

Persistence and stability of discrete vortices

Dmitry Pelinovsky

Department of Mathematics, McMaster University, Canada

$$\begin{aligned} i\dot{u}_{n,m} + \epsilon (u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1} - 4u_{n,m}) \\ + |u_{n,m}|^2 u_{n,m} = 0, \quad (n, m) \in \mathbb{Z}^2 \end{aligned}$$

**Joint work with P. Kevrekidis
(University of Massachusetts)**

**”Discrete and Continuous Models in Nonlinear Optics”
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■ *Experimental motivations*

- Bose-Einstein condensates in optical lattices
- Light-induced photonic lattices
- Coupled optical waveguides

■ *Persistence of localized solutions*

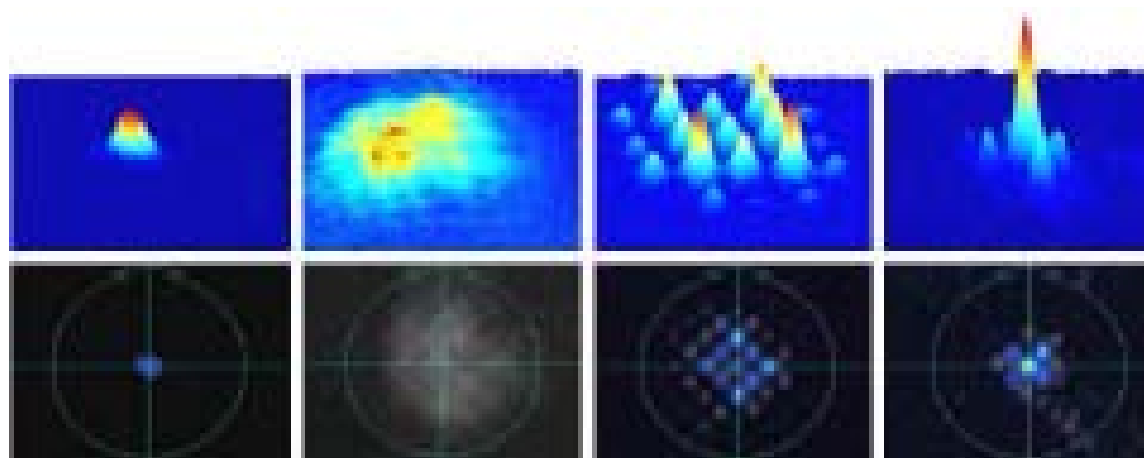
- Implicit Function Theorem
- Lyapunov–Schmidt reductions

■ *Stability of localized solutions*

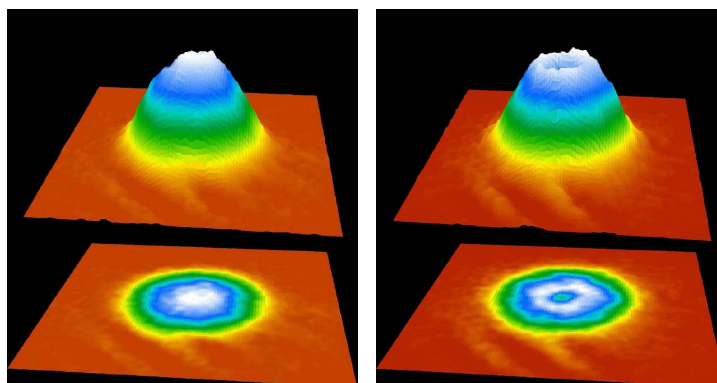
- Splitting of zero eigenvalues
- Negative index theory

Experimental pictures

- Discrete solitons



- Discrete vortices



Main Formalism

$$i\dot{u}_{n,m} + \epsilon (u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1} - 4u_{n,m}) + |u_{n,m}|^2 u_{n,m} = 0, \quad (n, m) \in \mathbb{Z}^2$$

- Vector space $\Omega = l^2(\mathbb{Z}^2, \mathbb{C})$ for $\{u_{n,m}\}_{(n,m) \in \mathbb{Z}^2}$:

$$(\mathbf{u}, \mathbf{w})_{\Omega} = \sum_{(n,m) \in \mathbb{Z}^2} \bar{u}_{n,m} w_{n,m}$$

- Hamiltonian formulation:

$$i\dot{u}_{n,m} = \frac{\partial H}{\partial \bar{u}_{n,m}},$$

where

$$H = \sum_{(n,m) \in \mathbb{Z}^2} \epsilon |u_{n+1,m} - u_{n,m}|^2 + |u_{n,m+1} - u_{n,m}|^2 - \frac{1}{2} |u_{n,m}|^4$$

