Stability of incompressible viscous fluid flows in a thin spherical shell

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Reference: Journal of Mathematical Fluid Mechanics, accepted

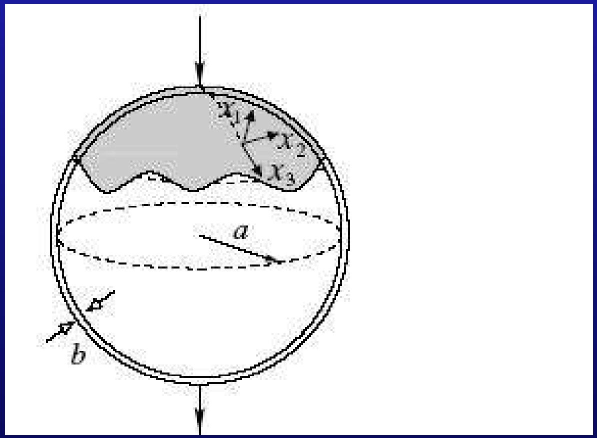
Plan of the talk:

- Background in fluid mechanics
- Analysis of eigenvalues
- Numerical illustrations

Background

Motivations:

- Navier–Stokes equations in spherical coordinates
- Thin spherical layer confining the fluid motion
- Exact stationary solution for incompressible viscous fluid flows
- Stability and evolution of the stationary solution



Possible applications:

- oil on a metal ball
- ice melting on the Earth surface

Stationary solution

Exact solution of the stationary NS equations

$$u_r = 0, \quad u_\theta = \frac{\alpha}{r \sin\theta}, \quad u_\phi = 0, \quad p = \beta - \frac{\alpha^2}{2r^2 \sin^2\theta},$$

where (α, β) are arbitrary parameters.

Properties:

- fluid flow from the North pole $\theta = 0$ to the South pole $\theta = \pi$
- azimuthal symmetry with respect to ϕ
- no flow along the radial coordinate r

Similar studies in fluid mechanics

- eigenfunction decompositions in spherical harmonics by Blinova (1943,1956)
- spectral approximations in spherical coordinates by Ben-Yu (1995), Furnier et al. (2004), Mohseni-Colonius (2000), Simonnet (2000), and many others
- vortex dynamics on a sphere by Boatto-Cabral (2003), Crowdy (2006)
- Darcy's law in curved spaces by Parision et al. (2001), Leandro-Miranda-Moraes (2006)
- averaging technique for Navier–Stokes equations in domains with a thin layer by Temam-Ziane (1997), Iftimie-Raugel (2001)

Averaging theorem

Averaging theorem [Temam-Ziane, 1997]: In the limit $\varepsilon \to 0$, when a thin spherical layer

$$\Omega = \{\mathbf{x} \in \mathbb{R}^3 : 1 < |\mathbf{x}| < 1 + \varepsilon\} \subset \mathbb{R}^3$$

converges to a sphere

$$S = \{(\theta, \phi) : \quad 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \phi < 2\pi\},$$

the strong global solution of the 3D NS equations $\mathbf{u}(r, \theta, \phi, t)$ converges to the strong unique global solution of the 2D NS equations on the sphere

$$\mathbf{v}(\theta, \phi, t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_1^{1+\varepsilon} r\mathbf{u}(r, \theta, \phi, t) dr = (0, v_\theta, v_\phi).$$

Navier–Stokes equations

Navier–Stokes equations on the sphere:

$$\frac{\partial v_\theta}{\partial t} - \frac{v_\phi \omega}{\sin\theta} + \frac{\partial q}{\partial \theta} = \nu \left(\Delta_S v_\theta - \frac{v_\theta}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\phi}{\partial \phi} \right),$$

$$\frac{\partial v_\phi}{\partial t} + \frac{v_\theta \omega}{\sin\theta} + \frac{1}{\sin\theta}\frac{\partial q}{\partial \phi} = \nu \left(\Delta_S v_\phi + \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\theta}{\partial \phi} - \frac{v_\phi}{\sin^2\theta} \right),$$

$$\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{\partial v_\phi}{\partial \phi} = 0,$$

where q is a static pressure, ω is the vorticity, ν is viscosity, and Δ_S is the Laplace–Beltrami operator on sphere S.

