Stability of periodic waves in integrable PDEs

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Outline of the lecture

1. Stability of critical points in Hamiltonian systems
2. Integrable PDEs
3. Periodic waves in the defocusing NLS equation
4. Periodic waves in the KP-II equation
5. Summary and open questions

Stability of critical points in Hamiltonian systems

Consider an abstract Hamiltonian dynamical system

$$\frac{du}{dt} = J\, H'(u), \quad u(t) \in X$$

where X is the phase space, $J : X \mapsto X$ is a skew-adjoint operator with a bounded inverse, and $H : X \to \mathbb{R}$ is the Hamilton function.

- Assume existence of a critical point $u_0 \in X$ such that $H'(u_0) = 0$.
- Perform linearization $u(t) = u_0 + v e^{\lambda t}$, where λ is the spectral parameter and $v \in X$ satisfies the spectral problem

  $$J H''(u_0) v = \lambda v,$$

  where $H''(u_0) : X \to X$ is a self-adjoint Hessian operator.

Main Question

Consider the spectral problem:

$$JH''(u_0)v = \lambda v, \quad v \in X.$$

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and eigenvalues of $H''(u_0)$?

Assumptions of the negative index theory:

- The spectrum of $H''(u_0)$ is strictly positive except for finitely many negative and zero eigenvalues of finite multiplicity.
- The spectrum of $JH''(u_0)$ is purely imaginary except for finitely many unstable eigenvalues.
- Multiplicity of the zero eigenvalue of $JH''(u_0)$ is given by the number of parameters in u_0 (symmetries).

Answer for gradient systems

For a gradient system:

$$\frac{du}{dt} = -F'(u) \quad \Rightarrow \quad \lambda v = -F''(u_0) v,$$

there exists the exact relation between unstable eigenvalues of $-F''(u_0)$ and negative eigenvalues of $F''(u_0)$.

Theorem

The number of unstable eigenvalues of $-F''(u_0)$ is equal to the number of negative eigenvalues of $F''(u_0)$.

What is about Hamiltonian systems?

$$\lambda v = JH''(u_0) v, \quad v \in X.$$

Quadruple Symmetry: If λ is an eigenvalue, so is $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$.

Stability Theorems for Hamiltonian Systems

Consider the spectral stability problem:

$$JH''(u_0)v = \lambda v, \quad v \in X,$$

under the same assumptions on J and $H''(u_0)$.

Stability Theorem [Grillakis–Shatah–Strauss (1990)]

- Assume no symmetries/zero eigenvalues of $H''(u_0)$. If $H''(u_0)$ has no negative eigenvalues, then $JH''(u_0)$ has no unstable eigenvalues and u_0 is linearly and nonlinearly stable.

- Assume zero eigenvalue of $H''(u_0)$ of multiplicity N and related N symmetries/conserved quantities. If $H''(u_0)$ has no negative eigenvalues under N constraints, then $JH''(u_0)$ has no unstable eigenvalues and u_0 is orbitally stable.

Negative Index Theorem [Kapitula–Promislow (2013)]

Assume no symmetries/zero eigenvalues of $H''(u_0)$. Then,

$$N_{\rm re}(JH''(u_0)) + 2N_{\rm c}(JH''(u_0)) + 2N_{\rm im}^-(JH''(u_0)) = N_{\rm neg}(H''(u_0)) < \infty,$$

where

- $N_{\rm re}$ - number of real unstable eigenvalues;
- $2N_{\rm c}$ - number of complex unstable eigenvalues;
- $2N_{\rm im}^-$ - number neutrally stable eigenvalues of negative Krein signature.

Definition

Suppose that $\lambda \in i\mathbb{R}$ is a simple isolated eigenvalue of $JH''(u_0)$ with the eigenvector v. Then, the sign of the quadratic form

$$\langle H''(u_0)v, v\rangle_{L^2} = \lambda \langle J^{-1}v, v\rangle_{L^2}$$

is called the Krein signature of the eigenvalue λ.

Example: two coupled oscillators

Consider energy

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2)$$

The quadratic form for H has four positive eigenvalues.