- Existence problem for time-periodic localized solutions

$$u_{n,m}(t) = \phi_{n,m} e^{i(1-4\epsilon)t + i\theta_0}, \quad \theta_0 \in \mathbb{R}$$

such that

$$(1 - |\phi_{n,m}|^2)\phi_{n,m} = \epsilon (\phi_{n+1,m} + \phi_{n-1,m} + \phi_{n,m+1} + \phi_{n,m-1}).$$

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$$u_{n,m}(t) = e^{i(1-4\epsilon)t + i\theta_0} \left(\phi_{n,m} + a_{n,m} e^{\lambda t} + \bar{b}_{n,m} e^{\bar{\lambda} t} \right)$$

such that

$$\begin{aligned} (1 - 2|\phi_{n,m}|^2) a_{n,m} - \phi_{n,m}^2 b_{n,m} - \epsilon (a_{n+1,m} + a_{n-1,m} + a_{n,m+1} + a_{n,m-1}) &= i\lambda a_{n,m} \\ -\bar{\phi}_{n,m}^2 a_{n,m} + (1 - 2|\phi_{n,m}|^2) b_{n,m} - \epsilon (b_{n+1,m} + b_{n-1,m} + b_{n,m+1} + b_{n,m-1}) &= -i\lambda b_{n,m} \end{aligned}$$

where $\lambda \in \mathbb{C}$ and $(\mathbf{a}, \mathbf{b}) \in \Omega \times \Omega$

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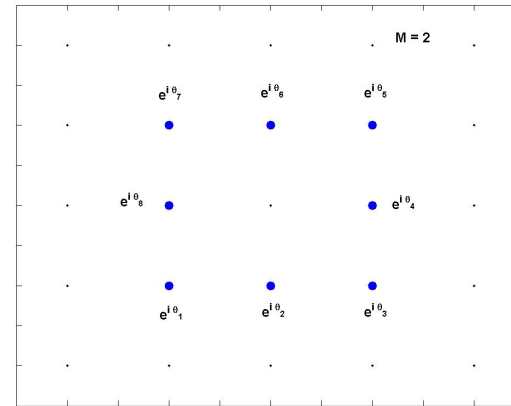
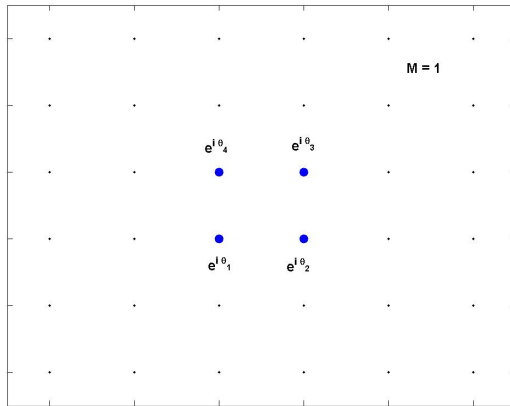
- Time-dependent nonlinear dynamics of localized solutions

Existence problem

$$(1 - |\phi_{n,m}|^2)\phi_{n,m} = \epsilon (\phi_{n+1,m} + \phi_{n-1,m} + \phi_{n,m+1} + \phi_{n,m-1})$$

Limiting solution:

$$\epsilon = 0 : \quad \phi_{n,m}^{(0)} = \begin{cases} e^{i\theta_{n,m}}, & (n, m) \in S, \\ 0, & (n, m) \in \mathbb{Z}^2 \setminus S, \end{cases}$$



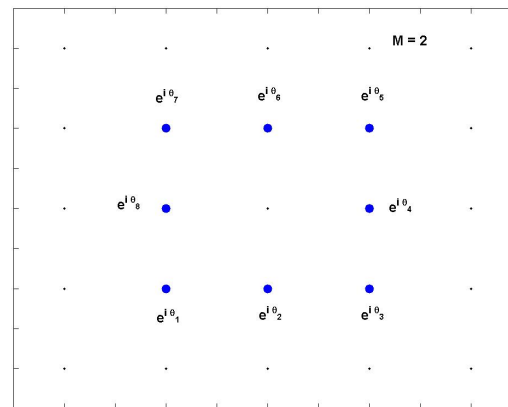
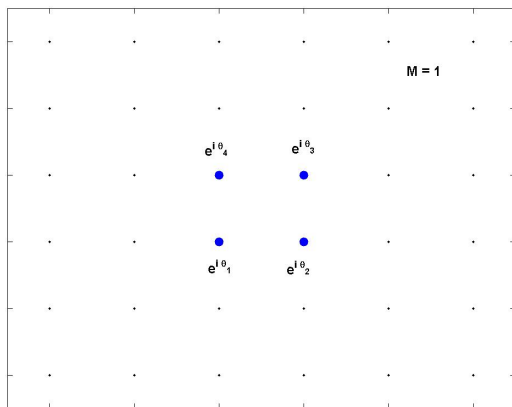
Examples of a square discrete contour S

