Count of unstable eigenvalues in linearized Hamiltonian systems

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Nonlinear Waves in Fluids in Honor of R. Grimshaw
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Thanks to contributions of

- Marina Chugunova (now at Claremont Graduate University)
- Scipio Cuccagna (now at University of Trieste)
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- Jianke Yang (still at University of Vermont)

Studies with Roger - J.Math.Phys. **36** (1995), 4203

A spectral transform for the intermediate nonlinear Schrödinger equation

Dmitry E. Pelinovsky[a] and Roger H. J. Grimshaw
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(Received 6 April 1995; accepted for publication 28 April 1995)

A new spectral transform system is introduced to solve the initial-value problem for the intermediate nonlinear Schrödinger (INLS) equation describing envelope waves in a deep stratified fluid. The spectral system is a combination of the Zakharov–Shabat linear system and a local Riemann–Hilbert problem in a strip of the complex plane. The inverse scattering transform technique is developed and the Bäcklund–Darboux transformation, soliton solutions and an infinite number of conservation laws are constructed. It is shown that all these properties of the INLS equation are closely related to those of the intermediate long-wave equation.

- Derived integrability of the INLS, which is similar to the ILW equation
- Similar integrability scheme was derived in a geometric context in the works of J. Zhang (1994)
- The INLS equation keeps Y. Matsuno busy for 1997-2007.

Studies with Roger - Physica D **98** (1996), 139

ELSEVIER

Physica D 98 (1996) 139–155

PHYSICA D

An asymptotic approach to solitary wave instability and critical collapse in long-wave KdV-type evolution equations

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Received 31 October 1995; accepted 7 April 1996

Communicated by A.C. Newell

- Predicted blow-up rate in the critical KdV equation.
- Was in contradiction with the numerical results from J.Bona *et al.* (1996)
- The blow-up rate was justified by F. Merle and Y. Martel in 2001.

ELSEVIER

12 May 1997

Physics Letters A 229 (1997) 165–172

PHYSICS LETTERS A

Structural transformation of eigenvalues for a perturbed algebraic soliton potential

Dmitry E. Pelinovsky, Roger H.J. Grimshaw

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Received 24 September 1996; revised manuscript received 30 January 1997; accepted for publication 21 February 1997
Communicated by A.R. Bishop

- Proposed a delicate asymptotic analysis prior to the studies of edge bifurcations and Evans functions for Lax operators.
- Breathers in the Gardner equation were reported in the explicit form.
- Draw attention of researchers to algebraic solitons.

D.E. Pelinovsky and R.H.J. Grimshaw, “Asymptotic Methods in Soliton Stability Theory", *Advances in Fluid Mechanics Series, 12: Nonlinear Instability Analysis*, edited by L.Debnath and S.R.Choudhury, (Computational Mechanics Publications, Southampton, Boston), 245–312 (1997)

- Was said to be the only article, for which the entire volume was purchased.
- Keep me busy for 2000-2010 with rigorous stability theory

Studies with Roger

D.E. Pelinovsky and R.H.J. Grimshaw, “Asymptotic Methods in Soliton Stability Theory", *Advances in Fluid Mechanics Series, 12: Nonlinear Instability Analysis*, edited by L.Debnath and S.R.Choudhury, (Computational Mechanics Publications, Southampton, Boston), 245–312 (1997)

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- and this topic is the subject of this lecture...

Stability of nonlinear waves in Hamiltonian systems

Consider an abstract Hamiltonian dynamical system

$$\frac{du}{dt} = J\nabla H(u), \quad u(t) \in X$$

where $X \subset L^2$ is a phase space, $J^+ = -J$ is the symplectic operator, and $H : X \to \mathbb{R}$ is the Hamiltonian function.

