Spectral stability of nonlinear waves in KdV equations

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Stability of nonlinear waves in Hamiltonian systems

Consider an abstract Hamiltonian dynamical system

$$\frac{du}{dt} = J\nabla H(u), \quad u(t) \in X$$

where $X \subset L^2$ is a phase space, $J^+ = -J$ is the symplectic operator, and $H : X \to \mathbb{R}$ is the Hamiltonian function.

- Assume existence of the stationary state (nonlinear wave) $u_0 \in X$ such that $\nabla H(u_0) = 0$.
- Perform linearization at the stationary solution

  $$u(t) = u_0 + ve^{\lambda t},$$

  where λ is the spectral parameter and $v \in X$ satisfies the spectral problem

  $$JD^2 H(u_0)v = \lambda v,$$

  associated with the self-adjoint Hessian operator $D^2 H(u_0)$.

Main Question

Consider the spectral stability problem:

$$JD^2H(u_0)v = \lambda v, \quad v \in X.$$

- Let stationary solutions u_0 decay exponentially as $|x| \to \infty$ (solitary waves, vortices, etc).
- Let the skew-symmetric operator J be invertible
- Let the self-adjoint operator $D^2H(u_0)$ have a positive essential spectrum and finitely many negative eigenvalues.

Question: Is there a relation between unstable eigenvalues of $JD^2H(u_0)$ and negative eigenvalues of $D^2H(u_0)$?

Remark: One-to-one correspondence clearly exists for the gradient system:

$$\frac{du}{dt} = -\nabla F(u) \quad \Rightarrow \quad \lambda v = -D^2F(u_0)v.$$

State of the art

For simplicity, assume a zero-dimensional kernel of $D^2H(u_0)$.
If λ is an eigenvalue, so is $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$.

- **Grillakis, Shatah, Strauss, 1990** Orbital Stability Theory:
  - If $D^2H(u_0)$ has no negative eigenvalue, then $JD^2H(u_0)$ has no unstable eigenvalues.
  - If $D^2H(u_0)$ has an odd number of negative eigenvalues, then $JD^2H(u_0)$ has at least one real unstable eigenvalue.

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- **Kapitula, Kevrekidis, Sandstede, 2004** Negative Index Theory:

$$N_{\mathrm{re}}(JD^2H(u_0))+2N_{\mathrm{c}}(JD^2H(u_0))+2N_{\mathrm{im}}^{-}(JD^2H(u_0)) = N_{\mathrm{neg}}(D^2H(u_0)),$$

where N_{re} is the number of positive real eigenvalues, N_{c} is the number of complex eigenvalues in the first quadrant, and N_{im}^{-} is the number of positive imaginary eigenvalues of negative Krein signature.

Remarks on Krein signature

- Suppose that $\lambda \in i\mathbb{R}$ is a simple isolated eigenvalue of $JD^2H(u_0)$ with the eigenvector v. Then, the sign of

  $$E''_\omega(v) = \langle D^2H(u_0)v, v\rangle_{L^2}$$

  is called the Krein signature of the eigenvalue λ.

- If λ is an eigenvalue of $JD^2H(u_0)$ with $\mathrm{Re}(\lambda) \neq 0$ and an eigenvector v, then

  $$E''_\omega(v) = \langle D^2H(u_0)v, v\rangle_{L^2} = 0.$$

- If λ is a multiple isolated eigenvalue of $JD^2H(u_0)$, then the number $N^-_{\mathrm{im}}(JD^2H(u_0))$ of eigenvalues of "negative Krein signature" has to be introduced via the number of negative eigenvalues of the quadratic form $E''_\omega(v)$ restricted at the invariant subspace of $JD^2H(u_0)$ associated with the eigenvalue λ.

Sharp stability results

Consider the spectral stability problem:

$$L_+ u = -\lambda w, \quad L_- w = \lambda u, \quad u, w \in X,$$

and assume again zero-dimensional kernels of L_+ and L_-.

- Hessian of the energy:

$$E''_\omega(v) = \langle L_+ u, u \rangle_{L^2} + \langle L_- w, w \rangle_{L^2}.$$

- **Pelinovsky, 2005** Sharp Negative Index Theory:

$$\begin{cases} N_{\rm re}^-(JD^2H(u_0)) + N_{\rm c}(JD^2H(u_0)) + N_{\rm im}^-(JD^2H(u_0)) = N_{\rm neg}(L_+), \\ N_{\rm re}^+(JD^2H(u_0)) + N_{\rm c}(JD^2H(u_0)) + N_{\rm im}^-(JD^2H(u_0)) = N_{\rm neg}(L_-), \end{cases}$$

where $N_{\rm re}^+$ ($N_{\rm re}^-$) is the number of positive eigenvalues with positive (negative) quadratic form $\langle L_+ u, u \rangle_{L^2}$.