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Examples of a square discrete contour S

What phase configurations $\theta_{n,m}$ can be continued for $\epsilon \neq 0$?

Lyapunov-Schmidt reductions

Proposition: Let $N = \dim(S)$ and \mathcal{T} be the torus on $[0, 2\pi]^N$. There exists a vector-valued function $\mathbf{g} : \mathcal{T} \mapsto \mathbb{R}^N$, such that the limiting solution is continued to $\epsilon \neq 0$ if and only if $\boldsymbol{\theta} \in \mathcal{T}$ is a root of $\mathbf{g}(\boldsymbol{\theta}, \epsilon) = \mathbf{0}$.

- The Jacobian of the nonlinear system:

$$\mathcal{H} = \begin{pmatrix} 1 - 2|\phi_{n,m}|^2 & -\phi_{n,m}^2 \\ -\bar{\phi}_{n,m}^2 & 1 - 2|\phi_{n,m}|^2 \end{pmatrix} - \epsilon \delta_{\pm 1, \pm 1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- \mathcal{H} is a self-adjoint Fredholm operator of index zero:

$$\dim(\ker(\mathcal{H}^{(0)})) = N$$

- Analytic functions:

$$\mathbf{g}(\boldsymbol{\theta}, \epsilon) = \sum_{k=1}^{\infty} \epsilon^k \mathbf{g}^{(k)}(\boldsymbol{\theta})$$

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- Let $\boldsymbol{\theta}_*$ be the root of $\mathbf{g}^{(1)}(\boldsymbol{\theta}) = \mathbf{0}$ and $\mathcal{M}_1 = \mathcal{D}\mathbf{g}^{(1)}(\boldsymbol{\theta}_*)$.
If $\dim(\ker(\mathcal{M}_1)) = 1$, there exists a unique continuation of the limiting solution for $\epsilon \neq 0$.

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If $\dim(\ker(\mathcal{M}_1)) = 1$, there exists a unique continuation of the limiting solution for $\epsilon \neq 0$.
- Let $\boldsymbol{\theta}_*$ be a $(1 + d)$ -parameter solution of $\mathbf{g}^{(1)}(\boldsymbol{\theta}) = \mathbf{0}$. The limiting solution can not be continued to $\epsilon \neq 0$ if $\mathbf{g}^{(2)}(\boldsymbol{\theta}_*)$ is not orthogonal to $\ker(\mathcal{M}_1)$.

First-order reductions : classification of solutions

$$\mathbf{g}_j^{(1)}(\boldsymbol{\theta}) = \sin(\theta_j - \theta_{j+1}) + \sin(\theta_j - \theta_{j-1}) = 0, \quad 1 \leq j \leq 4M$$

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- (1) Discrete solitons

$$\theta_j = \{0, \pi\}, \quad 1 \leq j \leq 4M$$

- (2) Symmetric vortices of charge L

$$\theta_j = \frac{\pi L(j-1)}{2M}, \quad 1 \leq j \leq 4M,$$

- (3) One-parameter asymmetric vortices of charge $L = M$

$$\theta_{j+1} - \theta_j = \left\{ \begin{array}{c} \theta \\ \pi - \theta \end{array} \right\} \text{ mod}(2\pi), \quad 1 \leq j \leq 4M$$

where

- M is number of nodes at each side of the square contour
- L is the vortex charge (winding number)

First-order reductions : persistence of solutions

$$\mathcal{M}_1 = \begin{pmatrix} a_1 + a_2 & -a_2 & 0 & \dots & a_1 \\ -a_2 & a_2 + a_3 & -a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -a_1 & 0 & 0 & \dots & a_{N-1} + a_N \end{pmatrix}, \quad a_j = \cos(\theta_{j+1} - \theta_j)$$

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- \mathcal{M}_1 has a simple zero eigenvalue if all $a_j \neq 0$ and

$$\left(\prod_{i=1}^N a_i \right) \left(\sum_{i=1}^N \frac{1}{a_i} \right) \neq 0.$$