The two oscillators are stable:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = y_2, \\ \dot{y}_1 = -\omega_1^2 x_1, \\ \dot{y}_2 = -\omega_2^2 x_2, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 + \omega_1^2 x_1 = 0, \\ \ddot{x}_2 + \omega_2^2 x_2 = 0. \end{cases}$$

No negative eigenvalues of H implies no unstable eigenvalues of JH, or

$$N_{\rm unstable}(JH) = 0 = N_{\rm neg}(H)$$

Example: two coupled oscillators

Consider energy

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 - \lambda_2^2 x_2^2)$$

The quadratic form for H has three positive and one negative eigenvalues.

One of the two oscillators is unstable:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = y_2, \\ \dot{y}_1 = -\omega_1^2 x_1, \\ \dot{y}_2 = \lambda_2^2 x_2, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 + \omega_1^2 x_1 = 0, \\ \ddot{x}_2 - \lambda_2^2 x_2 = 0. \end{cases}$$

Negative index count:

$$N_{\rm re}(JH) = 1 = N_{\rm neg}(H)$$

Example: two coupled oscillators

Consider energy

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(-\lambda_1^2 x_1^2 - \lambda_2^2 x_2^2)$$

The quadratic form for H has two positive and two negative eigenvalues.

Both oscillators are unstable:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = y_2, \\ \dot{y}_1 = \lambda_1^2 x_1, \\ \dot{y}_2 = \lambda_2^2 x_2, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 - \lambda_1^2 x_1 = 0, \\ \ddot{x}_2 - \lambda_2^2 x_2 = 0. \end{cases}$$

Negative index count:

$$N_{\rm re}(JH) = 2 = N_{\rm neg}(H)$$

Example: two coupled oscillators

Consider energy

$$H = \frac{1}{2}(y_1^2 - y_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 - \omega_2^2 x_2^2)$$

The quadratic form for H has two positive and two negative eigenvalues.

The two oscillators are nevertheless stable:

$$\left\{ \begin{array}{l} \dot{x}_1 = y_1, \\ \dot{x}_2 = -y_2, \\ \dot{y}_1 = -\omega_1^2 x_1, \\ \dot{y}_2 = \omega_2^2 x_2, \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} \ddot{x}_1 + \omega_1^2 x_1 = 0, \\ \ddot{x}_2 + \omega_2^2 x_2 = 0. \end{array} \right.$$

Negative index count:

$$2N_{\rm im}^-(JH) = 2 = N_{\rm neg}(H)$$

Example: two coupled oscillators

Consider energy

$$H = \frac{1}{2}(y_1^2 - y_2^2) + \omega^2 x_1 x_2$$

The quadratic form for H has two positive and two negative eigenvalues.

The two oscillators are unstable with a quadruplet of complex eigenvalues:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = -y_2, \\ \dot{y}_1 = -\omega^2 x_2, \\ \dot{y}_2 = -\omega^2 x_1, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 + \omega^2 x_2 = 0, \\ \ddot{x}_2 - \omega^2 x_1 = 0, \end{cases} \quad \Rightarrow \quad x_1^{(4)} + \omega^4 x_1 = 0.$$

Negative index count:

$$2N_{\rm c}(JH) = 2 = N_{\rm neg}(H)$$

Integrable PDEs

Many integrable PDEs can be formulated as Hamiltonian dynamical systems in the form:

$$\frac{du}{dt} = J\, H'(u), \quad u(t) \in X$$

where $X \subset L^2$ is the phase space, $J^* = -J$ represents the symplectic structure, and $H : X \to \mathbb{R}$ is the Hamilton function.

Example: Korteweg–de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Hamiltonian system in the form

$$\frac{du}{dt} = \frac{\partial}{\partial x}\,\frac{\delta H}{\delta u}, \quad \text{where} \quad H(u) = \frac{1}{2}\int_{\mathbb{R}} \left[\left(\frac{\partial u}{\partial x}\right)^2 - 2u^3\right] dx.$$

Nonlinear Schrödinger equation

Example: nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + 2|u|^2 u = 0, \quad u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$$

Hamiltonian system in the form

$$\frac{du}{dt} = \mathrm{i}\,\frac{\delta H}{\delta \bar{u}}, \quad \text{where} \quad H(u) = \frac{1}{2}\int_{\mathbb{R}} \left[\left| \frac{\partial u}{\partial x} \right|^2 - 2|u|^4 \right] dx.$$