Example: NLS equation

Consider the nonlinear Schrödinger equation

$$i\psi_t = -\psi_{xx} + V(x)\psi + |\psi|^2\psi,$$

where V is an external potential.

- The stationary state $\psi = \phi e^{-i\omega t}$ is a critical point of the energy:

$$E_\omega(u) = \int_{\mathbb{R}} \left(|u_x|^2 + V|u|^2 - \omega|u|^2 + \frac{1}{2}|u|^4 \right) dx.$$

- The Hessian of the energy is

$$D^2H(u_0) = \begin{bmatrix} -\partial_x^2 + V - \omega + 2|\phi|^2 & \phi^2 \\ \bar{\phi}^2 & -\partial_x^2 + V - \omega + 2|\phi|^2 \end{bmatrix}.$$

- The skew-symmetric operator J is

$$J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Example: KdV equation

Consider the Korteweg–De Vries equation

$$u_t + f'(u)u_x + u_{xxx} = 0,$$

where f is the nonlinear speed.

- The travelling state $u = \phi(x - ct)$ is a critical point of the energy:

$$E_c(u) = \int_{\mathbb{R}} \left(u_x^2 + cu^2 + \int_0^u f(u)du \right) dx.$$

- The Hessian of the energy is

$$D^2 H(u_0) = -\partial_x^2 + c - f'(u).$$

- The skew-symmetric operator $J = \partial_x$ is not invertible and hence violates assumptions of the theory.

Literature background

- Orbital stability theory: Bona–Souganidis–Strauss (1987); Angulo–Nataly (2008); Angula–Scialom–Banquet (2011)
- Evans function and asymptotic stability: Pego–Weinstein (1992); Pego–Weinstein (1994)
- Spectral stability of periodic waves: Haragus–Kapitula (2008); Deconinck–Kapitula (2010)
- Spectral stability of solitary waves: Lin (2008); Kapitula–Stefanov (2012).

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Main Claim: KdV-stability follows immediately from Pontryagin's Invariant Subspace Theorem used in M. Chugunova and D.P., "Count of eigenvalues in the generalized eigenvalue problem", J. Math. Phys. **51** 052901 (2010)

Main Result

Consider the spectral stability problem $\partial_x \mathcal{L} v = \lambda v$, where $\mathcal{L}$ is a self-adjoint operator with a dense domain $D(\mathcal{L})$ in $L^2(\mathbb{R})$. Assume:

- Real-valued $\mathcal{L}$: λ and $\bar{\lambda}$ are eigenvalues
- Hamiltonian symmetry: λ and $-\lambda$ are eigenvalues
- $\mathcal{L} = \mathcal{L}_0 + K_{\mathcal{L}}$, where
  - $\mathcal{L}_0$ is a strongly elliptic unbounded operator with constant coefficients
  - $K_{\mathcal{L}}$ is a relatively compact perturbation of $\mathcal{L}_0$
- There is $c_0 > 0$ such that $\sigma(\mathcal{L}_0) \geq c_0$.
- There are $n(\mathcal{L}) < \infty$ negative eigenvalues of $\mathcal{L}$.
- $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}\{f_0\}$ with $f_0 \in D(\mathcal{L}) \cap \dot{H}^{-1}(\mathbb{R})$.
- $\langle \mathcal{L}^{-1}\phi_0, \phi_0\rangle \neq 0$, where $\phi_0 = \partial_x^{-1} f \in L^2(\mathbb{R})$ and $\langle f_0, \phi_0\rangle = 0$.

Theorem

$$N_{\mathrm{re}}(\partial_x \mathcal{L}) + 2N_{\mathrm{c}}(\partial_x \mathcal{L}) + 2N_{\mathrm{im}}^{-}(\partial_x \mathcal{L}) = n(\mathcal{L}) - n_0,$$

where $n_0 = 1$ *if* $\langle \mathcal{L}^{-1}\phi_0, \phi_0\rangle < 0$ *and* $n_0 = 0$ *if* $\langle \mathcal{L}^{-1}\phi_0, \phi_0\rangle > 0$.