- Assume existence of the stationary state (nonlinear wave) $u_0 \in X$ such that $\nabla H(u_0) = 0$.
- Perform linearization at the stationary solution

$$u(t) = u_0 + ve^{\lambda t},$$

where λ is the spectral parameter and $v \in X$ satisfies the spectral problem

$$JD^2H(u_0)v = \lambda v,$$

associated with the self-adjoint Hessian operator $D^2H(u_0)$.

Main Questions

Consider the spectral stability problem:

$$JD^2H(u_0)v = \lambda v, \quad v \in X.$$

- Let stationary solutions u_0 decay exponentially as $|x| \to \infty$ (solitary waves, vortices, etc).
- Let the skew-symmetric operator J be invertible
- Let the self-adjoint operator $D^2H(u_0)$ have a positive essential spectrum and finitely many negative eigenvalues.

Question: Is there a relation between unstable eigenvalues of $JD^2H(u_0)$ and negative eigenvalues of $D^2H(u_0)$?

Remark: One-to-one correspondence clearly exists for the gradient system:

$$\frac{du}{dt} = -\nabla F(u) \quad \Rightarrow \quad \lambda v = -D^2F(u_0)v.$$

Example: NLS equation

Consider the nonlinear Schrödinger equation

$$i\psi_t = -\psi_{xx} + V(x)\psi + |\psi|^2\psi,$$

where V is an external potential.

- The stationary state $\psi = \phi e^{-i\omega t}$ is a critical point of the energy:

$$E_\omega(u) = \int_{\mathbb{R}} \left(|u_x|^2 + V|u|^2 - \omega|u|^2 + \frac{1}{2}|u|^4 \right) dx.$$

- The self-adjoint operator $D^2H(u_0)$ is the Hessian of the energy:

$$E''_\omega(v) = \langle [v, \bar{v}], D^2H(u_0)[v, \bar{v}]^T \rangle_{L^2},$$

with

$$D^2H(u_0) = \begin{bmatrix} -\partial_x^2 + V - \omega + 2|\phi|^2 & \phi^2 \\ \bar{\phi}^2 & -\partial_x^2 + V - \omega + 2|\phi|^2 \end{bmatrix}.$$

- The symplectic operator J is

$$J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Example: KdV equation

Consider the Korteweg–De Vries equation

$$u_t + f'(u)u_x + u_{xxx} = 0,$$

where f is the nonlinear speed.

- The travelling state $u = \phi(x - ct)$ is a critical point of the energy:

$$E_c(u) = \int_{\mathbb{R}} \left(u_x^2 + cu^2 + \int_0^u f(u)du \right) dx.$$

- The self-adjoint operator $D^2H(u_0)$ is the Hessian of the energy:

$$E_c''(v) = \langle v, D^2H(u_0)v \rangle_{L^2},$$

  with

$$D^2H(u_0) = -\partial_x^2 + c - f'(u).$$

- The symplectic operator $J = \partial_x$ is not invertible and hence violates assumptions of the theory.

Example: nonlinear Dirac equation

Consider the nonlinear Dirac equation

$$\begin{cases} i(u_t + u_x) + v + \partial_{\bar{u}} W(u,v) = 0, \\ i(v_t - v_x) + u + \partial_{\bar{v}} W(u,v) = 0, \end{cases}$$

where W is the nonlinear potential.

- The stationary state $u = \phi(x) e^{-i\omega t}$ and $v = \psi(x) e^{-i\omega t}$ is a critical point of the energy:

$$E_\omega(u,v) = \int_{\mathbb{R}} (u_x \bar{u} - u \bar{u}_x - v_x \bar{v} + v \bar{v}_x + v\bar{u} + u\bar{v})\, dx + \int_{\mathbb{R}} \left(\omega |u|^2 + \omega |v|^2 - W(u,v)\right) dx.$$

- The self-adjoint operator $D^2 H(u_0)$ is sign-indefinite at the continuous spectrum and hence violates assumptions of the theory.

