Stability of incompressible viscous fluid flows in a thin spherical shell

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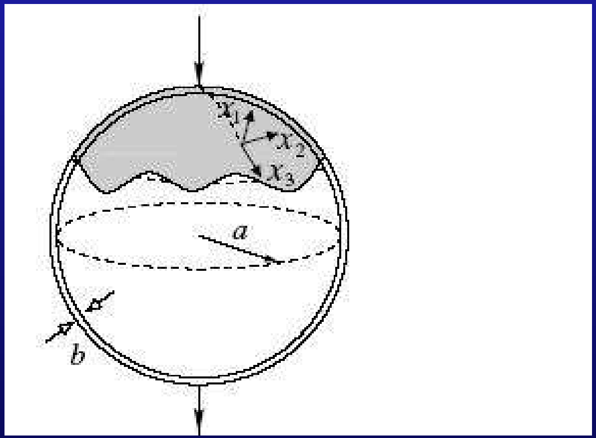
Background

Motivations:

- Navier–Stokes equations in spherical coordinates
- Thin spherical layer confining the fluid motion
- Exact stationary solution for incompressible viscous fluid flows
- Stability and evolution of the stationary solution



Possible applications:

- oil on a metal ball
- ice melting on the Earth surface

Stationary solution

Exact solution of the stationary NS equations

$$u_r = 0, \quad u_\theta = \frac{\alpha}{r \sin\theta}, \quad u_\phi = 0, \quad p = \beta - \frac{\alpha^2}{2r^2 \sin^2\theta},$$

where (α, β) are arbitrary parameters.

Properties:

- fluid flow from the North pole $\theta = 0$ to the South pole $\theta = \pi$
- azimuthal symmetry with respect to ϕ
- no flow along the radial coordinate r

The exact solution is related to Darcy's law on a sphere: see Leandro, Miranda, Moraes, J. Phys. A: Math.Gen. **39**, 1619 (2006)

Averaging theorem

Averaging theorem [Temam-Ziane, 1997]: In the limit $\varepsilon \to 0$, when a thin spherical layer

$$\Omega = \{\mathbf{x} \in \mathbb{R}^3 : 1 < |\mathbf{x}| < 1 + \varepsilon\} \subset \mathbb{R}^3$$

converges to a sphere

$$S = \{(\theta, \phi) : \quad 0 \leqslant \theta \leqslant \pi, \ 0 \leqslant \phi < 2\pi\},$$

the strong global solution of the 3D NS equations $\mathbf{u}(r, \theta, \phi, t)$ converges to the strong unique global solution of the 2D NS equations on the sphere

$$\mathbf{v}(\theta, \phi, t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_1^{1+\varepsilon} r\mathbf{u}(r, \theta, \phi, t) dr = (0, v_\theta, v_\phi).$$

Navier–Stokes equations

Navier–Stokes equations on the sphere:

$$\frac{\partial v_\theta}{\partial t} - \frac{v_\phi \omega}{\sin\theta} + \frac{\partial q}{\partial \theta} = \nu \left(\Delta_S v_\theta - \frac{v_\theta}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta} \frac{\partial v_\phi}{\partial \phi} \right),$$

$$\frac{\partial v_\phi}{\partial t} + \frac{v_\theta \omega}{\sin\theta} + \frac{1}{\sin\theta} \frac{\partial q}{\partial \phi} = \nu \left(\Delta_S v_\phi + \frac{2\cos\theta}{\sin^2\theta} \frac{\partial v_\theta}{\partial \phi} - \frac{v_\phi}{\sin^2\theta} \right),$$

$$\frac{\partial}{\partial \theta} (\sin\theta v_\theta) + \frac{\partial v_\phi}{\partial \phi} = 0,$$

where q is a static pressure, ω is the vorticity, ν is viscosity, and Δ_S is the Laplace–Beltrami operator on sphere S.

This model was used in hydrodynamics since the works of E. Blinova (1943) and Kochin, Kibel and Roze (1948).

