Heteroclinic orbits for travelling kinks in difference and nonlocal wave equations

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1D case:

\[ u_{tt} - u_{xx} + V'(u) = 0 \]

where \( V(u) \) is nonlinear potential (depends on a physical context)

Kink (domain walls) solutions (steady or moving):

\[ \lim_{x \to -\infty} u(x, t) = u_2, \quad \lim_{x \to \infty} u(x, t) = u_1; \]
Travelling waves: \( u(x, t) = u(x - ct) \equiv u(z) \).

ODE: \((1 - c^2)u_{zz} - V'(u) = 0\)
Example 1: the sine-Gordon equation

\[ u_{tt} - u_{xx} + \sin u = 0. \]

Travelling waves: \((1 - c^2)u_{zz} = \sin u.\)
Nonlinear wave equation

- Only \(2\pi\)-kink (antikink) solutions exist
- Solutions exist for arbitrary velocity \(c\) as long as \(c^2 < 1\)

\[
u(z) = 4 \arctan \exp \left\{ \pm \frac{z - z_0}{\sqrt{1 - c^2}} \right\}, \quad z = x - ct.
\]
Example 2: the double sine-Gordon equation

\[ u_{tt} - u_{xx} + \sin u - 2A \sin 2u = 0. \]

- Exact \( 2\pi \)-kink solution exist for \( 1 - 4A > 0 \):
  \[
  u(z) = \pi + 2 \arctan \left( \frac{\sinh(\sqrt{1 - 4A} (z - z_0))}{\sqrt{1 - 4A} \sqrt{1 - c^2}} \right),
  \]
  \[ z = x - ct \]

- Solution exist for arbitrary velocity \( c \) as long as \( c^2 < 1 \)
Example 3: the $\phi^4$ equation

$u_{tt} - u_{xx} - u + u^3 = 0$.

- Exact kink solution, exists for any $c^2 < 1$,

$$u(z) = \tanh \left( \frac{z - z_0}{\sqrt{2} \sqrt{1 - c^2}} \right), \quad z = x - ct$$
Example 4: the $\phi^4 - \phi^6$ equation

$$u_{tt} - u_{xx} - u(1 - u^2)(1 + \gamma u^2) = 0.$$  

- Exact kink solution, exists for any $c^2 < 1$ and $\gamma > -1$:

$$u(z) = \frac{\sqrt{18 + 6\gamma \tanh \left( \frac{1}{2} \sqrt{2(1 + \gamma)} (z - z_0) \right)}}{\sqrt{18(1 + \gamma) - 12\gamma \tanh^2 \left( \frac{1}{2} \sqrt{2(1 + \gamma)} (z - z_0) \right)}},$$

$$z = \frac{x - ct}{\sqrt{1 - c^2}}.$$
Nonlocal nonlinear wave equation

Generic form:

\[ u_{tt} - \mathcal{L}u + V'(u) = 0 \]

• \( \mathcal{L} \) is Fourier multiplier operator: \( \widehat{\mathcal{L}u(k)} = P(k)\hat{u}(k) \);

• \( P(k) \) is the symbol of the operator \( \mathcal{L} \);

• If \( P(k) = -k^2 \), we are back to the nonlinear wave equation.
Nonlocal nonlinear wave equation

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Applications of nonlocal wave equations:

- discrete models (e.g. lattice models of solid state physics);
- complex dispersion (e.g. nonlinear optics);
- long-range interaction (e.g. models in solid state physics);
- specific geometry (e.g. Josephson junction theory).
Nonlocal nonlinear wave equation

Symbols:

- \( P(k) = -\frac{4}{\lambda^2} \sin^2 \left( \frac{\lambda k}{2} \right) \) (Frenkel-Kontorova model, solid state physics);
- \( P(k) = -\frac{k^2}{1 + \lambda^2 k^2} \) (Kac-Baker model, spin systems);
- \( P(k) = -\frac{k^2}{\sqrt{1 + \lambda^2 k^2}} \) (Silin-Gurevich model, Josephson junctions);
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In all these cases: \( P(k) \equiv P_\lambda(k) \) depends on \( \lambda \) and

\[ P_\lambda(k) \to -k^2 \quad \text{as} \quad \lambda \to 0. \]