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- If all $a_j = a = \cos(\frac{\pi L}{2M})$, eigenvalues of \mathcal{M}_1 are:

$$\lambda_n = 4a \sin^2 \frac{\pi n}{4M}, \quad 1 \leq n \leq 4M$$

Family (2) persists for $\epsilon \neq 0$ and $L \neq M$

Second-order reductions : termination of solutions

- If all $a_j = \pm a = \cos \theta$, there are $2M - 1$ negative eigenvalues of \mathcal{M}_1 , 2 zero eigenvalues and $2M - 1$ positive eigenvalues of \mathcal{M}_1 .

- Persistence of family (3) depends on $\mathbf{g}^{(2)}(\boldsymbol{\theta})$

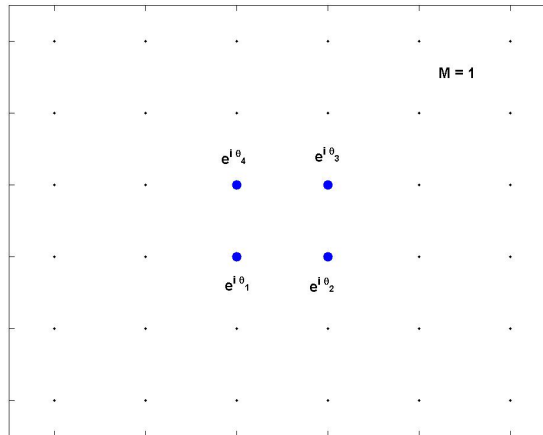
$$\begin{aligned} \mathbf{g}_j^{(2)} &= \frac{1}{2} \sin(\theta_{j+1} - \theta_j) [\cos(\theta_j - \theta_{j+1}) + \cos(\theta_{j+2} - \theta_{j+1})] \\ &\quad + \frac{1}{2} \sin(\theta_{j-1} - \theta_j) [\cos(\theta_j - \theta_{j-1}) + \cos(\theta_{j-2} - \theta_{j-1})] \end{aligned}$$

- If $\ker(\mathcal{M}_1) = \{\mathbf{p}_0, \mathbf{p}_1\}$, then $(\mathbf{g}^{(2)}, \mathbf{p}_1) \neq 0$.

- Family (3) terminates except for one symmetric configuration:

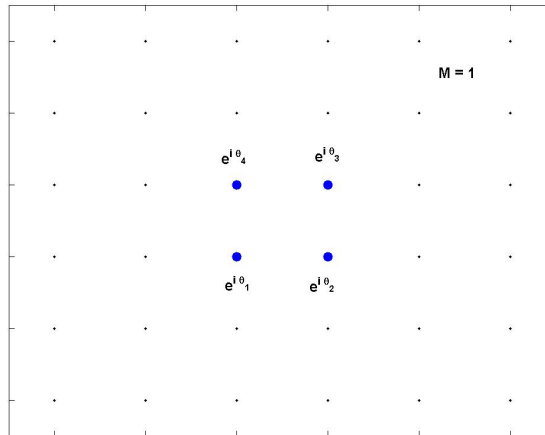
$$\theta_1 = 0, \quad \theta_2 = \theta, \quad \theta_3 = \pi, \quad \theta_4 = \pi + \theta,$$

Higher-order reductions : termination of the last family



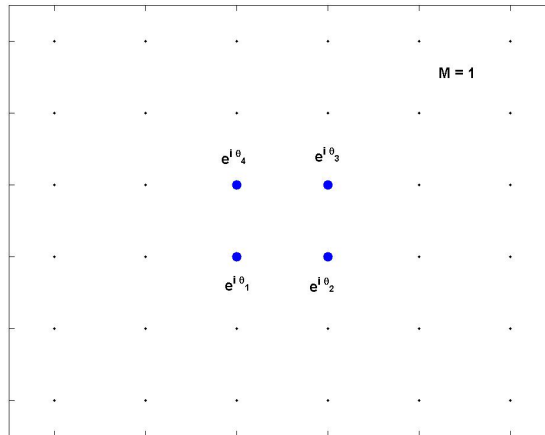
- Symbolic software algorithm is used on a squared domain of N_0 -by- N_0 lattice nodes, where $N_0 = 2K + 2M + 1$, and K is the order of the Lyapunov-Schmidt reductions.