Reduction to linearized problem

The stationary solution for the NS equations on the sphere:

$$v_\theta = \frac{\alpha}{\sin\theta}, \qquad v_\phi = 0, \qquad q = \beta.$$

Linearization

$$v_\theta = \frac{1}{\sin\theta} + U(\theta,\phi)e^{\lambda t}, \quad v_\phi = V(\theta,\phi)e^{\lambda t}, \quad q = Q(\theta,\phi)e^{\lambda t}$$

Fourier series in ϕ:

$$U(\theta,\phi) = \sum_{k\in\mathbb{Z}} U_k(\theta)e^{ik\phi}, \quad V(\theta,\phi) = \sum_{k\in\mathbb{Z}} V_k(\theta)e^{ik\phi}$$

Stream function

Stream function formulation for $k \neq 0$:

$$U_k = \frac{ik}{\sin\theta}\Psi_k(\theta), \qquad V_k = -\Psi_k'(\theta)$$

Spectral problem for $k \neq 0$:

$$\Phi_k = \Delta_k \Psi_k, \quad \nu\Delta_k\Phi_k - \frac{\Phi_k'}{\sin\theta} = \lambda\Phi_k,$$

where

$$\Delta_k = \frac{d^2}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d}{d\theta} - \frac{k^2}{\sin^2\theta}.$$

Main results

For energy formulation, we require that

$$\int_0^\pi \left(|U_k|^2 + |V_k|^2\right) \sin\theta d\theta < \infty.$$

Theorem 1: When $\nu > 0$, the stationary flow is *asymptotically stable* such that the spectrum of the linearized problem consists of a set of simple isolated negative eigenvalues λ.

Theorem 2: If the interval $\theta \in [0, \pi]$ is truncated by $\theta \in [\theta_0, \pi - \theta_0]$, the spectrum of the linearized problem consists of a set of isolated real negative eigenvalues for sufficiently large ν and any $0 < \theta_0 < \frac{\pi}{2}$.

Analysis of eigenvalues

- Let $x = \cos\theta$, $\epsilon = 1/\nu$ and $\mu = \lambda/\nu$. Then,

$$L_k\Psi_k = \Phi_k, \qquad L_k\Phi_k + \epsilon\Phi_k' = \mu\Phi_k,$$

where $L_k = \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] - \frac{k^2}{1-x^2}$ is the associated Legendre operator defined on $x \in [-1, 1]$.

- Function space

$$\|\Psi_k\|_{H_k}^2 = \int_{-1}^{1}\left[(1-x^2)|\Psi_k'(x)|^2 + \frac{k^2}{1-x^2}|\Psi_k(x)|^2\right]dx < \infty,$$

such that

$$\lim_{x\to\pm1}\Psi_k(x) = \lim_{x\to\pm1}(1-x^2)\Psi_k'(x) = 0.$$

Analysis of eigenvalues

- Since $(\Psi_k, L_k \Psi_k) = -\|\Psi_k\|^2_{H_k}$, the kernel of L is empty in H_k and $\Psi_k = L_k^{-1} \Phi_k$.

- Transformation $\Phi_k = \left(\frac{1-x}{1+x}\right)^{\epsilon/4} \varphi(x)$ brings the closed equation for $\varphi(x)$:

$$\frac{d}{dx}\left[(1-x^2)\frac{d\varphi}{dx}\right] - \frac{\sigma^2}{1-x^2}\varphi + s(s+1)\varphi = 0,$$

where $\sigma = \sqrt{k^2 + \epsilon^2/4} > 0$ and $\mu = -s(s+1)$.

- By the ODE theory in regular singular points $x = \pm 1$, $\Psi \in H_k$ if and only if $\varphi \to (1-x^2)^{\sigma/2}$ as $x \to \pm 1$. The singular components $(1-x^2)^{-\sigma/2}$ must be removed from the solution.

- The essential spectrum is empty (Dunford, Schwartz, 1963).