State of the art

Consider the spectral stability problem:

$$JD^2H(u_0)v = \lambda v, \quad v \in X.$$

For simplicity, assume a zero-dimensional kernel of $D^2H(u_0)$.
If λ is an eigenvalue, so is $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$.

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- **Grillakis, Shatah, Strauss, 1990** Orbital Stability Theory:
  - If $D^2H(u_0)$ has no negative eigenvalue, then $JD^2H(u_0)$ has no unstable eigenvalues.
  - If $D^2H(u_0)$ has an odd number of negative eigenvalues, then $JD^2H(u_0)$ has at least one real unstable eigenvalue.

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    - If $D^2H(u_0)$ has an odd number of negative eigenvalues, then $JD^2H(u_0)$ has at least one real unstable eigenvalue.
- **Kapitula, Kevrekidis, Sandstede, 2004** Negative Index Theory:

$$N_{\mathrm{re}}(JD^2H(u_0)) + 2N_{\mathrm{c}}(JD^2H(u_0)) + 2N_{\mathrm{im}}^-(JD^2H(u_0)) = N_{\mathrm{neg}}(D^2H(u_0)),$$

where N_{re} is the number of positive real eigenvalues, N_{c} is the number of complex eigenvalues in the first quadrant, and N_{im}^- is the number of positive imaginary eigenvalues of negative Krein signature.

Remarks on Krein signature

- Suppose that $\lambda \in i\mathbb{R}$ is a simple isolated eigenvalue of $JD^2H(u_0)$ with the eigenvector v. Then, the sign of
$$E''_\omega(v) = \langle D^2H(u_0)v, v\rangle_{L^2}$$
is called the Krein signature of the eigenvalue λ.
- If λ is an eigenvalue of $JD^2H(u_0)$ with $\mathrm{Re}(\lambda) \neq 0$ and an eigenvector v, then
$$E''_\omega(v) = \langle D^2H(u_0)v, v\rangle_{L^2} = 0.$$
- If λ is a multiple isolated eigenvalue of $JD^2H(u_0)$, then the number $N^-_{\mathrm{im}}(JD^2H(u_0))$ of eigenvalues of “negative Krein signature" has to be introduced via the number of negative eigenvalues of the quadratic form $E''_\omega(v)$ restricted at the invariant subspace of $JD^2H(u_0)$ associated with the eigenvalue λ.

Further decompositions

For the NLS equation,

$$i\psi_t = -\psi_{xx} + V(x)\psi + |\psi|^2\psi,$$

the linearization $\psi = \phi e^{-i\omega t} + v e^{-i\omega t + \lambda t}$ yields

$$JD^2H(u_0)\begin{bmatrix} v \\ \bar{v} \end{bmatrix} = \lambda \begin{bmatrix} v \\ \bar{v} \end{bmatrix}, \quad v \in X,$$

where

$$D^2H(u_0) = \begin{bmatrix} -\partial_x^2 + V - \omega + 2|\phi|^2 & \phi^2 \\ \bar{\phi}^2 & -\partial_x^2 + V - \omega + 2|\phi|^2 \end{bmatrix}, \quad J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

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If the stationary state ϕ is chosen to be real-valued, then rotation of coordinates for $v = u + iw$ gives the linear eigenvalue problem in the form

$$JD^2H(u_0)\begin{bmatrix} u \\ w \end{bmatrix} = \lambda \begin{bmatrix} u \\ w \end{bmatrix},$$

where

$$D^2H(u_0) = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and $L_\pm = -\partial_x^2 + V - \omega + (2 \pm 1)\phi^2$.

Sharp stability results

Consider the spectral stability problem:

$$L_+ u = -\lambda w, \quad L_- w = \lambda u, \quad u, w \in X,$$

and assume again zero-dimensional kernels of L_+ and L_-.