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- Symbolic software algorithm is used on a squared domain of N_0 -by- N_0 lattice nodes, where $N_0 = 2K + 2M + 1$, and K is the order of the Lyapunov-Schmidt reductions.
- Super-symmetric family (3) has $\mathbf{g}^{(k)}(\boldsymbol{\theta}) = 0$ for $k = 1, 2, 3, 4, 5$ but $\mathbf{g}^{(6)}(\boldsymbol{\theta}) \neq 0$, unless $\theta_{j+1} - \theta_j = \frac{\pi}{2}$.
- Moreover, $(\mathbf{g}^{(6)}, \mathbf{p}_1) \neq 0$.

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- Moreover, $(\mathbf{g}^{(6)}, \mathbf{p}_1) \neq 0$.
- All asymmetric vortices (3) terminate

Zero eigenvalues of the stability problem

- Matrix-vector Hamiltonian form of the stability problem:

$$\mathcal{H}\psi = i\lambda\sigma\psi,$$

where

- $\psi \in l^2(\mathbb{Z}^2, \mathbb{C}^2)$
- \mathcal{H} is the Jacobian (energy) operator
- σ is the diagonal matrix of $(1, -1)$

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Eigenvalues of \mathcal{H} at $\epsilon = 0$:

- $\gamma = -2$ of multiplicity N
- $\gamma = 0$ of multiplicity N
- $\gamma = +1$ of multiplicity ∞

Eigenvalues of \mathcal{JH} at $\epsilon = 0$:

- $\lambda = 0$ of multiplicity $2N$
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How do zero eigenvalues split?

Stability of solutions in Lyapunov-Schmidt reductions

- First-order splitting of zero eigenvalues of \mathcal{H} :

$$\mathcal{M}_1 \mathbf{c} = \gamma \mathbf{c}$$

- First-order splitting of zero eigenvalues of $\mathcal{J}\mathcal{H}$:

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- Second-order splitting of zero eigenvalues of \mathcal{H} :

$$\mathcal{M}_1 = 0, \quad \mathcal{M}_2 \mathbf{c} = \gamma \mathbf{c}$$

- Second-order splitting of zero eigenvalues of \mathcal{JH} :

$$\mathcal{M}_1 = 0, \quad \mathcal{M}_2 \mathbf{c} = \frac{\lambda^2}{2} \mathbf{c} + \lambda \mathcal{L}_2 \mathbf{c}$$

where $M_2^T = M_2$ and $L_2^T = -L_2$.