As \( \lambda \to 0 \)

\[ u_{tt} - \mathcal{L}_\lambda u + V'(u) = 0 \quad \Rightarrow \quad u_{tt} - u_{xx} + V'(u) = 0 \]
Main question:
What happens with kink solutions when switching from local case $\lambda = 0$ to nonlocal case $\lambda \neq 0$?
Example 5: the Frenkel-Kontorova model (1938)

\[ u_{tt}(x, t) - \frac{1}{\lambda^2} (u(x + \lambda, t) - 2u(x, t) + u(x - \lambda, t)) + \sin u(x, t) = 0. \]

describes a chain of particles with nearest-neighbours interactions.

\[ \lambda \] - a parameter of interaction between neighbours.
The Frenkel-Kontorova model

The symbol: \[ P(k) = -\frac{4}{\lambda^2} \sin^2 \left( \frac{\lambda k}{2} \right) \]

The results (well-known):

- There are at rest $2\pi$-kinks (on-site and inter-site) in this model.
- No travelling $2\pi$-kinks in this model.
- Infinitely many travelling $4\pi$-kinks in this model.
- A kink-like excitation launched at some nonzero velocity emits radiation, slows down, and eventually stops.
The Frenkel-Kontorova model

(from M.Peyrard, M.D.Kruskal, Physica D, 14, p.88 (1984), initial velocity =0.8.)
The Frenkel-Kontorova model

Why do kink solutions disappear?

Consider linearized version of the Frenkel-Kontorova model at zero equilibrium:

$$u_{tt}(x, t) - \frac{1}{\lambda^2}(u(x + \lambda, t) - 2u(x, t) + u(x - \lambda, t)) + u(x, t) = 0.$$ 

Dispersion relation for Fourier transform:

$$1 + \frac{4}{\lambda^2} \sin^2 \left(\frac{\lambda k}{2}\right) = c^2 k^2, \quad k \in \mathbb{R},$$

For every $c \neq 0$, there exists at least one pair of solutions at $k = \pm k_0$. 
Example 6: the sine-Gordon model with Kac-Baker interactions

\[ u_{tt} - \frac{1}{2\lambda} \frac{d}{dx} \int_{-\infty}^{\infty} \exp \left( \frac{|x - x'|}{\lambda} \right) u_{x'}(x', t) \, dx' + \sin u = 0. \]
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The trick:

\[ q(x, t) = \frac{1}{2\lambda} \int_{-\infty}^{\infty} \exp \left\{ - \frac{|x - x'|}{\lambda} \right\} u_{x'}(x', t) \, dx' \]

Then \( q(x, t) \) is a solution of:

\[ -\lambda^2 q_{xx} + q = u_x. \]

The symbol: \( P(k) = -\frac{k^2}{1 + \lambda^2 k^2} \)
Travelling waves: $u(z) = u(x - ct)$

$$c^2 u_{zz} + \sin u = q_z$$

$$-\lambda^2 q_{zz} + q = u_z$$
SG equation with Kac-Baker interactions

Travelling waves: \( u(z) = u(x - ct) \)

\[
\begin{align*}
  c^2 u_{zz} + \sin u &= q_z \\
  -\lambda^2 q_{zz} + q &= u_z
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Phase space: \( \{ u \ (\text{mod} \ 2\pi), u', q, q' \} \)
SG equation with Kac-Baker interactions

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Equilibrium points:
\( O_0 (u = u' = q = q' = 0) \), \( O_\pi (u = \pi, u' = q = q' = 0) \)
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Equilibrium points:
\( O_0( u = u' = q = q' = 0), \ O_\pi( u = \pi, u' = q = q' = 0) \)

\( O_0 \) is the saddle–center point:

\[
1 + \frac{k^2}{1 + \lambda^2 k^2} = c^2 k^2
\]

For every \( c \neq 0 \), there exists exactly one pair of solutions at \( k = \pm k_0 \).
Results:

- There are static $2\pi$-kinks for $0 < \lambda < 1$.
- No travelling $2\pi$-kinks in this model;
- Infinitely many $4\pi$-kinks for discrete set of velocities;
**Results:**

- There are static $2\pi$-kinks for $0 < \lambda < 1$.
- No travelling $2\pi$-kinks in this model;
- Infinitely many $4\pi$-kinks for discrete set of velocities;

**Summary:** switching from $\lambda = 0$ to $\lambda \neq 0$ results in disappearance of $2\pi$-kink solutions in classical models.