In real variables (p, q) for $u = p + iq$, the NLS is formulated as

$$\frac{d}{dt}\left[\begin{array}{c} p \\ q \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right] \left[\begin{array}{c} \frac{\delta H}{\delta p} \\ \frac{\delta H}{\delta q} \end{array}\right],$$

where

$$H(u) = \frac{1}{2}\int_{\mathbb{R}} \left[p_x^2 + q_x^2 - 2(p^2 + q^2)^4 \right] dx.$$

Stability of critical points

For the Hamiltonian system

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The defocusing nonlinear Schrödinger equation

The cubic NLS equation

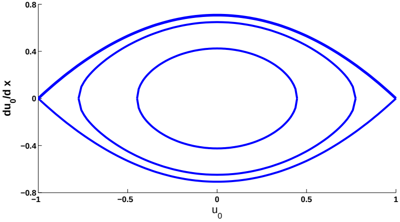
$$i\psi_t + \psi_{xx} - |\psi|^2\psi = 0$$

has long been known for modulational stability of periodic waves.

Periodic waves are of the form $\psi(x,t) = u_0(x)e^{-it}$, where

$$u_0''(x) + (1 - |u_0|^2)u_0 = 0$$

has the exact solution $u_0(x) = \sqrt{1-\mathcal{E}}\,\mathrm{sn}\left(x\frac{\sqrt{1+\mathcal{E}}}{\sqrt{2}}; \sqrt{\frac{1-\mathcal{E}}{1+\mathcal{E}}}\right)$ with $\mathcal{E} \in (0,1)$.



Orbital stability of periodic waves in $H^1_{\rm per}$

Periodic waves are constrained minimizers of energy in $H^1_{\rm per}$:

$$E(\psi) = \int \left[|\psi_x|^2 + \frac{1}{2}(1 - |\psi|^2)^2 \right] dx$$

under fixed values of

$$Q(\psi) = \int |\psi|^2 dx, \quad M(\psi) = \frac{i}{2} \int (\bar{\psi}\psi_x - \psi\bar{\psi}_x) dx,$$

if the period of perturbations coincides with the period of waves.
[Gallay–Haragus (2007)]

However, periodic waves are not constrained minimizers of E if the period of perturbations is N multiple to the period of waves. Moreover, the number of negative eigenvalues of E becomes unbounded if $N \to \infty$.

Main Question: how to justify rigorously modulational stability of the periodic waves with respect to long perturbations?

Orbital stability of periodic waves in $H^2_{\rm per}$

Periodic waves are critical points of the higher-order energy in $H^2_{\rm per}$:

$$R(\psi) = \int \left[|\psi_{xx}|^2 + 3|\psi|^2|\psi_x|^2 + \frac{1}{2}(\bar{\psi}\psi_x + \psi\bar{\psi}_x)^2 + \frac{1}{2}|\psi|^6 \right] dx.$$

They are constrained minimizers of R under fixed Q, M for co-periodic perturbations and are saddle points of R for N-multiple perturbations.

Bottman–Deconinck–Nivala (2011): there exists a range of values for parameter c such that the energy functional $\Lambda_c := R - cE$ is positively definite at the periodic wave u_0.

Gallay–Pelinovsky (2015): For all $\mathcal{E} \in (0,1)$, the second variation of Λ_c at the periodic wave u_0 is nonnegative for every perturbations in H^2 only if $c \in [c_-, c_+]$ with

$$c_\pm := 2 \pm \frac{2\kappa}{1+\kappa^2}, \quad \kappa = \sqrt{\frac{1-\mathcal{E}}{1+\mathcal{E}}}.$$

Moreover, it is strictly positive up to symmetries in (c_-, c_+) if $\mathcal{E}$ is small.

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Theorem (Gallay–P (2015))

For all $\mathcal{E} \in (0,1)$, the second variation of Λ_c at the periodic wave u_0 is nonnegative for every perturbation in H^2 only if $c \in [c_-, c_+]$. Moreover, it is strictly positive up to symmetries in (c_-, c_+) if $\mathcal{E}$ is small.