Analysis of eigenvalues

- There exists a reduction to the hypergeometric equation $\varphi = (1-x^2)^{\sigma/2}F(z)$ with $z = (1-x)/2$, where the hypergeometric function admits the power series at $z = 0$:

$$F(z;\alpha,\beta,\gamma) = 1 + \frac{\alpha\beta}{\gamma 1!}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!}z^2 + \ldots,$$

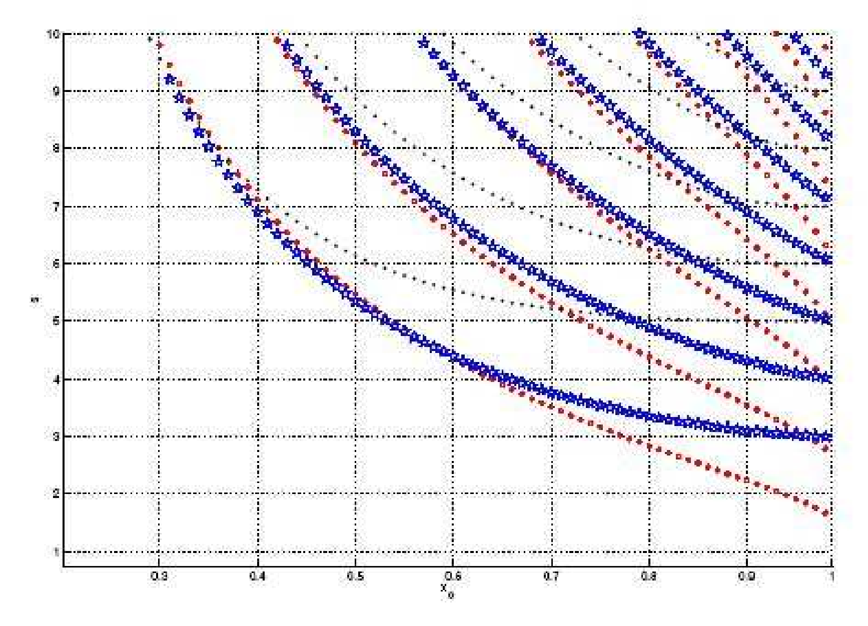
where $\alpha = \sigma - s$, $\beta = \sigma + s + 1$, and $\gamma = \sigma + 1$.

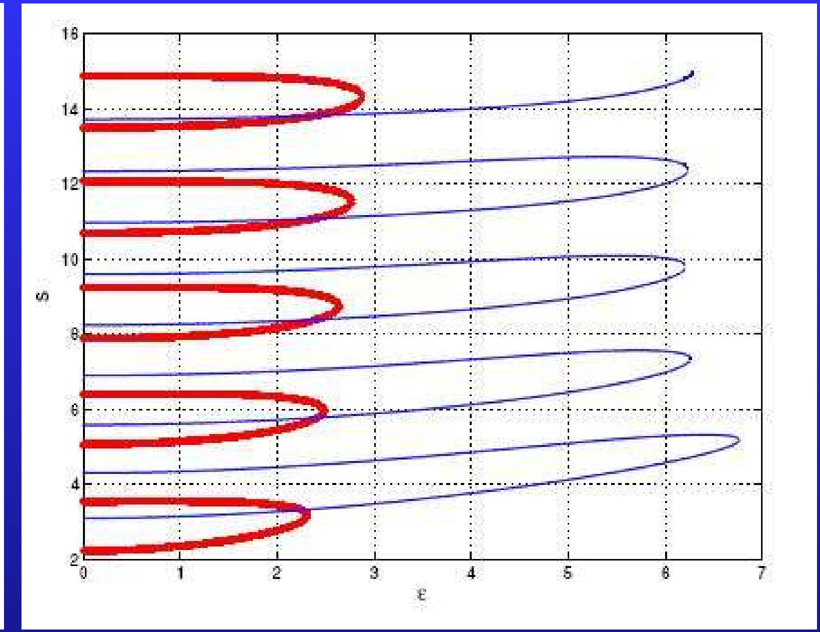
- The function $F(z;\alpha,\beta,\gamma)$ with $\alpha + \beta - \gamma = \sigma > 0$ diverges as $z \to 1$ unless the truncation of the power series occurs at $\alpha = -n$, $n \in \mathbb{Z}$, such that

$$\mu_n = -s_n(s_n+1), \quad s_n = \sigma + n, \quad n \in \mathbb{Z},$$

and $F_n(z)$ is a polynomial of degree n.

Approximations of eigenvalues for $k \neq 0$





$\mu = -s(s+1)$

For $\epsilon > \epsilon_0 > 0$ or $\nu < \nu_0 < \infty$, real eigenvalues coalesce and split to the complex domain with $\mathrm{Re}(\lambda) < 0$.