- Hessian of the energy:

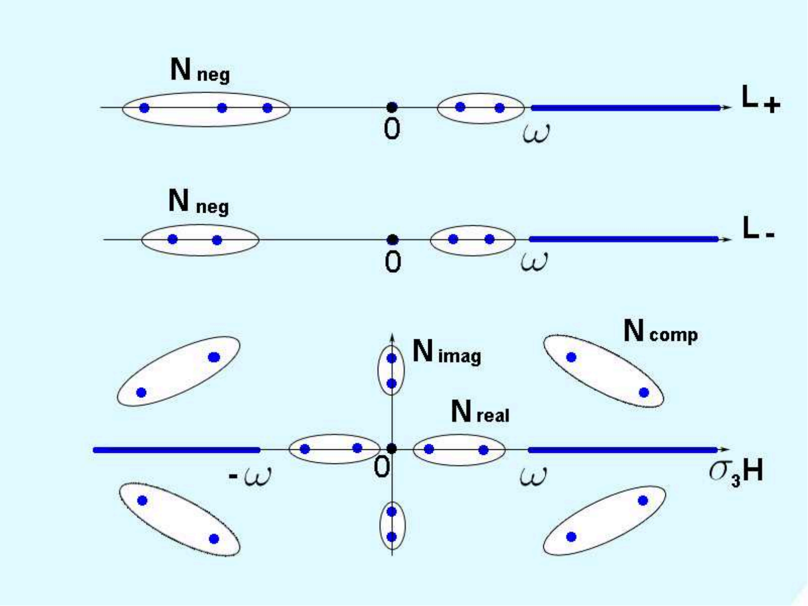
$$E''_\omega(v) = \langle D^2 H(u_0)[v, \bar{v}]^T, [v, \bar{v}]^T \rangle_{L^2} = \langle L_+ u, u \rangle_{L^2} + \langle L_- w, w \rangle_{L^2}.$$

- **Pelinovsky, 2005** Sharp Negative Index Theory:

$$\begin{cases} N_{\rm re}^-(JD^2H(u_0)) + N_{\rm c}(JD^2H(u_0)) + N_{\rm im}^-(JD^2H(u_0)) = N_{\rm neg}(L_+), \\ N_{\rm re}^+(JD^2H(u_0)) + N_{\rm c}(JD^2H(u_0)) + N_{\rm im}^-(JD^2H(u_0)) = N_{\rm neg}(L_-), \end{cases}$$

where $N_{\rm re}^+$ ($N_{\rm re}^-$) is the number of positive eigenvalues with positive (negative) quadratic form $\langle L_+ u, u \rangle_{L^2}$.

Graphical illustration



Remarks on the generalized eigenvalue problem

Consider the spectral stability problem:

$$L_+ u = -\lambda w, \quad L_- w = \lambda u, \quad u, w \in X,$$

- If L_- is invertible and the inverse of L_- is bounded, then we set $w = L_-^{-1} u$ and cast the linear stability problem as a generalized eigenvalue problem

$$L_+ u = \gamma L_-^{-1} u, \quad \gamma = -\lambda^2, \quad u \in X,$$

where both L_+ and L_-^{-1} are self-adjoint.

$$\lambda \in i\mathbb{R} \;\Rightarrow\; \gamma \in \mathbb{R}_+, \quad \lambda \in \mathbb{R} \;\Rightarrow\; \gamma \in \mathbb{R}_-, \quad \lambda \in \mathbb{C} \;\Rightarrow\; \gamma \in \mathbb{C}.$$

Remarks on the generalized eigenvalue problem

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- The generalized eigenvalue problem occurs in the stability of equilibrium states in the finite-dimensional Hamiltonian systems associated with

$$E = \frac{1}{2} \sum_{i,j=1}^{N} M_{i,j} \frac{dx_i}{dt} \frac{dx_j}{dt} + V(x_1, ..., x_N).$$

  If M is positive, then L_-^{-1} is positive and the Sharp Negative Index Theory yields

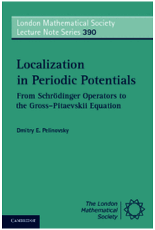
$$N_{\rm re}^-(JD^2H(u_0)) = N_{\rm neg}(L_+), \quad N_{\rm c}(JD^2H(u_0)) = N_{\rm im}^-(JD^2H(u_0)) = 0,$$

  which is the same as for the gradient systems.