Is this the only scenario?
Consider the bifurcation problem in the general form

\[ L_\lambda u = F(u). \]

- \( L_\lambda \) - a Fourier multiplier operator with an even symbol \( P_\lambda(k) \) such that
  \[ L_\lambda \to \frac{d^2}{dx^2} \text{ as } \lambda \to 0; \]

- \( F(u) \) - an odd function such that \( F(u_+) = F(u_-) = 0 \) with \( u_+ = -u_- \) and
  \[ F'(u_+) = F'(u_-) > 0 \]

- Dispersion equation \( P_\lambda(k) = F'(u_\pm) \) has one pair of roots \( k = \pm k_0(\lambda) \), such that \( k_0(\lambda) \to \infty \) as \( \lambda \to 0 \).
Let us consider the limiting equation $u''(z) = F(u(z))$ and assume:

- It has an odd kink solution $u_0(z)$ for $z \in \mathbb{R}$ such that $u_0(z) \to u_\pm$ as $z \to \pm \infty$.

- When $u_0(z)$ is continued for $z \in \mathbb{C}$, the closest to real axis singularities are located in quartets, e.g. in the upper half-plane at $z_\pm = \pm \alpha + i \beta$, $\alpha, \beta > 0$. 

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There exists an infinite set of values $\{\lambda_n\}_{n \in \mathbb{N}}$, such that for each $\lambda_n$, the nonlinear equation $L_{\lambda_n} u = F(u)$ admits a kink solution. Moreover, the sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ satisfies the asymptotic law

$$k_0(\lambda_n) \sim \frac{(n\pi + \varphi_0)}{\alpha}, \quad n \to \infty,$$

where $\varphi_0$ is uniquely defined constant. Hence, $\lambda_n \to 0$ as $n \to \infty$. 
Perturbation $v(z) = u(z) - u_0(z)$ satisfies the expanded equation

$$(L_\lambda - F'(u_0)) v = H_\lambda + N(v),$$

where $H_\lambda$ is explicitly computed from $u_0$ and $N(v)$ is $O(v^2)$. 
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- The homogeneous equation $(L_\lambda - F'(u_0)) v = 0$ has a pair of solutions that behave like $e^{\pm ik_0(\lambda)z}$.

- To satisfy the solvability condition at the leading order, we set

$$J_\pm(\lambda) := \int_{-\infty}^{\infty} e^{\pm ik(\lambda)z} H_\lambda(z) dz = 0$$
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  \[
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  \]

- By Darboux principle and asymptotic analysis (Murray, 1984), if $H_\lambda(z) \sim C_0\lambda^q e^{i\pi\kappa/2}(z - z_{\pm})^\kappa$, then
  \[
  J_{\pm}(\lambda) \sim \frac{4\pi\lambda^q|C_0|e^{-\beta k(\lambda)}}{\Gamma(-\kappa)|k(\lambda)|^{\kappa+1}} \cos(\alpha k(\lambda) + \pi/2 - \arg(C_0)).
  \]
Example 7: nonlocal double sine-Gordon model

\[ u_{tt} - \frac{1}{2\lambda} \frac{d}{dx} \int_{-\infty}^{\infty} \exp \left( \frac{|x-x'|}{\lambda} \right) u_{x'}(x') \, dx' = \sin(u) + 2a \sin(2u). \]


• As \( \lambda \to 0 \), the \( 2\pi \)-kinks are given by:

\[ u_0(z) = \pi + 2 \arctan \left[ \frac{1}{\sqrt{1+4a}} \sinh \left( \frac{\sqrt{1+4a}}{\sqrt{1-c^2}} z \right) \right]. \]

• Symmetric pairs of singularities exist for \( a > 0 \) at \( z_{\pm} = \pm \alpha + i \beta \):

\[ \alpha = \frac{\sqrt{1-c^2}}{2\sqrt{1+4a}} \cosh^{-1}(1+8a), \quad \beta = \frac{\pi \sqrt{1-c^2}}{2\sqrt{1+4a}}. \]

• For fixed \( a > 0 \), there exist a discrete set of curve in the \((c, \lambda)\) plane, along which the \( 2\pi \)-kinks exist.
Curves $c(\lambda)$ for $a = 1/8$. 
Nonlocal double SG model

The asymptotic law as $n \to \infty$:

$$2\alpha k_0(\lambda_n) \sim \pi(1 + 2n), \quad \Rightarrow \quad \pi(1 + 2n)\lambda_n = \delta(a, c),$$

with $\varphi_0 = \pi/2$.