Analysis of eigenvalues for $k = 0$

- In the same variables,

$$L_0\Psi_0 = \Phi_0, \qquad \Phi_0' + \frac{\epsilon}{1-x^2}\Phi_0 = \mu\Psi_0',$$

where $L_0 = \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right]$ is the Legendre operator.

- Function space H_0 with the norm

$$\|\Psi_0\|_{H_0}^2 = \int_{-1}^{1}(1-x^2)|\Psi_0'(x)|^2 dx < \infty.$$

- The eigenvalue $\mu = 0$ with the eigenfunctions $\Phi_0 = 0$ and $\Psi_0 = 1$ is algebraically simple.

Analysis of eigenvalues for $k = 0$

- Transformation $\Phi_0 = \left(\frac{1-x}{1+x}\right)^{\epsilon/4} \varphi(x)$ brings the closed equation for $\varphi(x)$:

$$\frac{d}{dx}\left[(1-x^2)\frac{d\varphi}{dx}\right] - \frac{\epsilon^2}{4(1-x^2)}\varphi + s(s+1)\varphi = 0,$$

with the relation

$$\mu\Psi_0'(x) = \left(\frac{1-x}{1+x}\right)^{\epsilon/4}\left(\frac{d\varphi}{dx} + \frac{\epsilon}{2(1-x^2)}\varphi\right),$$

where $\mu = -s(s+1)$.

- From the regular behavior at $x \to 1$, it follows that $\varphi = (1-x^2)^{\epsilon/2} F(z;\alpha,\beta,\gamma)$ with $z = (1-x)/2$, $\alpha = \epsilon/2 - s$, $\beta = \epsilon/2 + s + 1$, and $\gamma = \epsilon/2 + 1$.

Analysis of eigenvalues for $k = 0$

- To study the behavior at $x \to -1$, we use the relation

$$F(z;\alpha,\beta,\gamma) = (1-z)^{\gamma-\alpha-\beta} F(z;\gamma-\alpha,\gamma-\beta,\gamma),$$

  where $\gamma - \alpha - \beta = -\epsilon/2$.

- When the power series for $F(z;\gamma-\alpha,\gamma-\beta,\gamma)$ is truncated (for $s = n$), the solution $\Psi_0(x)$ is in H_0 if $\epsilon < 2$.

- When $\epsilon \geq 2$, the solution $\Psi_0(x)$ is not in H_0 for any solution $F(z;\alpha,\beta,\gamma)$.

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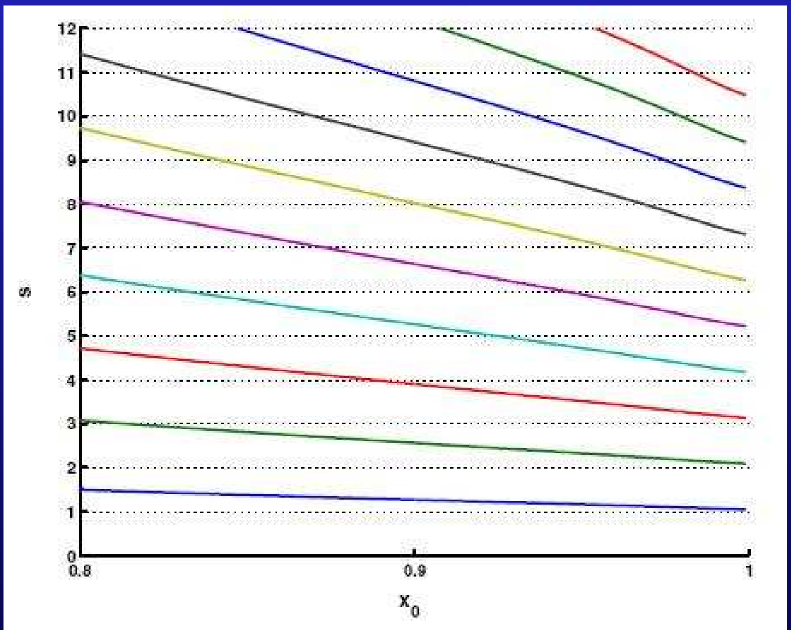
- When the power series for $F(z;\gamma-\alpha,\gamma-\beta,\gamma)$ is truncated (for $s = n$), the solution $\Psi_0(x)$ is in H_0 if $\epsilon < 2$.
- When $\epsilon \geq 2$, the solution $\Psi_0(x)$ is not in H_0 for any solution $F(z;\alpha,\beta,\gamma)$.