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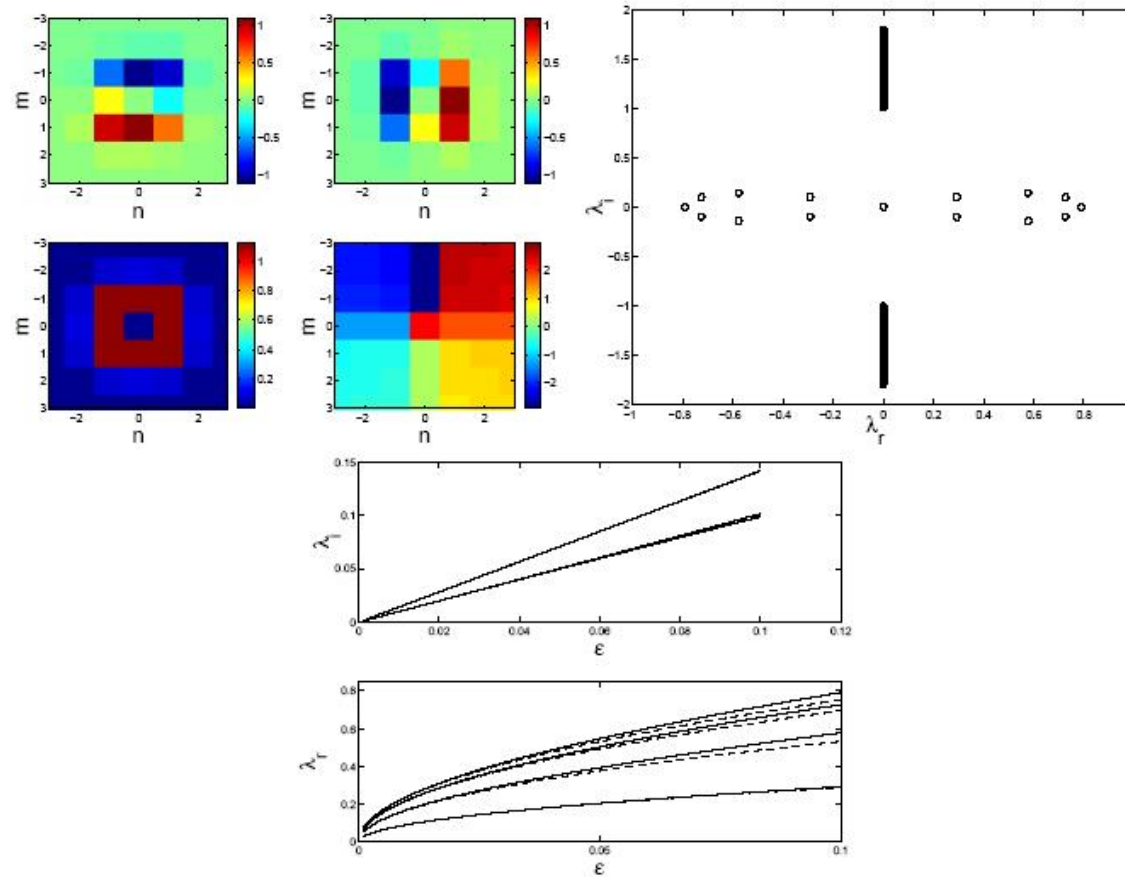
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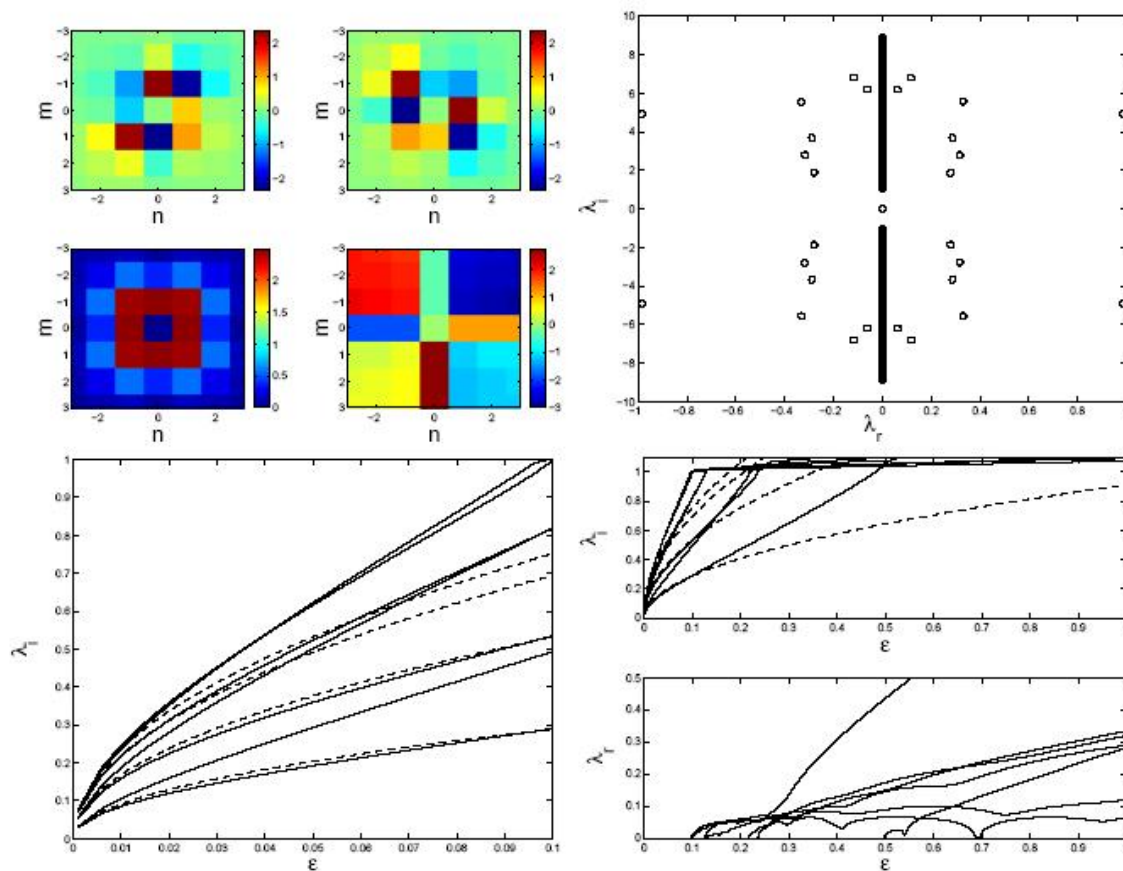
- Six-order splitting : symbolic software algorithm

