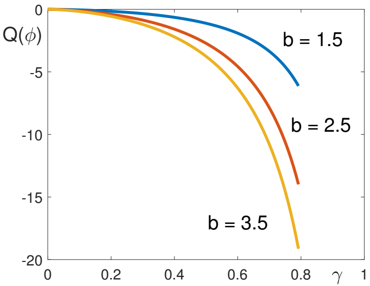
is strictly increasing.

Stability of smooth solitary waves: $b > 1$

Let $m(t,x) = \mu(x - ct)$ with $\mu \in X_k$. We say that the travelling wave is orbitally stable in X_k if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $m_0 \in X_k$ satisfying $\|m_0 - \mu\|_{H^1} < \delta$, there exists a unique solution $m \in C^0(\mathbb{R}, X_k)$ of the b-CH equation satisfying

$$\inf_{x_0 \in \mathbb{R}} \|m(t,\cdot) - \mu(\cdot - x_0)\|_{H^1} < \varepsilon, \quad t \in \mathbb{R}.$$

For general $b > 1$, we confirmed the stability criterioin numerically:



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$$\inf_{x_0 \in \mathbb{R}} \|m(t,\cdot) - \mu(\cdot - x_0)\|_{H^1} < \varepsilon, \quad t \in \mathbb{R}.$$

For $b = 2$ and $b = 3$, we proved monotonicity with explicit computation.

For every $b > 1$, monotonicity $k \mapsto Q(\phi)$ was proven in [Long & Liu, 2023] by using the period function for planar ODEs.

Proof of orbital stability of smooth solitary waves

1. We verify that the solitary wave $\mu \in X_k$ is a critical point of the augmented Hamiltonian

$$\Lambda_{\omega_1,\omega_2}(m) := \hat{M}(m) - \omega_1 \hat{E}(m) - \omega_2 \hat{F}(m),$$

for some (ω_1, ω_2) that depend on (b, c, k).

Proof of orbital stability of smooth solitary waves

1. We verify that the solitary wave $\mu \in X_k$ is a critical point of the augmented Hamiltonian

$$\Lambda_{\omega_1,\omega_2}(m) := \hat{M}(m) - \omega_1 \hat{E}(m) - \omega_2 \hat{F}(m),$$

for some (ω_1, ω_2) that depend on (b, c, k).

2. Expansion of the augmented Hamiltonian with small $\tilde{m} \in H^1(\mathbb{R})$ is

$$\Lambda_{\omega_1,\omega_2}(\mu + \tilde{m}) - \Lambda_{\omega_1,\omega_2}(\mu) = \langle \mathcal{L}\tilde{m}, \tilde{m} \rangle + \|\tilde{m}\|_{H^1}^3,$$

where $\mathcal{L}$ is the Sturm–Liouville operator in $L^2(\mathbb{R})$ with the dense domain $H^2(\mathbb{R})$. Since $\mathcal{L}\mu' = 0$ and $\mu'(x)$ has only one zero on $\mathbb{R}$, $\mathcal{L}$ admits exactly one simple negative eigenvalue and a simple zero eigenvalue.

Proof of orbital stability of smooth solitary waves

3. Since

$$b\hat{E}(m) - k^{\frac{1}{b}-1}\hat{M}(m)$$

is conserved in time, perturbations $\tilde{m}$ can be restricted to the class

$$\langle \mu^{\frac{1}{b}-1} - k^{\frac{1}{b}-1}, \tilde{m} \rangle = 0.$$

Proof of orbital stability of smooth solitary waves

3. Since
$$b\hat{E}(m) - k^{\frac{1}{b}-1}\hat{M}(m)$$
is conserved in time, perturbations $\tilde{m}$ can be restricted to the class
$$\langle \mu^{\frac{1}{b}-1} - k^{\frac{1}{b}-1}, \tilde{m} \rangle = 0.$$

4. $\mathcal{L}|_{\{v_0\}^\perp} \geq 0$ is coercive in the H^1 norm if and only if the mapping
$$k \mapsto Q(\phi) := \int_{\mathbb{R}} \left[b \left(\frac{c-k}{c-\phi} \right) - \left(\frac{c-k}{c-\phi} \right)^b - b + 1 \right] dx$$
is strictly increasing.

Section 6

Transverse stability of smooth solitary waves

2D generalization of the CH equation

The following 2D model was derived for fluids:

$$(u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx})_x + u_{yy} = 0$$

[R.M. Chen (2006)] [G. Gui, Y. Liu, W. Luo, Z. Yin (2021)]

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$$(u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx})_x + u_{yy} = 0$$

[R.M. Chen (2006)] [G. Gui, Y. Liu, W. Luo, Z. Yin (2021)]

In the small-amplitude and long-scale expansion,

$$u(x,y,t) = k + \varepsilon^2 v(\varepsilon(x-3kt), \varepsilon^2 y, \varepsilon^3 t), \quad \varepsilon > 0,$$

the 2D-CH equation formally reduces to the KP-II equation

$$v_T + 2kv_{XXX} + 3vv_X + \partial_X^{-1} v_{YY} = 0.$$

The line soliton $v(X,T) = \text{sech}^2\left(\frac{X-T}{2\sqrt{2k}}\right)$ is transversely stable in the KP-II equation. [T. Mizumachi & N. Tzvetkov (2012)] [T. Mizumachi (2015)]

Two theorems on transverse stability of line solitons

For the 2D-CH equation

$$(u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx})_x + u_{yy} = 0$$

we proved the following

- For every $\varepsilon > 0$, the linear stability problem contains a pair of resonances located in the left half-plane of the complex plane and no eigenvalues with $\mathrm{Re}(\lambda) \geq 0$ near $\lambda = 0$.

- For every small $\varepsilon > 0$, the line solitons are linearly stable with respect to transverse perturbations.

[A. Geyer, Y. Liu, & D.P., Journal de Mathématiques Pures et Appliquées (2024)]

Summary

I have reviewed traveling waves in the b-CH equation in 1D:

$$u_t - u_{txx} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}$$

which models unidirectional small-amplitude shallow water waves.

- Peaked traveling waves are unstable in $H^1 \cap W^{1,\infty}$
  - LWP only holds in $H^1 \cap W^{1,\infty}$.
  - For $b = 2$, perturbations are bounded in H^1 and growing in $W^{1,\infty}$.
  - Spectral instability of peakons holds for every b.
- Smooth traveling waves are stable in H^3 for $b > 1$
  - LWP and GWP hold for perturbations with $m = u - u'' > 0$
  - Hamiltonian formulation exists for every $b > 1$
  - TW is constrained minimizer of the augmented Hamiltonian.

MANY THANKS FOR YOUR ATTENTION!