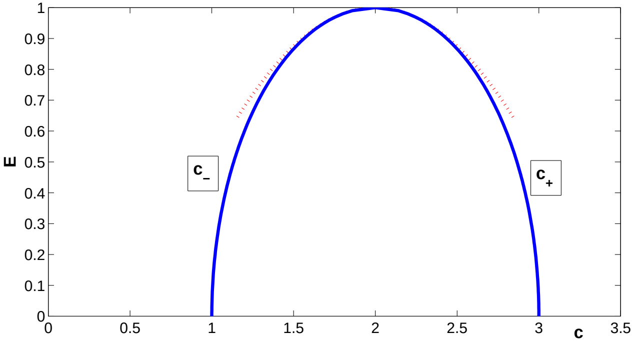

Figure: $(\mathcal{E}, c)$-plane for positivity of the second variation of Λ_c.

A simple perturbative argument

Using the decomposition $\psi = u_0 + u + iv$ with real-valued perturbation functions u and v, we can write

$$\Lambda_c(\psi) - \Lambda_c(u_0) = \langle K_+(c)u, u\rangle_{L^2} + \langle K_-(c)v, v\rangle_{L^2} + \text{cubic terms}$$

where

$$K_+(c)\partial_x u_0 = 0 \quad \text{and} \quad K_-(c)u_0 = 0.$$

If $\mathcal{E} = 1$, we have periodic wave of zero amplitude $u_0 = 0$, for which

$$\begin{aligned} \langle K_\pm(c)u, u\rangle_{L^2} &= \int_{\mathbb{R}} \left[u_{xx}^2 - cu_x^2 + (c-1)u^2\right] dx \\ &= \int \left(u_{xx} + \frac{c}{2}u\right)^2 dx - \left(1 - \frac{c}{2}\right)^2 \int u^2 dx. \end{aligned}$$

Then, $\langle K_\pm(c)u, u\rangle_{L^2} \geq 0$ if $c = 2$. By perturbative computations, one can find (c_-, c_+) near $c = 2$ for $\mathcal{E} < 1$.

Orbital stability of periodic waves in $H^2_{\rm Nper}$

Theorem (Gallay–P (2015))

Assume that $\psi_0 \in H^2_{\rm Nper}$ *and consider the global-in-time solution* ψ *to the cubic NLS equation with initial data* ψ_0*. For any* $\epsilon > 0$*, there is* $\delta > 0$ *s.t. if*

$$\|\psi_0 - u_0\|_{H^2_{\rm Nper}} \leq \delta,$$

then, for any $t \in \mathbb{R}$*, there exist numbers* $\xi(t)$ *and* $\theta(t)$ *such that*

$$\|e^{i(t+\theta(t))}\psi(\cdot + \xi(t), t) - u_0\|_{H^2_{\rm Nper}} \leq \epsilon.$$

Moreover, ξ*,* θ *are continuous and* $|\dot{\xi}(t)| + |\dot{\theta}(t)| \leq C\epsilon$*.*

Remark: N-periodic perturbations are considered to get coercivity of the Lyapunov function, since the gap between zero and first positive eigenvalue shrinks to zero as $N \to \infty$.

The Kadomtsev–Petviashvili (KP) equation

It is a 2D generalization of the Korteweg-de Vries (KdV) equation:

$$(u_t + 6uu_x + u_{xxx})_x = \pm u_{yy}.$$

The plus/minus sign corresponds to KP-I/KP-II equations.
KP stands for B. Kadomtsev and V.I. Petviashvili, who derived this equation in 1970 to study transverse stability of 1D travelling waves.

Each sign is applicable as a model for fluid dynamics:

- KP-I for high surface tension (e.g., oil);
- KP-II for low surface tension (e.g., water).

1D periodic travelling waves

1D wave satisfies the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

Periodic travelling waves $u = \phi(x + ct)$ are found from the ODE:

$$c\phi(x) + 3\phi(x)^2 + \phi''(x) = 0,$$

solutions are available with the Jacobian elliptic function *cn*.