Extended eigenvalue problem

Consider the spectral stability problem:

$$\partial_x \mathcal{L} v = \lambda v, \quad v \in D(\mathcal{L}) \cap \dot{H}^{-1}(\mathbb{R}).$$

Set $v = \partial_x w$, where $w \in D(\mathcal{L}\partial_x) \subset L^2(\mathbb{R})$. Then, the spectral stability problem is extended to the form

$$\mathcal{M} w = -\lambda v, \quad \mathcal{L} v = \lambda w,$$

where $\mathcal{M} := -\partial_x \mathcal{L} \partial_x$.

Lemma

The extended problem has a pair of simple eigenvalues $\pm\lambda_0 \neq 0$ with the eigenvectors

$$(v_0, \pm w_0) \in D(\mathcal{L}) \cap \dot{H}^{-1}(\mathbb{R}) \times D(\mathcal{L}\partial_x)$$

if and only if the KdV spectral problem has a pair of simple eigenvalues $\pm\lambda_0$ with the eigenvectors

$$v_\pm = v_0 \pm \partial_x w_0.$$

Generalized eigenvalue problem

Recall that $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}\{f_0\}$ and that zero eigenvalue of $\mathcal{L}$ is isolated from the rest of the spectrum.

- Let P be the orthogonal projection from $L^2(\mathbb{R})$ to $[\mathrm{span}\{f_0\}]^\perp \subset L^2(\mathbb{R})$.
- If $\lambda \neq 0$, then $w = Pw$, so that we can invert $\mathcal{L}$ and express v as

$$v = \lambda P \mathcal{L}^{-1} P w + v_0, \quad v_0 \in \mathrm{Ker}(\mathcal{L}).$$

- Substituting v, we split the other equation of the system into two parts

$$P\mathcal{M}Pw = -\lambda^2 P\mathcal{L}^{-1}Pw, \quad v_0 = -\frac{1}{\lambda}(I - P)\mathcal{M}Pw,$$

Lemma

The KdV spectral problem problem has a pair of simple eigenvalues $\pm\lambda_0 \neq 0$ with the eigenvectors $v_\pm$ if and only if the generalized eigenvalue problem

$$Aw = \gamma K w, \quad A := P\mathcal{M}P, \quad K := P\mathcal{L}^{-1}P,$$

has a double eigenvalue $\gamma_0 = -\lambda_0^2$ with the eigenvectors $w_\pm = \partial_x^{-1} v_\pm$ in space $\mathcal{H} := D(\mathcal{M}) \cap [\mathrm{span}(f_0)]^\perp \subset L^2(\mathbb{R})$.

Shifted generalized eigenvalue problem

Complication here is that the essential spectrum of $\mathcal{M} = -\partial_x \mathcal{L} \partial_x$ touches zero. As a result, the essential spectrum of $A = P\mathcal{M}P$ also touches zero in the generalized eigenvalue problem:

$$Aw = \gamma K w, \quad w \in \mathcal{H}, \quad \gamma = -\lambda^2,$$

Lemma

Let $A_\delta := A + \delta K$. For small positive values of δ, there is a positive δ-independent constant d_0 such that $\sigma_e(A_\delta) \geq d_0 \delta$.

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Lemma

Let $A_\delta := A + \delta K$. For small positive values of δ, there is a positive δ-independent constant d_0 such that $\sigma_e(A_\delta) \geq d_0 \delta$.

For small positive δ, we obtain a shifted generalized eigenvalue problem

$$(A + \delta K)w = (\gamma + \delta)Kw, \qquad u \in \mathcal{H},$$

and zero is not in the spectrum of neither K nor $A + \delta K$.

Since A and K are self-adjoint in Hilbert space, for small positive δ, we have the orthogonal decomposition:

$$\mathcal{H} = \mathcal{H}_K^- \oplus \mathcal{H}_K^+ = \mathcal{H}_{A_\delta}^- \oplus \mathcal{H}_{A_\delta}^+.$$

Count of negative eigenvalues

Theorem (Chugunova, P.; 2010)

For small positive δ, eigenvalues of the shifted generalized eigenvalue problem are counted as follows:

$$N_p^- + N_n^0 + N_n^+ + N_{c+} = \dim(\mathcal{H}^-_{A_\delta}), \quad (1)$$
$$N_n^- + N_n^0 + N_n^+ + N_{c+} = \dim(\mathcal{H}^-_K), \quad (2)$$

where

- N_p^- (N_n^-) *is the number of negative eigenvalues* γ *whose (generalized) eigenvectors are associated to the non-negative (non-positive) values of the quadratic form* $\langle K\cdot, \cdot\rangle$.
- N_p^+ (N_n^+) *is the number of positive eigenvalues* γ *whose (generalized) eigenvectors are ...*
- N_p^0 (N_n^0) *is the multiplicity of zero eigenvalue whose (generalized) eigenvectors are ...*
- N_{c+} (N_{c-}) *is the number of complex eigenvalues* γ *in the upper (lower) half-plane.*