The stationary solution persists the asymptotic reduction

$$v_\theta = \frac{\alpha}{\sin\theta}, \qquad v_\phi = 0, \qquad q = \beta.$$

Reduction to linearized problem

Linearization

$$v_\theta = \frac{1}{\sin\theta} + U(\theta,\phi)e^{\lambda t}, \quad v_\phi = V(\theta,\phi)e^{\lambda t}, \quad q = Q(\theta,\phi)e^{\lambda t}$$

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Fourier series in ϕ:

$$U(\theta,\phi) = \sum_{k\in\mathbb{Z}} U_k(\theta)e^{ik\phi}, \quad V(\theta,\phi) = \sum_{k\in\mathbb{Z}} V_k(\theta)e^{ik\phi}$$

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Stream function formulation $U_k = \frac{ik}{\sin\theta}\Psi_k(\theta)$ and $V_k = -\Psi_k'(\theta)$ for $k \neq 0$ results in the spectral problem

$$\Phi_k = \Delta_k \Psi_k, \quad \nu\Delta_k\Phi_k - \frac{\Phi_k'}{\sin\theta} = \lambda\Phi_k.$$

Function spaces

For $k = 0$, $U_0 = Q_0 = 0$ (translations in α and β are excluded) and $V_0 = -\Psi_0'(\theta)$ solves the third-order spectral problem

$$\frac{d}{d\theta}\left(\nu\Delta_0\Psi_0 - \lambda\Psi_0\right) - \frac{1}{\sin\theta}\Delta_0\Psi_0 = 0,$$

where $\Delta_k = \frac{d^2}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d}{d\theta} - \frac{k^2}{\sin^2\theta}$.

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For applications, we require that

$$\int_0^\pi \left(|U_k|^2 + |V_k|^2\right)\sin\theta d\theta < \infty,$$

or, in the stream function formulation, for any k

$$\int_0^\pi \left(|\Psi_k'|^2 + \frac{k^2|\Psi_k|^2}{\sin^2\theta}\right)\sin\theta d\theta < \infty.$$

Main results

Theorem 1: When $\nu > 0$, the stationary flow is *asymptotically stable* with respect to *symmetry-breaking* perturbations in the sense that the spectrum of the linearized problem with $k \neq 0$ consists of a set of isolated real negative eigenvalues λ of finite multiplicities.

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Theorem 2: The stationary flow is *asymptotically stable* with respect to *symmetry-preserving* perturbations with $k = 0$ for $\nu \geq \frac{1}{2}$. The spectrum of the linearized problem is empty for $0 < \nu < \frac{1}{2}$.

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Theorem 2: The stationary flow is *asymptotically stable* with respect to *symmetry-preserving* perturbations with $k = 0$ for $\nu \geq \frac{1}{2}$. The spectrum of the linearized problem is empty for $0 < \nu < \frac{1}{2}$.

Theorem 3: If the interval $\theta \in [0, \pi]$ is truncated by $\theta \in [\theta_0, \pi - \theta_0]$, the spectrum of the linearized problem consists of a set of isolated real negative eigenvalues for sufficiently large ν and any $0 < \theta_0 < \frac{\pi}{2}$.

Analysis: $k \neq 0$

- Let $x = \cos\theta$, $\epsilon = 1/\nu$ and $\mu = \lambda/\nu$. Then,

$$L_k \Psi_k = \Phi_k, \qquad L_k \Phi_k + \epsilon \Phi_k' = \mu \Phi_k,$$

  where $L_k = \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] - \frac{k^2}{1-x^2}$ is the associated Legendre operator defined on $x \in [-1, 1]$.