KdV cnoidal waves are linearly and nonlinearly stable:

- N. Bottman, B. Deconinck, DCDS A (2009)
- B. Deconinck, T. Kapitula, Physics Letters A (2010)
- M. Nivala, B. Deconinck, Physica D (2010)

Transverse stability of periodic waves

Transverse stability of periodic waves is determined for small 2D perturbations w:

$$(w_t + cw_x + 6(\phi(x)w)_x + w_{xxx})_x = \pm w_{yy}.$$

or for $w(x,y,t) = W(x)e^{\lambda t + ipy}$ by the spectral problem

$$\lambda W_x + cW_{xx} + 6(\phi(x)W)_{xx} + W_{xxxx} \pm p^2 W = 0.$$

Functional-analytic results in the recent literature:

KP-I: Periodic and solitary waves are transversely unstable [Johnson–Zumbrun (2010); Rousset–Tzvetkov (2011); Hakkaev (2012)]

KP-II: Solitary waves are transversely stable [Mizumachi–Tzvetkov (2012); T. Mizumachi (2015) (2017)]

KP-II: Stability of periodic waves is open [M. Haragus (2010)].

Main result for KP-II

Rewrite the spectral problem as $A_{c,p}(\lambda)W = 0$, where

$$A_{c,p}(\lambda)W := \lambda W_x + cW_{xx} + 6(\phi(x)W)_{xx} + W_{xxxx} - p^2 W.$$

Theorem (M.Haragus–J.Li–D.P, 2017)

For every $p \neq 0$, the linear operator $A_{c,p}(\lambda)$ is invertible in $C_b(\mathbb{R})$ for any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$. Consequently, the periodic travelling wave is transversely spectrally stable with respect to 2D bounded perturbations.

Forgotten results on spectral transverse stability of periodic waves in KP-II:

- E.A. Kuznetsov, M.D. Spector, and G. E. Falkovich, Physica D (1984).
- M.D. Spector, Sov. Phys. JETP (1988).

Eigenfunctions of spectral problem are computed explicitly and completeness of eigenfunction is analyzed formally.

Conserved quantities for KP-II equation

KP-II

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0.$$

is a Hamiltonian system with conserved momentum

$$Q(u) = \frac{1}{2} \int u^2 dxdy$$

and energy

$$E(u) = \frac{1}{2} \int \left[u_x^2 - 2u^3 - (\partial_x^{-1} u_y)^2 \right] dxdy.$$

In particular,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \frac{\delta E}{\delta u}, \quad \text{where} \quad \frac{\delta E}{\delta u} = -u_{xx} - 3u^2 - \partial_x^{-2} u_{yy}.$$

$E(u)$ is sign-indefinite near $u = 0$ $\Rightarrow$ the energy method does not work for global well-posedness of KP-II in energy space.

Transverse spectral stability for periodic perturbations

Let $\phi(x + 2\pi) = \phi(x)$, $c > 1$ be the periodic wave of KdV. Then, it is a critical point of $E(u) - cQ(u)$. Consider the spectral problem

$$A_{c,p}(\lambda)W = \lambda W_x + cW_{xx} + 6(\phi(x)W)_{xx} + W_{xxxx} - p^2 W = 0,$$

for $p \neq 0$ and $\mathrm{Re}(\lambda) > 0$. If $W \in L^2_{\mathrm{per}}(0, 2\pi)$ is a solution for $p \neq 0$, then $W \in \dot{L}^2_{\mathrm{per}}(0, 2\pi)$, the zero-mean subspace of $L^2_{\mathrm{per}}(0, 2\pi)$.

Recall that ∂_x^{-1} is a bounded operator from $\dot{L}^2_{\mathrm{per}}(0, 2\pi)$ to $\dot{L}^2_{\mathrm{per}}(0, 2\pi)$ and rewrite $A_{c,p}(\lambda)W = 0$ formally as

$$\lambda W = \partial_x L_{c,p} W, \quad L_{c,p} := -\partial_x^2 - c - 6\phi(x) + p^2 \partial_x^{-2}.$$

The operator $L_{c,p} : H^2_{\mathrm{per}}(0, 2\pi) \to L^2(0, 2\pi)$ is self-adjoint,
In fact, $L_{c,p}$ is the Hessian operator of $E(u) - cQ(u)$.