Elements of analysis

1. **Calculus of constrained Hilbert spaces** to deal with the symmetries of the Hamiltonian system (kernels of the linearized operators).
2. **Sylvester's Law of Inertia** to count negative eigenvalues of the Hessian operators.
3. **Pontryagin's Invariant Subspace Theorem** to close the count.

Main references:
S. Cuccagna, D.P., V. Vougalter, Comm. Pure Appl. Math. **58**, 1 (2005)
M. Chugunova, D.P., J. Math. Phys. **51**, 052901 (2010)
D.P., *Localization in Periodic Potentials* (Cambridge University Press, 2011)



Null space of the spectral stability problem

Consider again the spectral stability problem for the NLS equation:

$$L_+ u = -\lambda w, \quad L_- w = \lambda u, \quad u, w \in X,$$

and $L_\pm = -\partial_x^2 + V - \omega + (2 \pm 1)\phi^2$.

Because of the gauge invariance (a symmetry of the Hamiltonian system), L_- has a nontrivial kernel as $L_- \phi = 0$ is equivalent to the stationary equation

$$-\phi'' + V(x)\phi + \phi^3 = \omega\phi.$$

The one-dimensional kernel of $D^2 H(u_0)$ produces a two-dimensional generalized kernel of $J D^2 H(u_0)$ as the generalized eigenvector $L_+ \partial_\omega \phi = \phi$ is equivalent to the derivative equation

$$\left(-\partial_x^2 + V(x) - \omega + 3\phi^2(x)\right) \frac{\partial \phi}{\partial \omega} = \phi.$$

Moreover, if $V(x) \equiv 0$, then L_+ has also a nontrivial kernel: $L_+ \phi'(x) = 0$.

Calculus of constrained Hilbert spaces

Question: How to invert operators L_+ and L_- and to cast the spectral stability problem as the generalized eigenvalue problem?

$$L_+ u = -\lambda w, \quad L_- w = \lambda u, \quad \Rightarrow \quad L_+ u = (-\lambda^2) L_-^{-1} u.$$

Answer: Use constrained Hilbert spaces for $\lambda \neq 0$. For instance, if $\mathrm{Ker}(L_-) = \mathrm{span}\{\phi\}$, then $u \in L^2_c$, where

$$L^2_c = \left\{ u \in L^2 : \quad \langle u, \phi \rangle_{L^2} = 0 \right\}.$$

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$$L^2_c = \left\{ u \in L^2 : \quad \langle u, \phi \rangle_{L^2} = 0 \right\}.$$

Theorem

Let $L : X \to L^2$ be a self-adjoint operator with $\sigma_e(L) > 0$ and $\dim \sigma_p(L) < \infty$. Assume non-degeneracy conditions $0 \notin \sigma_p(L)$ and $\langle L^{-1} \phi, \phi \rangle_{L^2} \neq 0$. Then,

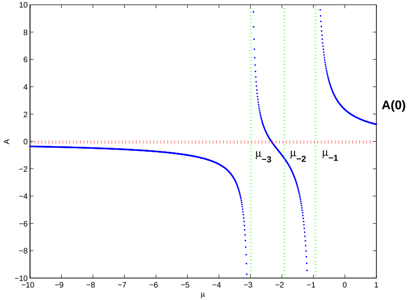
$$N_{\mathrm{neg}}(L|_{L^2_c}) = N_{\mathrm{neg}}(L) - N_{\mathrm{neg}}(\langle L^{-1} \phi, \phi \rangle_{L^2}).$$

Note that if $L = L_+$, then $\langle L_+^{-1} \phi, \phi \rangle_{L^2} = \langle \partial_\omega \phi, \phi \rangle_{L^2} = \frac{1}{2} \frac{d}{d\omega} \|\phi\|^2_{L^2}$.