- Function space

$$\|\Psi_k\|_{H_k}^2 = \int_{-1}^{1} \left[(1-x^2)|\Psi_k'(x)|^2 + \frac{k^2}{1-x^2}|\Psi_k(x)|^2\right] dx < \infty,$$

  such that

$$\lim_{x \to \pm 1} \Psi_k(x) = \lim_{x \to \pm 1} (1-x^2)\Psi_k'(x) = 0.$$

Analysis: $k \neq 0$

- Since $(\Psi_k, L_k \Psi_k) = -\|\Psi_k\|^2_{H_k}$, the kernel of L is empty in H_k and $\Psi_k = L_k^{-1}\Phi_k$.
- Transformation $\Phi_k = \left(\frac{1-x}{1+x}\right)^{\epsilon/4} \varphi(x)$ brings the closed equation for $\varphi(x)$:

$$\frac{d}{dx}\left[(1-x^2)\frac{d\varphi}{dx}\right] - \frac{\sigma^2}{1-x^2}\varphi + s(s+1)\varphi = 0,$$

where $\sigma = \sqrt{k^2 + \epsilon^2/4} > 0$ and $\mu = -s(s+1)$.

- By the ODE theory in regular singular points $x = \pm 1$, $\Psi \in H_k$ if and only if $\varphi \to (1-x^2)^{\sigma/2}$ as $x \to \pm 1$. The singular components $(1-x^2)^{-\sigma/2}$ must be removed from the solution.
- The essential spectrum is empty (Dunford, Schwartz, 1963).

Analysis: $k \neq 0$

- There exists a reduction to the hypergeometric equation $\varphi = (1 - x^2)^{\sigma/2} F(z)$ with $z = (1 - x)/2$, where the hypergeometric function admits the power series at $z = 0$:

$$F(z; \alpha, \beta, \gamma) = 1 + \frac{\alpha\beta}{\gamma 1!} z + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{\gamma(\gamma + 1)2!} z^2 + ...,$$

  where $\alpha = \sigma - s$, $\beta = \sigma + s + 1$, and $\gamma = \sigma + 1$.

- The function $F(z; \alpha, \beta, \gamma)$ with $\alpha + \beta - \gamma = \sigma > 0$ diverges as $z \to 1$ unless the truncation of the power series occurs at $\alpha = -n$, $n \in \mathbb{Z}$, such that

$$\mu_n = -s_n(s_n + 1), \quad s_n = \sigma + n, \quad n \in \mathbb{Z},$$

  and $F_n(z)$ is a polynomial of degree n.

Analysis: $k = 0$

- In the same variables,

$$L_0 \Psi_0 = \Phi_0, \qquad \Phi_0' + \frac{\epsilon}{1 - x^2} \Phi_0 = \mu \Psi_0',$$

where $L_0 = \frac{d}{dx} \left[(1 - x^2) \frac{d}{dx} \right]$ is the Legendre operator.

- Function space H_0 with the norm

$$\|\Psi_0\|_{H_0}^2 = \int_{-1}^{1} (1 - x^2) |\Psi_0'(x)|^2 dx < \infty.$$

- The eigenvalue $\mu = 0$ with the eigenfunctions $\Phi_0 = 0$ and $\Psi_0 = 1$ is algebraically simple.
- The essential spectrum is empty (Dunford, Schwartz, 1963).

Analysis: $k = 0$

- Transformation $\Phi_0 = \left(\frac{1-x}{1+x}\right)^{\epsilon/4} \varphi(x)$ brings the closed equation for $\varphi(x)$:

$$\frac{d}{dx}\left[(1-x^2)\frac{d\varphi}{dx}\right] - \frac{\epsilon^2}{4(1-x^2)}\varphi + s(s+1)\varphi = 0,$$

with the relation

$$\mu \Psi_0'(x) = \left(\frac{1-x}{1+x}\right)^{\epsilon/4} \left(\frac{d\varphi}{dx} + \frac{\epsilon}{2(1-x^2)}\varphi\right),$$

where $\mu = -s(s+1)$.