Application of the count

Count (2) is written as follows

$$N_n^- + N_n^0 + N_n^+ + N_{c^+} = \dim(\mathcal{H}_K^-),$$

- By construction of K, we have $\dim(\mathcal{H}_K^-) = n(\mathcal{L})$
- By definition of N_n^0 as the multiplicity of zero eigenvalue whose eigenvectors are associated to the non-positive values of the quadratic form $\langle K\cdot,\cdot\rangle$, we have $N_n^0 = n_0$, where $n_0 = 1$ if $\langle \mathcal{L}^{-1}\phi_0, \phi_0\rangle < 0$ and $n_0 = 0$ if $\langle \mathcal{L}^{-1}\phi_0, \phi_0\rangle > 0$.
- By symmetries of the spectral stability problem,

$$N_n^- = N_{\rm re}(\partial_x \mathcal{L}), \quad N_n^+ = 2N_{\rm im}^-(\partial_x \mathcal{L}), \quad N_{c^+} = 2N_{\rm c}(\partial_x \mathcal{L}),$$

  which yields the assertion of the main theorem.

Note that count (1) is not used. To use it, we would need to characterize the spectrum of $M = -\partial_x \mathcal{L} \partial_x$, the negative spectrum of A, and the negative spectrum of A_δ. If this is done, it produces the same eigenvalue count.

Example of the fifth-order KdV equation

Consider the fifth-order KdV equation,

$$u_t + u_{xxx} - u_{xxxxx} + 2uu_x = 0,$$

where the energy functional $E(u)$ is defined in $H^2(\mathbb{R})$,

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} \left(u_x^2 + u_{xx}^2 + u^3\right) dx,$$

and the momentum functional is $P(u) = \|u\|^2$.

Travelling waves $u = \phi(x - ct)$ exist as critical points of $E(u) + cP(u)$ with speed c.

Assumption $\sigma(\mathcal{L}_0) \geq c_0 > 0$ is satisfied because

$$c_{\text{wave}}(k) = k^2 + k^4 \geq 0, \quad k \in \mathbb{R}.$$

Then, $c > 0$ for travelling solitary waves and $c + c_{\text{wave}}(k) \geq c > 0$.

Reference: M. Chugunova, D.P., “Two-pulse solutions in the fifth-order KdV equation", DCDS B **8**, 773-800 (2007).

$$\frac{d^4\phi}{dx^4} - \frac{d^2\phi}{dx^2} + c\phi = \phi^2.$$

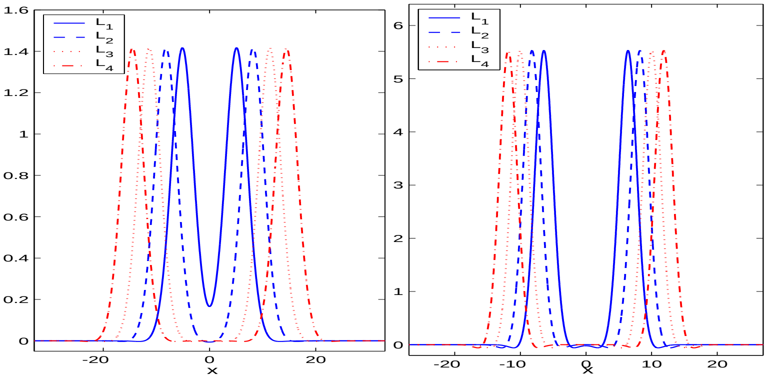

Figure : Numerical approximation of the first four two-pulse solutions.

Counts of eigenvalues:

- One-pulse solutions

$$n(\mathcal{H}) = 1, \quad n_0 = 1, \quad \langle \mathcal{L}_c^{-1}\phi, \phi \rangle = -\frac{1}{2}\frac{d}{dc}\|\phi\|^2 < 0.$$

The one-pulse solution is a ground state (Levandosky, 1999).