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Spectral problem for periodic perturbations

The spectral problem is defined in $\dot{L}^2_{\rm per}(0,2\pi)$,

$$\lambda W = \partial_x L_{c,p} W, \quad L_{c,p} := -\partial_x^2 - c - 6\phi(x) + p^2 \partial_x^{-2}.$$

hence, strictly speaking, we shall write $\Pi_0 L_{c,p} \Pi_0$, where
$\Pi_0 : L^2_{\rm per}(0,2\pi) \to \dot{L}^2_{\rm per}(0,2\pi)$ is the orthogonal projection operator.

Theorem (J.Bronski–M.Johnson–T.Kapitula, 2011)

If $\sigma(\Pi_0 L_{c,p} \Pi_0) \geq 0$, then no $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$ exists.

Let us check the case $c = 1$, when $\phi = 0$. The spectrum of $\Pi_0 L_{c=1,p} \Pi_0$ is

$$\sigma(\Pi_0 L_{c=1,p} \Pi_0) = \{n^2 - 1 - p^2 n^{-2}, \quad n \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$, there is a sufficiently large $p \in \mathbb{R}$ such that
$n^2 - 1 - p^2 n^{-2} < 0$. The theorem above can not be applied.

Spectral problem for periodic perturbations

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Orbital stability for periodic waves of KdV in 1D

The same problem happens in 1D when periodic waves are perturbed by long perturbations. The way to prove orbital stability is to consider the higher-order energy

$$R(u) = \int \left[u_{xx}^2 - 10uu_x^2 + 5u^4\right] dx.$$

which is constant for solutions of the KdV in H^2. The periodic wave ϕ is also a critical point of $R(u) - c^2 Q(u)$ with the associated Hessian operator

$$M_{c,p=0} = \partial_x^4 + 10\partial_x \phi(x) \partial_x - 10c\phi(x) - c^2.$$

Unfortunately, $M_{c,p=0}$ is not positive either. However,...

Proposition (B.Deconinck–T.Kapitula, 2010)

For every $c > 1$, the operator $M_{c,p=0} - bL_{c,p=0}$ is positive for every $b \in (b_-(c), b_+(c))$, where

$$b_-(c) = \left[\frac{5}{3} + \frac{1-2k^2}{3\sqrt{1-k^2+k^4}}\right] c, \quad b_+(c) = \left[\frac{5}{3} + \frac{1+k^2}{3\sqrt{1-k^2+k^4}}\right] c,$$

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A simple perturbative argument

For $c = 1$ and $\phi = 0$, we have

$$\begin{aligned} L_{c=1,p=0} &= -\partial_x^2 - 1, \\ M_{c=1,p=0} &= \partial_x^4 - 1. \end{aligned}$$

Therefore, the linear combination of the two Hessian operators

$$M_{c,p=0} - bL_{c,p=0} = \partial_x^4 + b\partial_x^2 + b - 1 = \left(\partial_x^2 + \frac{b}{2}\right)^2 - \left(1 - \frac{b}{2}\right)^2$$

is positive if $b = 2$. By perturbative computations, one can find a nonempty interval $(b_-(c), b_+(c))$ near $b = 2$ for $c > 1$.

From positivity of the combined Hessian operator and energy conservation of

$$\Lambda_b(u) := [R(u) - c^2 Q(u)] - b[E(u) - cQ(u)], \quad \text{e.g. } b = 2c,$$

orbital stability of 1D periodic waves in the KdV holds in Sobolev space $H^2_{\rm Nper}$ for any N-periodic perturbation.

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Higher-order energy for KP-II equation

Recall the momentum and energy for KP-II:

$$Q(u) = \int u^2 dxdy, \quad E(u) = \int \left[u_x^2 - 2u^3 - (\partial_x^{-1} u_y)^2\right] dxdy.$$

Periodic wave ϕ is a critical point of $E(u) - cQ(u)$.

Proposition (L.Molinet–J-C.Saut–N.Tzvetkov, 2007)

KP-II conserves the higher-order energy in H^2:

$$R(u) = \int \left[u_{xx}^2 - 10uu_x^2 + 5u^4 - \frac{10}{3}u_y^2 + \frac{5}{9}(\partial_x^{-2}u_{yy})^2 + \frac{10}{3}u^2\partial_x^{-2}u_{yy} + ...\right] dxdy.$$

Periodic wave ϕ is a critical point of $R(u) - c^2 Q(u)$. However, no b exists so that ϕ is a minimum of $[R(u) - c^2 Q(u)] - b[E(u) - cQ(u)]$.