Consider the quantity:

$$A(\mu) := \langle (\mu - L)^{-1}\phi, \phi \rangle_{L^2}, \quad \mu \notin \sigma(L).$$

- $A(\mu) \to -0$ as $\mu \to -\infty$.
- $A'(\mu) = -\|(\mu - L)^{-1}\phi\|^2_{L^2} < 0$ for all $\mu \notin \sigma(L)$
- Let $\mu = \mu_0 \in \sigma_p(L)$ with the eigenfunction v_0.
  - If $v_0 \notin L^2_c$, then $A(\mu)$ has a simple pole at μ_0.
  - If $v_0 \in L^2_c$, then $A(\mu)$ is continuous at μ_0.
- If $A(\mu_*) = 0$ for any $\mu_* \notin \sigma_p(L)$, then there is $v_0 \in L^2_c$ such that $(L - \mu_*)v_0 = \phi$ (μ_* is an eigenvalue of L under the constraint).
- $A(0) = -\langle L^{-1}\phi, \phi \rangle_{L^2} \neq \{0, \infty\}$ under non-degeneracy conditions.



Consider again the spectral stability problem

$$L_+ u = -\lambda w, \quad L_- w = \lambda u, \quad \Rightarrow \quad L_+ u = (-\lambda^2) L_-^{-1} u, \quad u \in X \subset L_c^2.$$

Assume that L_- is strictly positive, hence L_-^{-1} is a bounded operator.

Let us represent $L_- := SS^*$, where $S : \mathrm{Dom}(S) \to L^2$ is an invertible operator such that $\langle Mu, u \rangle_{L^2} = \|S^* u\|_{L^2}^2 > 0$. The generalized eigenvalue problem becomes now

$$u = Sv \quad \Rightarrow \quad S^* L_+ S v = \gamma v, \quad \gamma = -\lambda^2,$$

where $\tilde{L}_+ = S^* L_+ S$ is self-adjoint.

Sylvester's Law of Inertia

Consider again the spectral stability problem

$$L_+ u = -\lambda w, \quad L_- w = \lambda u, \quad \Rightarrow \quad L_+ u = (-\lambda^2) L_-^{-1} u, \quad u \in X \subset L^2_c.$$

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where $\tilde{L}_+ = S^* L_+ S$ is self-adjoint.

Theorem

Let $L : X \to L^2$ be a self-adjoint operator with $\sigma_e(L) > 0$ and $\mathrm{dim}\sigma_p(L) < \infty$. Let $S : \mathrm{Dom}(S) \to L^2$ be invertible. Then,

$$N_{\rm neg}(S^* L S) = N_{\rm neg}(L).$$

Proof of the Theorem

If λ_k is an eigenvalue of L with the eigenfunction u_k, then

$$Q_L(u) := \langle Lu, u\rangle_{L^2} = \sum_{k=1}^{n(L)} \lambda_k |c_k|^2 + Q_L(P_c u)$$

where $n(L) = N_{\text{neg}}(L)$ and $Q_L(P_c u) > 0$ for any $u \neq 0$.

If γ_k is an eigenvalue of $S^* LS$ with the eigenfunction v_k, then

$$Q_L(u) = \langle Lu, u\rangle_{L^2} = \langle S^* LSv, v\rangle_{L^2} = \sum_{k=1}^{n(S^* LS)} \gamma_k |\tilde{c}_k|^2 + Q_L(\tilde{P}_c u)$$

where $n(S^* LS) = N_{\text{neg}}(S^* LS)$ and $Q_L(\tilde{P}_c u) > 0$ for any $u \neq 0$.

The rest of the proof follows Sylvester's Law of Inertia: if the quadratic form is diagonalized by two different ways, the number of negative terms in the two sums of squares is the same.