- From the regular behavior at $x \to 1$, it follows that $\varphi = (1-x^2)^{\epsilon/2} F(z; \alpha, \beta, \gamma)$ with $z = (1-x)/2$, $\alpha = \epsilon/2 - s$, $\beta = \epsilon/2 + s + 1$, and $\gamma = \epsilon/2 + 1$.

Analysis: $k = 0$

- To study the behavior at $x \to -1$, we use the relation

$$F(z; \alpha, \beta, \gamma) = (1 - z)^{\gamma-\alpha-\beta} F(z; \gamma - \alpha, \gamma - \beta, \gamma),$$

  where $\gamma - \alpha - \beta = -\epsilon/2$.

- When the power series for $F(z; \gamma - \alpha, \gamma - \beta, \gamma)$ is truncated (for $s = n$), the solution $\Psi_0(x)$ is in H_0 if $\epsilon < 2$.

- When $\epsilon \geq 2$, the solution $\Psi_0(x)$ is not in H_0 for any solution $F(z; \alpha, \beta, \gamma)$.

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- When $\epsilon \geq 2$, the solution $\Psi_0(x)$ is not in H_0 for any solution $F(z;\alpha,\beta,\gamma)$.

$\Rightarrow$ Where are eigenvalues for $\epsilon \geq 2$???

Convergence results for $k \neq 0$

The rescaled eigenvalue problem is

$$L_k \Psi_k = \Phi_k, \qquad L_k \Phi_k + \epsilon \Phi_k' = \mu \Phi_k,$$

on $x \in [-x_0, x_0]$ for $0 < x_0 < 1$ subject to the boundary conditions

$$\Psi_k(\pm x_0) = \Psi_k'(\pm x_0) = 0$$

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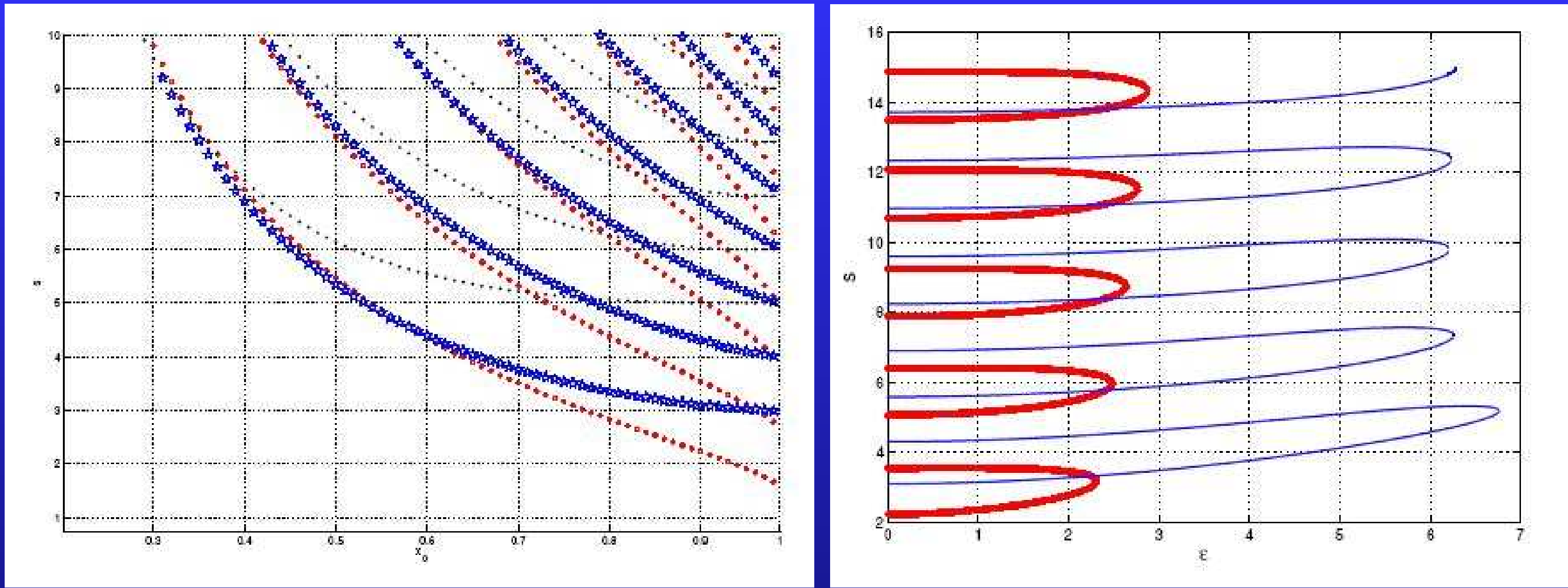
- The spectrum for $\epsilon = 0$ consists of real negative isolated eigenvalues μ with at most two independent eigenfunctions.
- By the perturbation theory, the spectrum remains on the real axis for sufficiently small $\epsilon \neq 0$.
- Eigenvalues μ for $x_0 < 1$ converges to simple eigenvalues μ as $x_0 \to 1$ for sufficiently small ϵ (Bailey et al., 1993).