New approach - commuting linear operators

Recall the spectral problem in $\dot{L}^2_{\rm per}(0, 2\pi)$:

$$\lambda W = \partial_x L_{c,p} W, \quad L_{c,p} := -\partial_x^2 - c - 6\phi(x) + p^2 \partial_x^{-2}.$$

Let us search for a self-adjoint operator $M_{c,p}$ in $\dot{L}^2_{\rm per}(0, 2\pi)$ such that

$$L_{c,p} \partial_x M_{c,p} = M_{c,p} \partial_x L_{c,p}.$$

Theorem (M.Haragus–J.Li–D.P, 2017)

Assume that $M_{c,p} \geq 0$ and the kernel of $M_{c,p}$ is contained in the kernel of $L_{c,p}$. The spectrum of $\partial_x L_{c,p}$ in $\dot{L}^2_{\rm per}(0, 2\pi)$ is purely imaginary.

Algorithmic search of the commuting operator

From the existence of the higher-order variational problem $R(u) - c^2 Q(u)$ associated with the higher-order energy of KP-II, we have one option for operator $M_{c,p}$:

$$
\begin{aligned}
M_{c,p} &= \partial_x^4 + 10\partial_x \phi(x)\partial_x - 10c\phi(x) - c^2 \\
&\quad - \frac{10}{3} p^2 \left(1 + \phi(x)\partial_x^{-2} + \partial_x^{-1}\phi(x)\partial_x^{-1} + \partial_x^{-2}\phi(x)\right) + \frac{5}{9} p^4 \partial_x^{-4}.
\end{aligned}
$$

Then, $L_{c,p}\partial_x M_{c,p} = M_{c,p}\partial_x L_{c,p}$. However,

Proposition

For every $p \neq 0$, no value of $b \in \mathbb{R}$ exists such that $M_{c,p} - bL_{c,p}$ is positive in $L^2(\mathbb{R})$.

This outcome is related to sign-indefinite properties of $E(u)$ and $R(u)$ at $u = 0$.

Algorithmic search of the commuting operator

Let us search for another operator $M_{c,p}$ to satisfy the commutability relation

$$L_{c,p}\partial_x M_{c,p} = M_{c,p}\partial_x L_{c,p}.$$

By using symbolic computations, we have found

$$M_{c,p} = \partial_x^4 + 10\partial_x \phi(x)\partial_x - 10c\phi(x) - c^2 + \frac{5}{3}p^2\left(1 + c\partial_x^{-2}\right).$$

Then,

$$M_{c,p} - bL_{c,p} = M_{c,p=0} - bL_{c,p=0} + \frac{5}{3}p^2 - \left(b - \frac{5c}{3}\right)p^2\partial_x^{-2}.$$

Proposition

The operator $M_{c,p} - 2cL_{c,p}$ is positive in $L^2(\mathbb{R})$ for every $p \in \mathbb{R}$.

From positivity of $M_{c,p} - 2cL_{c,p}$, we get spectral stability of the periodic travelling wave in the KP-II equation.

Summary

- Spectral stability theory is well-developed for critical points in Hamiltonian systems, when the Hessian operators have finitely many negative eigenvalues.
- Orbital stability holds in Hamiltonian systems if the critical point is a non-degenerate minimum of energy under constraints of fixed mass and momentum.
- For many integrable PDEs (NLS, KdV), one can use higher-order Hamiltonians to conclude on orbital stability of nonlinear waves.
- For the KP-II equation, one can find positive-definite operator unrelated to conserved quantities in order to conclude on spectral stability of nonlinear waves.

Open questions

For defocusing NLS:

- Is a regular way to prove positivity of a linear combination of Hessian operators at the periodic wave?
- Will the approach work with the other higher-order conserved quantities of the integrable hierarchy?

For KP-II:

- How is $M_{c,p}$ related to conserved quantities of the KP-II?
- Can we extend the proof to nonlinear orbital stability of periodic waves in the KP-II?
- Can we find commuting linear operators for non-integrable versions of nonlinear evolution equations?

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