Numerical analysis: symmetric vortex with $L = 1$ and $M = 2$



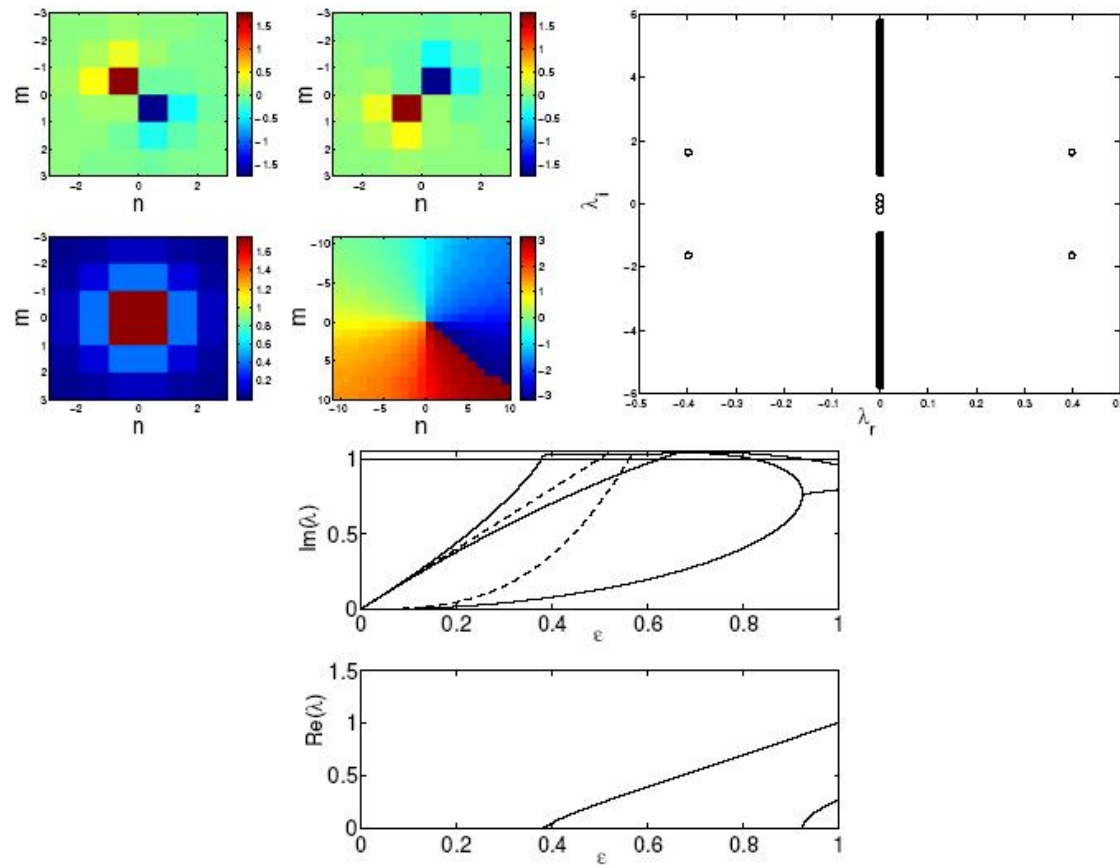
$$\mathcal{M}_1 \mathbf{c} = \gamma \mathbf{c} : \quad n(\mathcal{M}_1) = 0, \quad z(\mathcal{M}_1) = 1, \quad p(\mathcal{M}_1) = 7$$