Approximations of eigenvalues for $k \neq 0$



$\mu = -s(s+1)$

For $\epsilon > \epsilon_0 > 0$ or $\nu < \nu_0 < \infty$, real eigenvalues coalesce and split to the complex domain with $\mathrm{Re}(\lambda) < 0$.

Convergence results for $k = 0$

The transformation

$$\varphi(x) = \sqrt{1-x^2}\chi'(x) - \frac{\epsilon + 2x}{2\sqrt{1-x^2}}\chi(x),$$

brings the associated Legendre equation to the form

$$\frac{d}{dx}\left[(1-x^2)\frac{d\chi}{dx}\right] - \frac{\epsilon^2 + 4 + 4\epsilon x}{4(1-x^2)}\chi = \mu\chi,$$

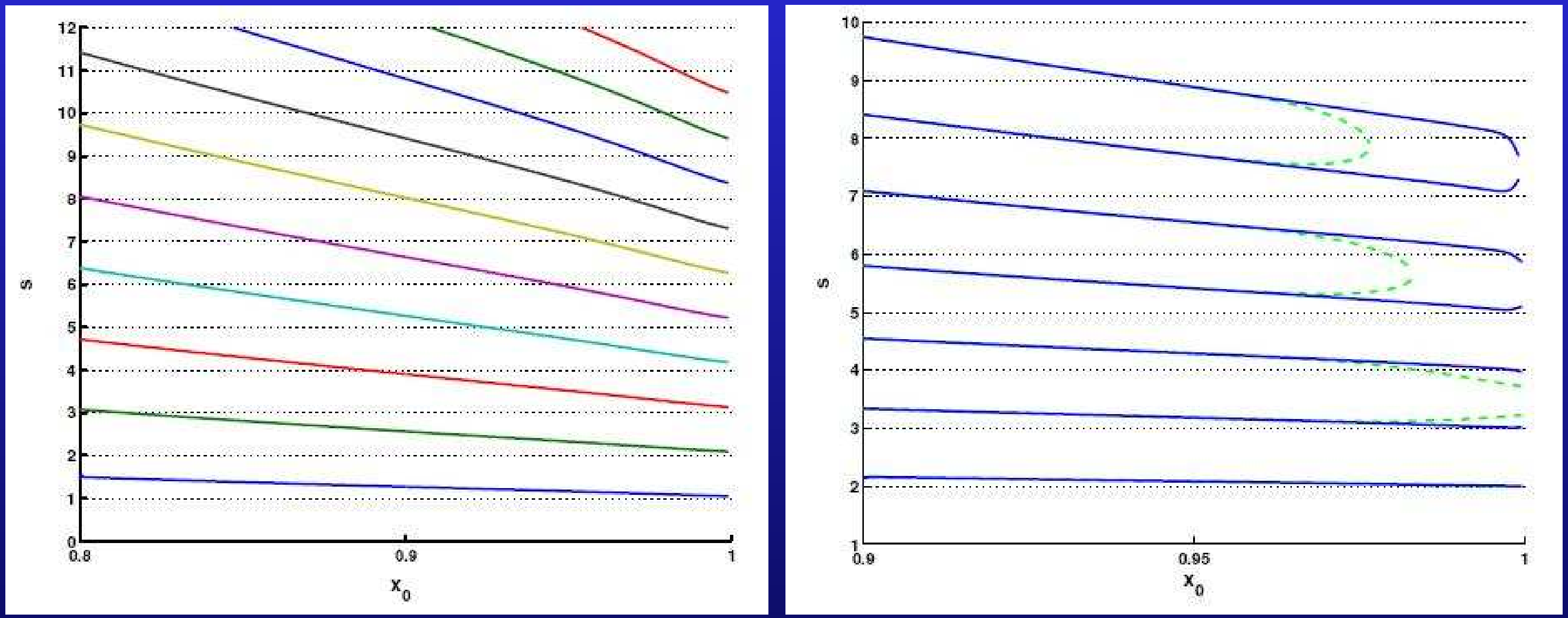
where

$$\Psi_0'(x) = \left(\frac{1-x}{1+x}\right)^{\epsilon/4} \frac{\chi(x)}{\sqrt{1-x^2}}.$$

If $\Psi_0(x)$ satisfies Neumann boundary conditions, then $\chi(x)$ satisfies Dirichlet boundary conditions on $x \in [-x_0, x_0]$ with $0 < x_0 < 1$.

Approximations of eigenvalues for $k = 0$

If $\mu_n = -s_n(s_n + 1)$ are eigenvalues of the Dirichlet problem for $x_0 < 1$, then $\lim_{x_0 \to 1} s_n = 1 + n$ for $0 \le \epsilon \le 2$ and $\lim_{x_0 \to 1} s_n = \frac{\epsilon}{2} + n$ for $\epsilon \ge 2$.



$\mu = -s(s+1)$: $\epsilon = 1$ (left) and $\epsilon = 4$ (right)

Open problem for $k = 0$

Consider the time-dependent heat equation