<table>
<thead>
<tr>
<th>$1 + 2n$</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
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<tbody>
<tr>
<td>$\delta/(\pi\lambda_n)$</td>
<td>3.7168</td>
<td>4.9763</td>
<td>6.3699</td>
<td>7.8595</td>
<td>9.4541</td>
<td>11.1396</td>
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**Table:** The values of $\delta/(\pi\lambda_n)$ for $a = 1/8$ and $c = 0.1$. 
Nonlocal double SG model

Stability experiment 1

Evolution of kink-like excitation (high energy).
Nonlocal double SG model

Stability experiment 2

Evolution of kink-like excitation (low energy).
Example 8: discrete $\phi^4-\phi^6$ model

$$u_{tt} - \lambda^{-2}(u(x + \lambda) - 2u(x) + u(x - \lambda)) + u(1 - u^2)(1 + \gamma u^2) = 0.$$  


- As $\lambda \to 0$, the kinks are given by:

  $$u_0(z) = \frac{\sqrt{3+\gamma} \tanh(\eta z)}{\sqrt{3(1+\gamma) - 2\gamma \tanh^2(\eta z)}}, \quad \eta = \frac{\sqrt{1+\gamma}}{\sqrt{2(1-c^2)}}.$$  

- Symmetric pairs of singularities exist for $\gamma > 0$ at $z_\pm = \pm \alpha + i\beta$:

  $$\alpha = \frac{\sqrt{1-c^2}}{2\sqrt{1+\gamma}} \cosh^{-1} \left( \frac{3 + 5\gamma}{3 + \gamma} \right), \quad \beta = \frac{\pi \sqrt{1-c^2}}{\sqrt{2(1+a)}}.$$  

- For fixed $\gamma > 0$, there exist a discrete set of curve in the $(c, \lambda)$ plane, along which the kinks exist.
The asymptotic law as $n \to \infty$:

$$4\alpha k_0(\lambda_n) \sim \pi(3 + 4n), \quad \Rightarrow \quad \pi(3 + 4n)\lambda_n = \chi(\gamma, c),$$

with $\varphi_0 = 3\pi/4$.

<table>
<thead>
<tr>
<th>$3 + 4n$</th>
<th>3</th>
<th>7</th>
<th>11</th>
<th>15</th>
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<tbody>
<tr>
<td>$\chi/({\pi}\lambda_n)$</td>
<td>3.5303</td>
<td>7.3547</td>
<td>11.1520</td>
<td>15.0329</td>
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**Table:** The values of $\chi/(\pi\lambda_n)$ for $\gamma = 5$ and $c = 0.6$. 
Example 9: discrete $\phi^4$ model

\[ u_{tt} - \lambda^{-2}(u(x + \lambda) - 2u(x) + u(x - \lambda)) + u(x)(1 - u(x)^2) = 0. \]


- As $\lambda \to 0$, the kinks are given by:
  \[ u_0(z) = \tanh(\eta z), \quad \eta = \frac{1}{2\sqrt{1-c^2}}. \]

- Singularity exists at $z = i\pi\sqrt{1 - c^2}$.

- No kinks exist for any $c \neq 0$. 
Example 10: another discrete $\phi^4$ model

\[ u_{tt} - \lambda^{-2}(u(x + \lambda) - 2u(x) + u(x - \lambda)) \]
\[ + \frac{1}{2}(u(x + \lambda) + u(x - \lambda)) \left( 1 - \frac{1}{2}u(x + \lambda)^2 - \frac{1}{2}u(x - \lambda)^2 \right) = 0. \]


- As $\lambda \to 0$, the kinks are still given by:
  \[ u_0(z) = \tanh(\eta z), \quad \eta = \frac{1}{2\sqrt{1-c^2}}. \]

- Singularity exists at $z = i\pi \sqrt{1-c^2}$.

- Three moving kinks exist for three values of $c \neq 0$ at fixed $\lambda \neq 0$. 
**Summary:** in Examples 7-8, switching from $\lambda = 0$ to $\lambda \neq 0$ results in selecting a **countable set of velocities** for radiationless kink propagation.

- The first ideas about existence of such countable sets go back to the works of V.G. Gelfreich (1990, 2008).

- No analytical proof of the main claim exists for now.

- It has been checked for several other models: triple sine-Gordon model, fifth-order Korteweg-de Vries equation, saturable discrete nonlinear Schrödinger equation, ...

- Apparently, it applies to more sophisticated examples, such as diatomic Toda lattice