Numerical analysis: symmetric vortex with $L = 3$ and $M = 2$



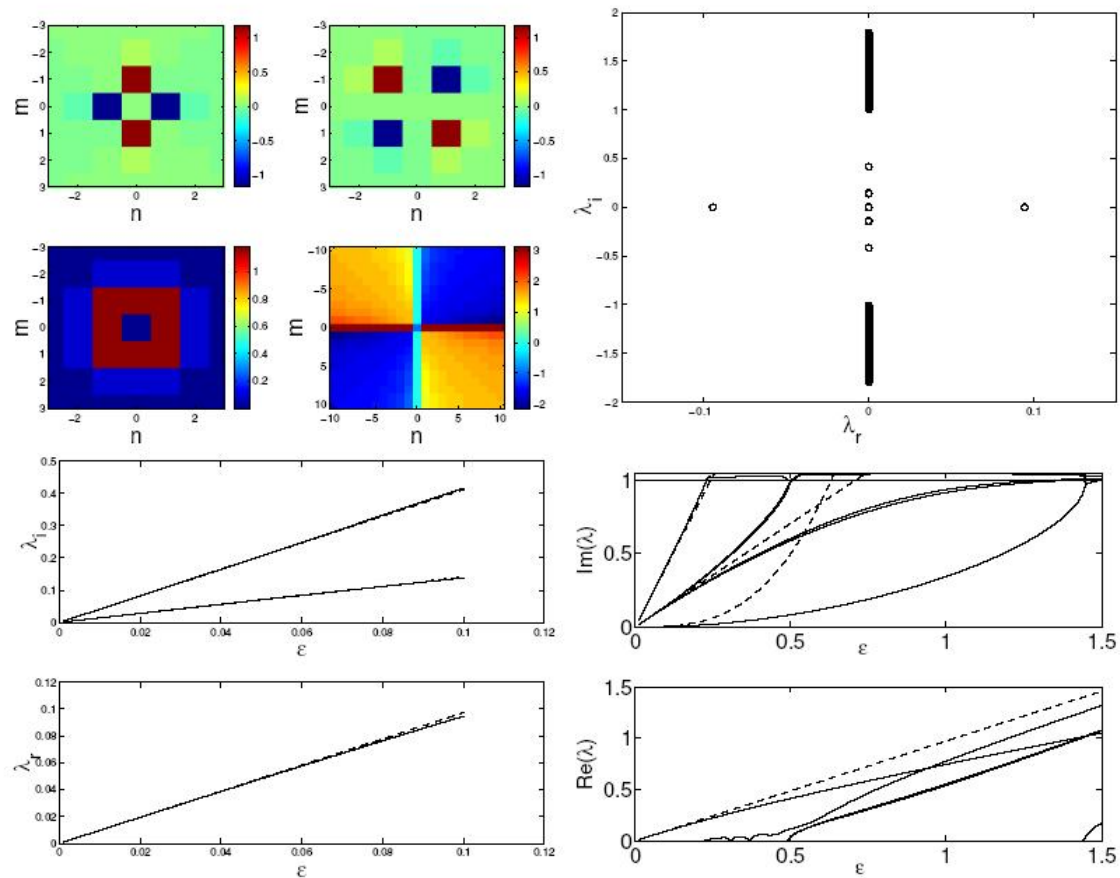
$$\mathcal{M}_1 \mathbf{c} = \gamma \mathbf{c} : \quad n(\mathcal{M}_1) = 7, z(\mathcal{M}_1) = 1, p(\mathcal{M}_1) = 0$$

Numerical analysis: symmetric vortex with $L = M = 1$



$$\mathcal{M}_2 \mathbf{c} = \gamma \mathbf{c} : \quad n(\mathcal{M}_2) = 0, \quad z(\mathcal{M}_2) = 2, \quad p(\mathcal{M}_2) = 2$$

Numerical analysis: symmetric vortex with $L = M = 2$



$$\mathcal{M}_2 \mathbf{c} = \gamma \mathbf{c} : \quad n(\mathcal{M}_2) = 1, \quad z(\mathcal{M}_2) = 2, \quad p(\mathcal{M}_2) = 5$$

Summary:

- Systematic classification of discrete vortices
- Rigorous study of their existence and stability
- Predictions of stable and unstable vortices

contour S_M	vortex of charge L	linearized energy H	stable and unstable eigenvalues
$M = 1$	symmetric $L = 1$	$n(H) = 5, p(H) = 2$	$N_r = 0, N_i^+ = 1, N_i^- = 2, N_c = 0$
$M = 2$	symmetric $L = 1$	$n(H) = 8, p(H) = 7$	$N_r = 1, N_i^+ = 0, N_i^- = 0, N_c = 3$
$M = 2$	symmetric $L = 2$	$n(H) = 10, p(H) = 5$	$N_r = 1, N_i^+ = 2, N_i^- = 4, N_c = 0$
$M = 2$	symmetric $L = 3$	$n(H) = 15, p(H) = 0$	$N_r = 0, N_i^+ = 0, N_i^- = 7, N_c = 0$
$M = 2$	asymmetric $L = 1$	$n(H) = 9, p(H) = 6$	$N_r = 6, N_i^+ = 0, N_i^- = 1, N_c = 0$
$M = 2$	asymmetric $L = 3$	$n(H) = 14, p(H) = 1$	$N_r = 1, N_i^+ = 0, N_i^- = 6, N_c = 0$