Further Extensions

Consider again the spectral stability problem

$$L_+ u = -\lambda w, \quad L_- w = \lambda u, \quad \Rightarrow \quad L_+ u = (-\lambda^2) L_-^{-1} u, \quad u \in X \subset L^2_c.$$

Assume that L_- is not positive but L_-^{-1} is a bounded operator.

- Complex eigenvalues $\gamma = -\lambda^2$ may exist. Each simple eigenvalue increases the number of negative terms in the sum of squares for $Q_{L_+}(u)$ by one.
- Positive eigenvalues $\gamma = -\lambda^2$ may correspond to the negative terms in the sum of squares for $Q_{L_+}(u)$.
- Multiple (defective) eigenvalues $\gamma = -\lambda^2$ may exist, which also increases the number of negative terms in the sum of squares for $Q_{L_+}(u)$.

All together, we obtain the count formula:

$$N^-_{\rm re}(JD^2H(u_0)) + N_{\rm c}(JD^2H(u_0)) + N^-_{\rm im}(JD^2H(u_0)) = N_{\rm neg}(L_+)$$

The hardest part in the proof of this formula is to prove that $Q_{L_+}(\tilde{P}_c u) > 0$.

Example of two coupled NLS equations

Consider the system of two coupled NLS equations:

$$\begin{aligned} iu_t + u_{xx} + \left(|u|^2 + \chi|v|^2\right) u &= 0, \\ iv_t + v_{xx} + \left(\chi|u|^2 + |v|^2\right) v &= 0, \end{aligned}$$

where $\chi > 0$ is the coupling constant.

Soliton solutions are given by

$$u = U(x)e^{it}, \quad v = V(x)e^{i\omega t},$$

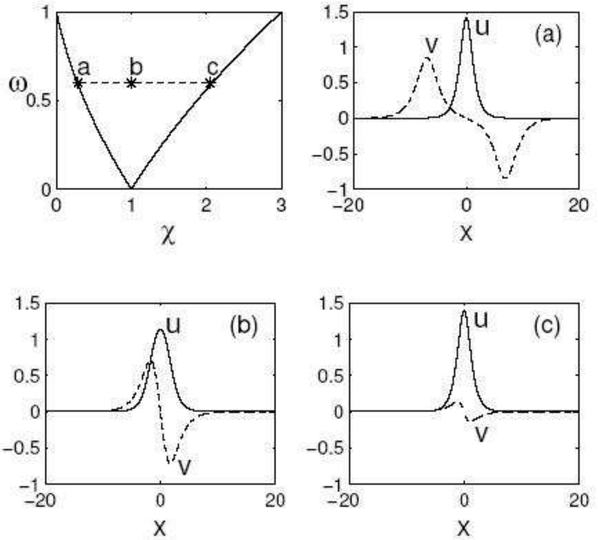
where $\omega > 0$ is the soliton propagation constant.

Consider families of solutions, for which $U(x) > 0$ for all $x \in \mathbb{R}$ and

$$V(x) \ \text{ has } \ n \ \text{ zeros on } \ \mathbb{R}.$$

Reference: D.P., J. Yang, Stud. Appl. Math. **115** (2005), 109–137.

Example $n = 1$



Stability of solution families

The stationary system:

$$\begin{aligned} U'' - U + \left(U^2 + \chi V^2\right) U &= 0, \\ V'' - \omega V + \left(\chi U^2 + V^2\right) v &= 0. \end{aligned}$$