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- Two-pulse solutions (even numbers)

$$n(\mathcal{H}) = 2, \quad n_0 = 1, \quad N_{\rm re}(\partial_x \mathcal{L}) = 1.$$

The two-pulse solution is spectrally unstable.

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- Two-pulse solutions (odd numbers)

$$n(\mathcal{H}) = 3, \quad n_0 = 1, \quad N_{\rm im}^-(\partial_x \mathcal{L}) = 1.$$

The two-pulse solution is spectrally stable and the embedded eigenvalue of negative Krein signature persists with respect to perturbations.

First two-pulse solution

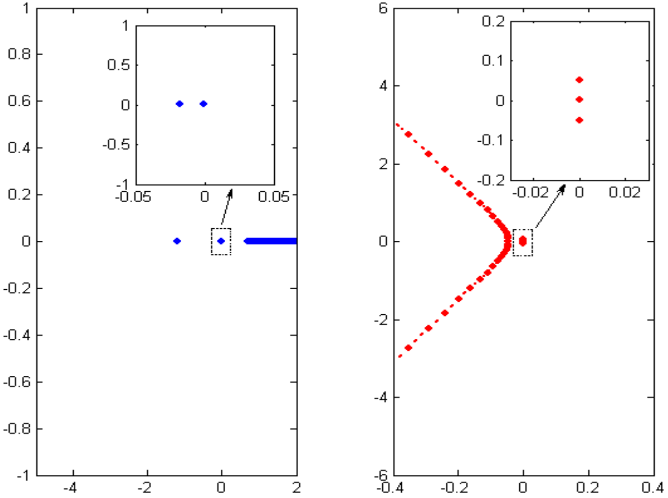

Figure : Numerical approximations of the spectra of operators $\mathcal{L}$ and $\partial_x \mathcal{L}$ for the two-pulse solution with $c = 1$ under an exponential weight $\alpha = 0.04$.

Second two-pulse solution

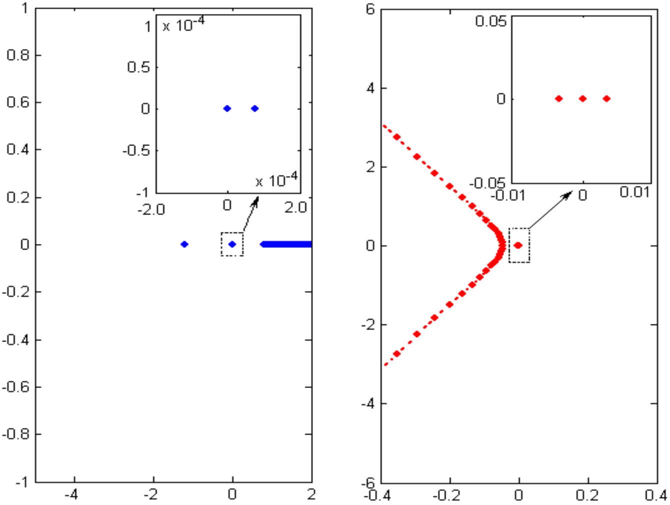

Figure : The same for the second two-pulse solution.

Time-evolution of two-pulse solutions

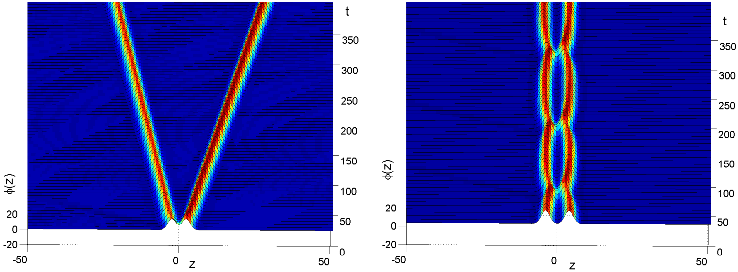

Figure : Initial conditions have different initial separations between the two pulses.

Spectral stability of nonlinear waves - what's next?

- Boussinesq equations with non-invertible J (Yin, 2009); (Stanislavova, Stefanov, 2012);

$$u_{tt} - u_{xx} - u_{ttxx} - (u^2)_{xx} = 0$$

- Dirac equations with sign-indefinite continuous spectrum of $D^2 H(u_0)$ (Comech, 2012); (Boussaid & Comech, 2012)

$$\begin{cases} i(u_t + u_x) + v + \partial_{\bar{u}} W(u, v) = 0, \\ i(v_t - v_x) + u + \partial_{\bar{v}} W(u, v) = 0, \end{cases}$$

where W is the nonlinear potential.