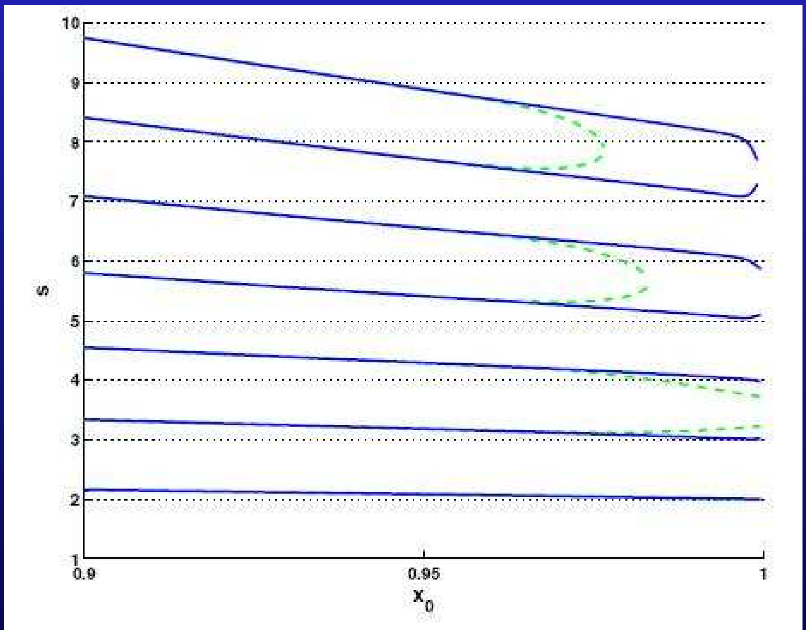
$\Rightarrow$ Where are eigenvalues for $\epsilon \geq 2$???

Approximations of eigenvalues for $k = 0$

If $\mu_n = -s_n(s_n+1)$ are eigenvalues of the Dirichlet problem for $x_0 < 1$, then

$$\lim_{x_0\to 1} s_n = 1+n \quad \text{for} \quad 0 \le \epsilon \le 2, \qquad \lim_{x_0\to 1} s_n = \frac{\epsilon}{2} + n \quad \text{for} \quad \epsilon \ge 2.$$





$\mu = -s(s+1)$: $\epsilon = 1$ (left) and $\epsilon = 4$ (right)

Open problem for $k = 0$

Consider the time-dependent heat equation

$$\frac{\partial v}{\partial t} + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}(\sin\theta\ v) = \nu\left(\frac{\partial^2 v}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial v}{\partial\theta} - \frac{v}{\sin^2\theta}\right)$$

- The nonlinear Navier–Stokes equations on the sphere reduces *exactly* to the linear heat equation for

$$v_\theta = \frac{1}{\sin\theta}, \qquad v_\phi = v(\theta,t), \qquad q = q(\theta,t).$$

- When $v(\theta,t) = -\Psi_0'(\theta)e^{\lambda t}$, the spectrum of the linear operator is empty in space with $\int_0^\pi |v(\theta,t)|^2 \sin\theta d\theta < \infty$ for $\nu \le \frac{1}{2}$.
- Is the Cauchy problem well-posed for $\nu \le \frac{1}{2}$?

Summary

- Navier–Stokes equations on the sphere arise *asymptotically* from the three-dimensional NS equations for a thin layer.
- Exact stationary solutions of the NS equations in spherical coordinates are available.
- Stability of exact stationary solutions can be understood from the linearized problem, which admits reduction to the associated Legendre and hypergeometric equations.
- Convergence of eigenvalues of the associated Legendre equations on the truncated domains may have interesting limiting features.
- Well-posedness of time-dependent linearized equations is an interesting further direction of studies.