- Lyapunov-Schmidt reductions near local bifurcation boundary

  $$U(x) = \sqrt{2}\,\mathrm{sech} x + \mathcal{O}(\epsilon^2), \quad V(x) = \epsilon \phi_n(x) + \mathcal{O}(\epsilon^3), \quad \omega = \omega_n + \mathcal{O}(\epsilon^2),$$

  where (ω_n, ϕ_n) is an eigenvalue–eigenfunction pair of

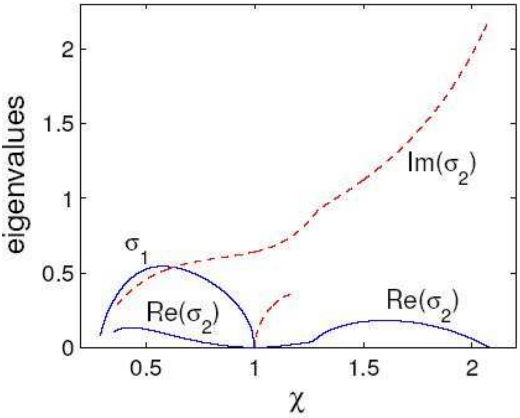
  $$\left(-\frac{d^2}{dx^2} - 2\chi\, \mathrm{sech}^2(x) \right) \phi_n = -\omega_n \phi_n.$$

  The eigenvalue exists for $\chi > \chi_n = \frac{n(n+1)}{2}$.

- For small positive ϵ, one can compute exactly:

  $$\left\{ \begin{array}{l} N_{\mathrm{re}}^-(JD^2H(u_0)) + N_{\mathrm{c}}(JD^2H(u_0)) + N_{\mathrm{im}}^-(JD^2H(u_0)) = n, \\ N_{\mathrm{re}}^+(JD^2H(u_0)) + N_{\mathrm{c}}(JD^2H(u_0)) + N_{\mathrm{im}}^-(JD^2H(u_0)) = n. \end{array} \right.$$

Example $n = 1$



$$N_c(JD^2H(u_0)) = \begin{cases} 1, & \chi_1 < \chi < 1, \\ 1, & 1 < \chi < \chi_2, \end{cases} \qquad N_{re}^-(JD^2H(u_0)) = \begin{cases} 1, & \chi_1 < \chi < 1, \\ 0, & 1 < \chi < \chi_2. \end{cases}$$

Symmetry for $\chi = 1$

If $\chi = 1$, the coupled NLS equations yield the Manakov system

$$\begin{aligned} iu_t + u_{xx} + \left(|u|^2 + |v|^2\right) u &= 0, \\ iv_t + v_{xx} + \left(|u|^2 + |v|^2\right) v &= 0, \end{aligned}$$

which has an additional symmetry of rotations of u relative to v.

If soliton solutions are given by

$$u = U(x)e^{it}, \quad v = V(x)e^{i\omega t},$$

with $\omega \neq 1$, then the linearized stability problem has the exact solution

$$\mathbf{u} = \begin{bmatrix} -V \\ U \end{bmatrix}, \quad \mathbf{w} = \mp i \begin{bmatrix} V \\ U \end{bmatrix}, \quad \lambda = \pm i(1 - \omega^2),$$

which have negative Krein signature if

$$\langle L_+ \mathbf{u}, \mathbf{u} \rangle_{L^2} = -(1 - \omega^2) \left(\|U\|^2_{L^2} - \|V\|^2_{L^2} \right) < 0.$$

As a result,

$$N_c(JD^2H(u_0)) = 0, \quad N^-_{\text{im}}(JD^2H(u_0)) = 1, \quad \text{for} \;\; \chi = 1,$$

which explains stability of soliton solutions in the Manakov system.

Stability and spectrum of nonlinear waves - what's next?

- KdV and Boussinesq equations with non-invertible J
  (Yin, 2009); (Stanislavova, Stefanov, 2012);
- Dirac equations with sign-indefinite continuous spectrum of $D^2H(u_0)$
  (Comech, 2012); (Boussaid & Cuccagna, 2012)
- Asymptotic stability via inverse scattering
  (Deift & Park, 2011); (Mizumachi & Pelinovsky, 2012)
- Asymptotic stability and scattering via functional analysis
  (Bambussi & Cuccagna, 2010); (Cuccagna, 2012)