$$\frac{\partial v}{\partial t} + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}(\sin\theta\ v) = \nu\left(\frac{\partial^2 v}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial v}{\partial\theta} - \frac{v}{\sin^2\theta}\right)$$

- The nonlinear Navier–Stokes equations on the sphere reduces *exactly* to the linear heat equation for

$$v_\theta = \frac{1}{\sin^2\theta}, \qquad v_\phi = v(\theta, t), \qquad q = q(\theta, t).$$

- When $v(\theta, t) = -\Psi_0'(\theta)e^{\lambda t}$, the spectrum of the linear operator is empty in space with $\int_0^\pi |v(\theta, t)|^2 \sin\theta d\theta < \infty$ for $\nu \leq \frac{1}{2}$.
- Is the Cauchy problem well-posed for $\nu \leq \frac{1}{2}$?

Summary

- Navier–Stokes equations on the sphere arise *asymptotically* from the three-dimensional NS equations for a thin layer.
- Exact stationary solutions of the NS equations in spherical coordinates are available.
- Stability of exact stationary solutions can be understood from the linearized problem, which admits reduction to the associated Legendre and hypergeometric equations.
- Convergence of eigenvalues of the associated Legendre equations on the truncated domains may have interesting limiting features.
- Well-posedness of time-dependent linearized equations is an interesting further directions of studies.